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On distributions invariant with respect to some linear transformations

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Abstract. The present paper deals with distributions invariant with respect to the identity component (denoted by G) of the group of linear transformations preserving an arbitrary non-degenerate quadratic form.

There are two different characterizations of such distributions presented in this paper. One of them expresses the invariance in differential terms; i.e., G-invariant distributions are represented as solutions of certain systems of first order linear differential equations.

The other characterization given by Theorem 5 establishes a one-to-one correspondence between any G-invariant distribution defined outside the origin and a certain number (determined by the signature of the quadratic form in question) of distributions on the real line satisfying some compatibility conditions.

The special case of Theorem 5 in which G is the group of proper Lorentz transformations is known in the literature as Methée theorem.

This paper is a result of lectures on Fourier transformation and linear differential equations held in the years 1974/1975 by Zofia Szmydt at the Warsaw University. While attending this lecture I became interested in one of the problems sugested to me by Z. Szmydt: take task of finding a fundamental solution of the wave operator using methods to some extent similar to those presented during the lectures and applied there to the Laplace operator(1).

Soon, the problem of characterization of distributions invariant with respect to the group G of proper Lorentz transformations (G-invariant) became important. In the literature such a characterization is known as Methée theorem [2]. The proof given by Methée is not elementary; there is also another version of Methée theorem given by L. Gårding and J. E. Roos. An outline of a proof of this theorem is contained in [1].

The present paper consists of two parts. In Part 1, dealing with Lorentz invariant distributions, I present an elementary proof of Methée

(1) See [3], § 37 and 38 or [4].

theorem, emphasizing its geometrical interpretation([2]). I also give another characterization of G-invariant distributions. It is covered by Lemma 2 and Theorem 3. In Theorem 3 differential conditions are determined under which any rotation invariant distribution([3]) is G-invariant. This theorem finds an important application in [5].

In the second part of this paper we are concerned with distributions invariant with respect to the identity component of the group of linear transformations preserving the quadratic form:

$$\sum_{i=1}^{m} t_i^2 - \sum_{j=1}^{n} x_j^2.$$

In Theorem 5 we present a characterization of such distributions. Then, applying this theorem, we give a complete description of distributions invariant with respect to the identity component of the group of linear transformations which preserve an arbitrary non-degenerate symmetric quadratic form on E^k, $k \geqslant 2$. As I have mentioned at the begining, the starting point for this paper was to give basis for a natural construction of a fundamental solution of the wave operator. This construction is presented in paper [5] written by Z. Szmydt and me.

This aim, entirely covered by Part 1 of the present paper, explains the introduction of some facts([4]) in Part 1 which are not directly connected with the problem of characterization of G-invariance, but still they are useful in [5]. On the other hand, not the whole of the material presented in Part 1 is necessary for reading paper [5]. An adequate choice is made in [6].

In the sequel we use the notation introduced in [4], Section 1.

I wish to thank Z. Szmydt for her help in the preparation of this paper.

1. Definition 1. Let G denote the set of all linear transformations of E^{n+1} preserving the quadratic form $t^2 - |x|^2$ and such that, if $(g_{ij})_{i,j=0,\ldots,n}$ is the matrix of such a transformation, then $g_{00} > 0$ and $\det(g_{ij}) = 1$. It is not difficult to see that G together with the operation of superposition of mappings forms a group (also denoted by G). This group is called the *group of proper Lorentz transformations*([5]).

([2]) See Lemma 1 (on the rectification of orbits) and footnote ([17]).

([3]) See Definition 2.

([4]) See Property 3, p. 268.

([5]) G is the identity component of the group of Lorentz transformations, i.e., the group of all linear transformations of E^{n+1} preserving $t^2 - |x|^2$, with the topology induced from the space of $(n+1) \times (n+1)$ matrices.

The following are some special transformations belonging to G:

DEFINITION 2. By σ_β (for $|\beta| < 1$) we denote the transformation given by the formula:

$$\sigma_\beta(t, x) = \left(\frac{t+\beta x_1}{\sqrt{1-\beta^2}}, \frac{\beta t + x_1}{\sqrt{1-\beta^2}}, x_2, \ldots, x_n\right) \text{(}^6\text{)}, \qquad (t, x) \in E^{n+1}.$$

Linear transformations of the form $(\mathrm{id}_t \times \tilde{R})$, where $\tilde{R}$ is an orientation preserving linear transformation of E^n given by an orthonormal matrix, will be called *rotations in* E^n (shortly: *rotations*) and denoted by R.

It is not difficult to prove the following two propositions:

PROPOSITION 1. *Let* σ_α, σ_β *be as in Definition* 2. *Then* $\sigma_\alpha \circ \sigma_\beta = \sigma_\gamma$, *where* $\gamma = \tanh(\tanh^{-1}\alpha + \tanh^{-1}\beta)$.

PROPOSITION 2. *Every element of* G *can be represented as the composite* $R_1 \circ \sigma_\beta \circ R_2$ *of suitably chosen special transformations.*

Notice, that, by Proposition 2, the group G can be defined equivalently as the group generated by special transformations defined by Definition 2.

Since σ_β is symmetric, we have by Proposition 2:

PROPOSITION 3. *Let* $(g_{ij})_{i,j=0,\ldots,n}$ *be the matrix of* $g \in G$. *Then the transformation given by the transposed matrix* $(g_{ji})_{i,j=0,\ldots,n}$ *also belongs to* G.

DEFINITION 3. (i) Let Λ be an open set in E^{n+1}. Λ is called *G-invariant* if it is invariant with respect to every transformation $g \in G$.

(ii) A distribution $u \in D'(\Lambda)$ (where Λ is G-invariant) is called *invariant* with respect to G (G-invariant) if it is invariant with respect to every $g \in G$.

(iii) Let Λ be an open G-invariant subset of E^{n+1}. An operation P: $C_0^\infty(\Lambda) \to C_0^\infty(E^1)$ is said to be *G-invariant* if $P(\varphi \circ g) = P(\varphi)$ for $\varphi \in C_0^\infty(\Lambda)$ and $g \in G$.

Let us distinguish the following G-invariant sectors in E^{n+1}:

$V_- = \{(t, x) \colon |x|^2 < t^2, t < 0\}$ the backward light cone,

$V_+ = \{(t, x) \colon |x|^2 < t^2, t > 0\}$ the forward light cone,

$$\Omega_1 = E^{n+1} \setminus \overline{V}_-, \qquad \Omega_2 = E^{n+1} \setminus \overline{V}_+.$$

PROPOSITION 4. *Let* $n > 3$. *The following are minimal G-invariant subsets called the orbits of* G:

$O_s = \{(t, x) \colon t^2 - |x|^2 = s\}$ *for* $s \in E^1$, $s < 0$.

$O_s^+ = \{(t, x) \colon t^2 - |x|^2 = s,\ t > 0\}$ *for* $s > 0$,

$O_s^- = \{(t, x) \colon t^2 - |x|^2 = s, t < 0\}$ *for* $s > 0$,

$O_0^- = \{(t, x) \colon t^2 - |x|^2 = 0, t < 0\}$,

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(6) The distinguished role of x_1 is of no importance. All the considerations remain valid with x_1 replaced by any x_i, $i = 2, 3, \ldots, n$.

$$O_0^+ = \{(t, x): t^2 - |x|^2 = 0, t > 0\}.$$

Proof. Since all the transformations in G preserve the form $t^2 - |x|^2$ and do not interchange the upper and the lower sheets of the two sheet hyperboloids, the above sets must be invariant. In order to prove that they are minimal, take two arbitrary points $p_1 = (t^1, x^1)$ and $p_2 = (t^2, x^2)$ belonging to the same set. Choose rotations R_1 and R_2 as in Definition 2 such that $R_1(p_1) = (t^1, y_1^1, 0, \ldots, 0) = q^1$, $R_2(p_2) = (t^2, y_1^2, 0, \ldots, 0) = q^2$. It is easy to see that, since the points q^1, q^2 also belong to the same set as p_1, p_2, there is a transformation σ_β such that $\sigma_\beta(q^1) = q^2$.

The superposition $g = R_2^{-1} \circ \sigma_\beta \circ R_1$ is a transformation in G such that $g(p_1) = p_2$.

PROPOSITION 5. *Let $u \in D'(\Lambda)$ (where Λ is G-invariant) be a G-invariant distribution and let $H = \operatorname{supp} u$. Then H is G-invariant, i.e., $g(H) = H$ for every transformation $g \in G$.*

Proof. Since all the transformations $g \in G$ are invertible and onto, it is sufficient to show $g(H') \subset H'$, where $H' = E^{n+1} \setminus H$. To prove this we need only to show $u[\varphi] = 0$ for an arbitrary $\varphi \in C_0^\infty(g(H'))$. This follows as a result of the G-invariance of u, since $u[\varphi] = u[\varphi \circ g]$ and $\varphi \circ g \in C_0^\infty(H')$.

As an illustration to the above considerations we shall prove the following

PROPOSITION 6. *If $u \in D'(E^{n+1})$ is a G-invariant distribution and $\operatorname{supp} u \subset \overline{E_+^{n+1}}$, then $\operatorname{supp} u \subset \overline{V}_+$.*

Proof. Suppose, conversely, that $p_0 \in (\overline{E_+^{n+1}} \setminus \overline{V}_+) \cap \operatorname{supp} u$. Then there is $p_1 \in E_-^{n+1}$ belonging to the same orbit as p_0. By Proposition 5, $p_1 \in \operatorname{supp} u$. This contradiction ends the proof.

Suppose that[7] $f: \Omega_1 \to E^1$ is a continuous G-invariant function. Then f must be constant on the orbits of G. In other words, there is a continuous function $h: E^1 \to E^1$ such that:

$$f(t, x) = h(t^2 - |x|^2). \tag{1}$$

In order to extend this characterization of G-invariant functions to the case of G-invariant distributions we write formula (1) in terms of distributions:

$$f[\varphi] = \int h(t^2 - |x|^2)\varphi(t, x)\,dt\,dx \quad \text{for } \varphi \in C_0^\infty(\Omega_1). \tag{2}$$

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[7] All the considerations concerning Ω_1 will be also valid for Ω_2 with adequate changes, see Table of changes footnote [10]. To underline this, in some places two versions will be given, the other one in brackets.

To perform further operations we need the following:

LEMMA 1 (on the rectification of orbits). *Let μ (μ^*) (8) denote the mapping from E_+^{n+1} (E_-^{n+1}) defined by the formulae:*

$$\mu\ (\mu^*)\colon s = t^2 - |x|^2, \quad y_i = x_i, \quad i = 1, \ldots, n.$$

This mapping is invertible and its Jacobian is equal to $2t > 0$ ($2t < 0$). It carries the orbits of G, or just their portions contained in E_+^{n+1} (E_-^{n+1}) onto hyperplanes $s =$ const or their subsets. The image of E_+^{n+1} (E_-^{n+1}) under μ (μ^*) is the sector:

$$\Omega^1 = \mu(E_+^{n+1}) = \{(s, y)\colon s + |y|^2 > 0\},$$
$$(\Omega^2 = \mu^*(E_-^{n+1}) = \{(s, y)\colon s + |y|^2 > 0\}).$$

Also notice that if $\varphi \in C_0^\infty(E_+^{n+1})$ ($\varphi \in C_0^\infty(E_-^{n+1})$), then

$$\varphi \circ \mu^{-1} \in C_0^\infty(\Omega^1) \qquad (\varphi \circ \mu^{*-1} \in C_0^\infty(\Omega^2)).$$

If $\psi \in C_0^\infty(\Omega^1)$ ($\psi \in C_0^\infty(\Omega^2)$), then $\psi \circ \mu \in C_0^\infty(E_+^{n+1})$ ($\psi \circ \mu^* \in C_0^\infty(E_-^{n+1})$).

It follows by Lemma 1 that for $\varphi \in C_0^\infty(E_+^{n+1})$ formula (2) takes the form

$$f[\varphi] = \int_{E^1} h(s) \left(\int_{E^n} \frac{\varphi(\sqrt{s + |y|^2}, y)}{2\sqrt{s + |y|^2}}\, dy \right) ds. \tag{3}$$

Formula (3) gives an idea for introducing the following

DEFINITION 4. By $\tilde{J}$ ($\tilde{J}^*$) we denote an operation from the space $C_0^\infty(E_+^{n+1})$ ($C_0^\infty(E_-^{n+1})$) into $C_0^\infty(E^1)$ defined as follows:

$$\big(\tilde{J}(\varphi)\big)(s) = \int_{E^n} \frac{\varphi(\sqrt{s + |y|^2}, y)}{2\sqrt{s + |y|^2}}\, dy \qquad \text{for } s \in E^1, \tag{4}$$

$$\big(\tilde{J}^*(\varphi)\big)(s) = \int_{E^n} \frac{\varphi(\sqrt{s + |y|^2}, y)}{2\sqrt{s + |y|^2}}\, dy. \tag{4'}$$

In Proposition 7 below we shall prove that $\tilde{J}$ ($\tilde{J}^*$) can be extended to the space of C_0^∞ functions on Ω_1 (Ω_2).

PROPOSITION 7. *Let $\tilde{J}(\varphi)$ ($\tilde{J}^*(\varphi)$) be given by formula (4) ((4′)) for $\varphi \in C_0^\infty(E_+^{n+1})$ ($\varphi \in C_0^\infty(E_-^{n+1})$). Then $\tilde{J}$ ($\tilde{J}^*$) can be extended in a unique way to a continuous(9) G-invariant operation J (J^*) from $C_0^\infty(\Omega_1)$ ($C_0^\infty(\Omega_2)$) into $C_0^\infty(E^1)$.*

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(8) See footnote (7).

(9) An operation $P\colon C_0^\infty(\Lambda) \to C_0^\infty(E^1)$, Λ an open subset of E^{n+1}, is called *continuous* if the condition $C_0^\infty(\Lambda) \ni \varphi_\nu \to 0$ in $D(\Lambda)$ implies $P(\varphi_\nu) \to 0$ in $D(E^1)$.

Proof[10]. Let $\varphi_0 \in C_0^\infty(\Omega_1)$. By Proposition 4, for every point $p \in \operatorname{supp}\varphi_0$ there is a point $q \in E_+^{n+1}$ belonging to the same orbit as p and a transformation $g_p \in G$ such that $g_p(p) = q$. Let $U_p \subset E_+^{n+1}$ be an open neighbourhood of q; then $V_p = g_p^{-1}(U_p)$ is an open neighbourhood of p. Let $V_{p_1}, \ldots, V_{p_k}$ be a finite subcovering of the covering $\{V_p\}_{p\in\operatorname{supp}\varphi_0}$ and let $\{a_i\}_{i=1,\ldots,k}$ be a partition of unity subordinated to the subcovering $\{V_{p_i}\}_{i=1,\ldots,k}$. Define:

$$J(\varphi_0) = \sum_{i=1}^{k} \tilde{J}\big((a_i\cdot\varphi_0)\circ g_{p_i}^{-1}\big). \tag{5}$$

To prove that definition (5) is correct, we have to show that it does not depend on the choice of a subcovering and the partition of unity subordinated to this subcovering. To show this, let $W_{q_1}, \ldots, W_{q_m}$ be another subcovering of $\operatorname{supp}\varphi_0$ with the corresponding transformations $\tilde{g}_{q_1}, \ldots, \tilde{g}_{q_m}$. Let $\{\beta_j\}$, $j = 1, \ldots, m$, be a partition of unity subordinated to $\{W_{q_j}\}$, $j = 1, \ldots, m$. For an arbitrary continuous function $h\colon E^1 \to E^1$ define $f\colon \Omega_1 \to E^1$ by formula (1). Since f is G-invariant, we have:

$$\begin{aligned} f[\varphi_0] &= \sum_{i=1}^{k} \int_{E^{n+1}} h(t^2-|x|^2)(a_i\cdot\varphi_0)(t,x)\,dt\,dx \\ &= \sum_{i=1}^{k} \int_{E^1} h(s)\Big(\tilde{J}\big((a_i\cdot\varphi_0)\circ g_{p_i}^{-1}\big)\Big)(s)\,ds; \end{aligned} \tag{6}$$

similarily

$$f[\varphi_0] = \sum_{j=1}^{m} \int_{E^1} h(s)\Big(\tilde{J}\big((\beta_j\cdot\varphi_0)\circ g_{q_j}^{-1}\big)\Big)(s)\,ds. \tag{7}$$

Combining (6) and (7) we arrive at:

$$\int_{E^1} h\cdot\Big(\sum_{i=1}^{k} \tilde{J}\big((a_i\cdot\varphi_0)\circ g_{p_j}^{-1}\big)\Big)\,ds = \int_{E^1} h\cdot\Big(\sum_{j=1}^{m} \tilde{J}\big((\beta_j\cdot\varphi_0)\circ\tilde{g}_{q_j}^{-1}\big)\Big)\,ds \tag{8}$$

for every continuous function h. Since both the functions

$$\sum_{i=1}^{k} \tilde{J}\big((a_i\cdot\varphi_0)\circ g_{p_i}^{-1}\big) \quad \text{and} \quad \sum_{j=1}^{m} \tilde{J}\big((\beta_j\cdot\varphi_0)\circ\tilde{g}_{q_j}^{-1}\big)$$

are continuous, they must be equal on E^1, and this ends the proof of the correctness of (5). A proof that J is a continuous G-invariant operation is easy and therefore omitted. To prove that J is a unique G-invariant extension of $\tilde{J}$, suppose that P is another such extension. Let $\varphi_0 \in C_0^\infty(\Omega_1)$ and let $\{a_i\}_{i=1,\ldots,k}$ and g_{p_i}, $i = 1, \ldots, k$, be as in formula (5). Then we

[10] All the considerations concerning Ω_1 are also valid for Ω_2 after the following changes: Table of changes: Replace Ω_1 by Ω_2, E_+^{n+1} by E_-^{n+1}, J by J^*, μ by μ^*.

have:

$$P(\varphi_0) = \sum_{i=1}^{k} P(a_i \cdot \varphi_0) = \sum_{i=1}^{k} \tilde{J}\big((a_i \cdot \varphi_0) \circ g_{p_i}^{-1}\big) = J(\varphi_0).$$

In our further consideration J (J^*) will always stand for the extension of $\tilde{J}$ $(\tilde{J}^*)$ given by Proposition 7, unless otherwise stated.

Notice that, by applying (5) and (6), formula (2) can be written in the form

$$(9) \qquad f[\varphi] = h[J(\varphi)] \quad \text{for } \varphi \in C_0^\infty(\Omega_1).$$

Our aim is to extend (9) to the case of distributions, but first we shall list some properties of $J(\varphi)$ $(J^*(\varphi))$ depending on how $\operatorname{supp}\varphi$ is situated.

PROPERTY 1. *If* $\varphi \in C_0^\infty(\Omega_1)$ $(\varphi \in C_0^\infty(\Omega_2))$, *then* $J(\varphi) \in C_0^\infty(E^1)$ $(J^*(\varphi) \in C_0^\infty(E^1))$.

If $\varphi \in C_0^\infty(V_+)$ $(\varphi \in C_0^\infty(V_-))$, *then* $J(\varphi) \in C_0^\infty(E^1_+)$ $(J^*(\varphi) \in C_0^\infty(E^1_+))$.

If $\varphi \in C_0^\infty(\Omega_1 \setminus \overline{V}_+)$ (11), *then* $J(\varphi) = J^*(\varphi) \in C_0^\infty(E^1_-)$.

PROPERTY 2. *If* $a \in C_0^\infty(E^1_+)$, *then there are* $\varphi \in C_0^\infty(V_+)$ *and* $\psi \in C_0^\infty(V_-)$ *such that* $J(\varphi) = a$, $J^*(\psi) = a$.

If $a \in C_0^\infty(E^1)$, *then there are* $\varphi \in C_0^\infty(\Omega_1)$ (12) *and* $\psi \in C_0^\infty(\Omega_2)$ *such that* $J(\varphi) = a$, $J^*(\psi) = a$.

If $a \in C_0^\infty(E^1_-)$, *then there is* $\varphi \in C_0^\infty(\Omega_1 \cap \Omega_2)$ *such that* $J(\varphi) = J^*(\varphi) = a$.

Proof. Let $a \in C_0^\infty(E^1_-)$ and choose $f \in C_0^\infty(E^n)$ such that $\int_{E^n} f dy = 1$ and $a \otimes f \in C_0^\infty(\Omega^1 \setminus \overline{E^{n+1}_+})$. Then the formula (13)

$$(10) \qquad \varphi(t, x) = 2t \cdot a(t^2 - |x|^2) \cdot f(x) \quad \text{for } (t, x) \in E^{n+1}_+$$

defines φ such that $\operatorname{supp}\varphi \subset \Omega_1 \cap \Omega_2$ and $J(\varphi) = a$. By Property 1 we have $J^*(\varphi) = J(\varphi) = a$.

We omit similar proofs for $a \in C_0^\infty(E^1_+)$ and $a \in C_0^\infty(E^1)$, noting only that in the case $a \in C_0^\infty(E^1_+)$, φ can be defined by (10) and ψ by the formula

$$(10^*) \qquad \psi(t, x) = 2t \cdot a(t^2 - |x|^2) \cdot f(x) \quad \text{for } (t, x) \in E^{n+1}_-$$

with an arbitrary $f \in C_0^\infty(E^n)$ satisfying $\int_{E^n} f dy = 1$.

Before we conclude the part dealing with the properties of J (J^*) we introduce the operation P^* defined on $C_0^\infty(E^{n+1})$ which to every function $\varphi \in C_0^\infty(E^{n+1})$ assignes the function $P^*(\varphi)$ defined for $s \geqslant 0$ by the right-hand side of (4) (14).

(11) Notice that $\Omega_1 \setminus \overline{V}_+ = \Omega_2 \setminus \overline{V}_- = \Omega_1 \cap \Omega_2$.

(12) It follows by formula (5) that there is $\tilde{\varphi} \in C_0^\infty(E^{n+1}_+)$ such that $\tilde{J}(\tilde{\varphi}) = J(\varphi) = a$.

(13) We set $\varphi = 0$ $(\psi = 0)$ on $\overline{E^{n+1}_-}$ (on $\overline{E^{n+1}_+}$) in formula (10) ((10*)).

(14) The operation P^* and its properties, although of no significance for the present paper will find an application in [5].

We shall prove the following property of P^*.

PROPERTY 3. *Let* $n = 2m$, $m \in N$.

(i) *If* $\varphi \in C_0^\infty(E^{n+1})$, *then* $P^*(\varphi) \in C^{m-1}(\overline{E_+^1}) \cap C^\infty(E_+^1)$.

(ii) *If* $\lim_{\nu\to\infty} \varphi_\nu = \varphi$ *in* $D(E^{n+1})$, *then*

$$\lim_{\nu\to\infty} \frac{d^{m-1}}{ds^{m-1}} \big(P^*(\varphi_\nu)\big)(s) = \frac{d^{m-1}}{ds^{m-1}} \big(P^*(\varphi)\big)(s)$$

uniformly on $\overline{E_+^1}$.

Proof. To prove (i) take $\varphi \in C_0^\infty(E^{n+1})$. It is easy to see that the function defined by (4) is continuous on $\overline{E_+^1}$. Also, there exist partial derivatives of order up to $m-1$ of the functions

$$E_+^1 \ni s \mapsto \frac{\varphi(\sqrt{s+|y|^2}, y)}{2\sqrt{s+|y|^2}}.$$

For $s \to 0$ they are $O\big((s+|y|^2)^{(1-n)/2}\big)$ functions, from which immediately follows the continuity of λ_k, where

$$\lambda_k\colon \overline{E_+^1} \ni s \mapsto \int_{E^n} \frac{d^k}{ds^k} \frac{\varphi(\sqrt{s+|y|^2}, y)}{2\sqrt{s+|y|^2}}\, dy, \quad k = 1, \ldots, m-1.$$

As a consequence of the relations:

$$\frac{d^k}{ds^k} P^*(\varphi)(s) = \lambda_k(s) \quad \text{for } s > 0,\ k = 1, \ldots, m-1,$$

we obtain $P^*(\varphi) \in C^{m-1}(\overline{E_+^1})$. The proof of (ii) is omitted.

Let us note that P^* is not an extension of J to $C_0^\infty(E^{n+1})$; $P^*(\varphi) = J(\varphi)|_{\overline{E_+^1}}$ for $\varphi \in C_0^\infty(E_+^{n+1})$.

LEMMA 2. *Let* Λ *be an open* G*-invariant set and let* $u \in D'(\Lambda)$ *be a* G*-invariant distribution. Then* u *satisfies the following system of differential equations*:

$$t\frac{\partial u}{\partial x_i} + x_i \frac{\partial u}{\partial t} = 0 \quad \textit{for } i = 1, \ldots, n \tag{11}$$

on every open subset $\Lambda_1 \subset \Lambda$.

Proof. It is sufficient[15] to prove (11) in the case $i = 1$. Let $\varphi \in C_0^\infty(\Lambda_1)$. Since u is G-invariant, we have $u[\varphi \circ \sigma_\beta] = u[\varphi]$ for all β, $|\beta| < 1$, where σ_β is the special transformation from G defined by Definition 2. In other words, the function $\Phi\colon (-1, 1) \ni \beta \mapsto u[\varphi \circ \sigma_\beta]$ is constant. In particular, this means that $\left.\frac{d\Phi}{d\beta}\right|_{\beta=0} = 0$. Hence, by the theorem on

([15]) See footnote ([6]).

differentiation of distributions depending on a parameter, we obtain $u\left[\frac{d}{d\beta}(\varphi\circ\sigma_\beta)|_{\beta=0}\right] = 0$. After a simple computation we get

$$\frac{d}{d\beta}(\varphi\circ\sigma_\beta)(t, x)|_{\beta=0} = t\frac{\partial\varphi}{\partial x_1} + x_1\frac{\partial\varphi}{\partial t}. \tag{12}$$

Combining the above we obtain:

$$0 = u\left[t\frac{\partial\varphi}{\partial x_1} + x_1\frac{\partial\varphi}{\partial t}\right] = u\left[\frac{\partial}{\partial x_1}(t\varphi) + \frac{\partial}{\partial t}(x_1\varphi)\right]$$
$$= -\left(t\frac{\partial u}{\partial x_1} + x_1\frac{\partial u}{\partial t}\right)[\varphi] \quad \text{for an arbitrary function } \varphi\in C_0^\infty(\Lambda_1),$$

which ends the proof of (11).

LEMMA 3. *Let* $\Sigma = (a, b)\times(a_1, b_1)\times(a_2, b_2)\times\ldots\times(a_n, b_n)\subset E^{n+1}$, *where* $-\infty\leqslant a<b\leqslant+\infty$, $-\infty\leqslant a_i<b_i\leqslant+\infty$ *for* $i = 1, \ldots, n$.

If $w\in D'(\Sigma)$ *satisfies the system of equations*: $\partial w/\partial y_1 = 0, \ldots, \partial w/\partial y_n = 0$, *then there exists a unique distribution* $S\in D'\big((a, b)\big)$ *such that* $w[\varphi] = S\big[\int_{E^n}\varphi dy\big]$ *for* $\varphi\in C_0^\infty(\Sigma)$.

Proof. This is done by induction, applying the theorem on distributions independent of one variable.

Now, we can prove a key theorem which gives a full characterization of all G-invariant distributions on Ω_1.

THEOREM 1. *Let* $v\in D'(\Omega_1)$ *be a G-invariant distribution. Then there exists a unique distribution* $T\in D'(E^1)$ *such that*:

$$v[\varphi] = T[J(\varphi)] \quad \textit{for } \varphi\in C_0^\infty(\Omega_1), \tag{13}$$

where $J(\varphi)$ *is defined by formula* (5).

Conversely, if $T\in D'(E^1)$ *is an arbitrary distribution, then* (13) *defines a G-invariant distribution* $v\in D'(\Omega_1)$.

Proof. Let μ be the transformation defined in Lemma 1. Set

$$v^1 = v\circ\mu^{-1}. \tag{14}$$

It is not difficult to see([16]) that the linear functional given by (14) is a distribution on Ω^1. At first we prove that v^1 satisfies one of the assumptions of Lemma 3, i.e., that $\partial v^1/\partial y_i = 0$ for $i = 1, \ldots, n$ ([17]). To prove this let $\psi\in C_0^\infty(\Omega^1)$. It is sufficient to show $v^1[\partial\psi/\partial y_i] = 0$ for $i = 1, \ldots, n$.

([16]) See [4], Definition 1.

([17]) In the case where v is a function constant on the orbits (as in our case) it is not surprising that after the rectification of the orbits by μ, v^1 is constant on their images, i.e., the portions of the hyperplanes s = const contained in Ω^1. This means that v^1 satisfies the equations $\partial v^1/\partial y_i = 0$ for $i = 1, \ldots, n$.

Applying Lemma 1 (on the rectification of orbits), we see that there is $\varphi \in C_0^\infty(E_+^{n+1})$ such that $\psi(s, y) = (\varphi\circ\mu^{-1})(s, y) = \varphi(\sqrt{s+|y|^2}, y)$. From the above, in view of (14) and Lemma 2, it follows:

$$v^1\left[\frac{\partial\psi}{\partial y_i}\right] = v^1\left[\frac{y_i}{\sqrt{s+|y|^2}}\cdot\frac{\partial\varphi}{\partial t}\circ\mu^{-1}+\frac{\partial\varphi}{\partial x_i}\circ\mu^{-1}\right]$$

$$= v\left[2t\left(\frac{\partial\varphi}{\partial t}\cdot\frac{x_i}{t}+\frac{\partial\varphi}{\partial x_i}\right)\right] = -2\left(x_i\frac{\partial v}{\partial t}+t\frac{\partial v}{\partial x_i}\right)[\varphi] = 0,$$

$$i = 1, \dots, n.$$

Now, in order to apply Lemma 3 to the distribution v^1, let $A_a = (a^a, b^a)\times(a_1^a, b_1^a)\times\dots\times(a_n^a, b_n^a)$ and let $\{A_a\}$ be the covering of Ω^1 by the cubes A_a. Then by Lemma 3 there exist uniquely determined distributions $T^a \in D'\big((a^a, b^a)\big)$ such that $v^1[\psi] = T^a\big[\int_{E^n}\psi dy\big]$ for $\psi \in C_0^\infty(A_a)$. If $A_a\cap A_\beta \neq \emptyset$, then $T^a = T^\beta$ on $(a^a, b^a)\cap(a^\beta, b^\beta)$. Because $\{(a^a, b^a)\}$ is a covering of E^1, by the theorem on gluing of distributions there is a unique distribution $T \in D'(E^1)$ which restricted to every (a^a, b^a) is identical with T^a. We shall prove that

$$v^1[\psi] = T\Big[\int_{E^n}\psi(s, y)\,dy\Big] \quad \text{for } \psi \in C_0^\infty(\Omega^1). \tag{15}$$

To prove (15), take a $\psi \in C_0^\infty(\Omega^1)$. There is a finite number of A_a's say: A_{a_i}, $i = 1, \dots, k$, such that $\operatorname{supp}\psi \subset \bigcup_{i=1}^{k} A_{a_i}$.

Denote by γ_i a partition of unity subordinated to the covering $\{A_{a_i}\}$. Then $\psi = \sum_{i=1}^{k}\psi_i$, where $\psi_i = \gamma_i\cdot\psi$ and $\operatorname{supp}\psi_i \subset A_{a_i}$ for $i = 1, \dots, k$. The relations

$$v^1[\psi_i] = T^{a_i}\Big[\int_{E^n}\psi_i(s, y)\,dy\Big] = T\Big[\int_{E^n}\psi_i(s, y)\,dy\Big]$$

yield

$$v^1[\psi] = \sum_{i=1}^{k} v^1[\psi_i] = \sum_{i=1}^{k} T\Big[\int_{E^n}\psi_i(s, y)\,dy\Big]$$

which immediately gives (15).

From (14) and (15) it follows that:

$$v[\varphi] = v^1\left[\frac{\varphi(\sqrt{s+|y|^2}, y)}{2\sqrt{s+|y|^2}}\right] = T\left[\int_{E^n}\frac{\varphi(\sqrt{s+|y|^2}, y)}{2\sqrt{s+|y|^2}}\,dy\right]$$

$$= T[\tilde{J}(\varphi)] \quad \text{for } \varphi \in C_0^\infty(E_+^{n+1}).$$

Let $\varphi_0 \in C_0^\infty(\Omega_1)$ and $J(\varphi_0)$ be given by formula (5) in the form $J(\varphi_0) = \sum_{i=1}^{k} \tilde{J}\big((\gamma_i\cdot\varphi_0)\circ g_i^{-1}\big)$, where g_i are chosen so that there exists a covering $\{V_i\}_{i=1,\ldots,k}$ of $\operatorname{supp}\varphi_0$ such that $g_i(V_i) \subset E_+^{n+1}$ for $i = 1, \ldots, k$, and $\{\gamma_i\}$, $i = 1, \ldots, k$, is a partition of unity subordinated to this covering. Now, we obtain:

$$v[\varphi_0] = v\Big[\sum_{i=1}^{k}\gamma_i\cdot\varphi_0\Big] = \sum_{i=1}^{k} v[(\gamma_i\cdot\varphi_0)\circ g_i^{-1}]$$
$$= \sum_{i=1}^{k} T\big[\tilde{J}\big((\gamma_i\cdot\varphi_0)\circ g_i^{-1}\big)\big] = T[J(\varphi_0)].$$

To prove that T is uniquely determined by formula (13), suppose that there is $T_1 \in D'(E^1)$ such that $v[\varphi] = T_1[J(\varphi)]$. From (13) and Property 2 we obtain that $T_1 = T$. Since J is a continuous G-invariant operation, the second part of Theorem 1 is obvious.

COROLLARY 1. *If $v \in D'(\Omega_1)$ is a G-invariant distribution such that $v = 0$ on $E_+^{n+1}\setminus \bar{V}_+$, then the distribution T defined by Theorem 1 is equal to zero on E_-^1.*

Proof. This follows from Theorem 1 and Property 2.

It is not difficult to see that after some changes[18] the proof of Theorem 1 transforms to the proof of the following.

THEOREM 1*. *Let $v^* \in D'(\Omega_2)$ be a G-invariant distribution. Then there exists a unique distribution $T^* \in D'(E^1)$ such that*

$$v^*[\varphi] = T^*[J^*(\varphi)] \quad \text{for } \varphi \in C_0^\infty(\Omega_2). \tag{13*}$$

And conversely, if $T^* \in D'(E^1)$ is an arbitrary distribution, then (13) defines a G-invariant distribution $v^* \in D'(\Omega_2)$.*

Combining Theorem 1 and Theorem 1* we obtain:

THEOREM 2 (Methée). *Let $v \in D'(E^{n+1}\setminus\{0\})$ be a G-invariant distribution. Then there exist unique distributions T and $T^* \in D'(E^1)$ such that:*

$$\begin{aligned} v[\varphi] &= T[J(\varphi)] &&\text{for } \varphi \in C_0^\infty(\Omega_1),\\ v[\varphi] &= T^*[J^*(\varphi)] &&\text{for } \varphi \in C_0^\infty(\Omega_2),\end{aligned} \tag{13**}$$

and $T = T^*$ on E_-^1.

*Conversely, to every pair of distributions $T, T^* \in D'(E^1)$ such that $T = T^*$ on E_-^1 there corresponds a unique G-invariant distribution $v \in D'(E^{n+1}\setminus\{0\})$ given by formulae (13**).*

Proof. On account of Theorems 1 and 1*, in order to prove the first part of Theorem 2 it is enough to show $T = T^*$ on E_-^1, i.e. $T[a] = T^*[a]$

(18) See Table of changes, footnote (10).

for $\alpha \in C_0^\infty(E_-^1)$. Choose an arbitrary $\alpha \in C_0^\infty(E_-^1)$. By Property 2, there is $\varphi \in C_0^\infty(\Omega_1 \cap \Omega_2)$ such that $J(\varphi) = J^*(\varphi) = \alpha$. Hence

$$T[\alpha] = T[J(\varphi)] = v[\varphi] = T^*[J^*(\varphi)] = T^*[\alpha].$$

The second part follows by the theorem on gluing of distributions and by Theorems 1 and 1^*.

THEOREM 3. (i) *Every distribution $u \in D'(\Lambda)$ (where Λ is G-invariant) satisfying on Λ the equation*

$$t\frac{\partial}{\partial x_1}u + x_1\frac{\partial}{\partial t}u = 0 \tag{16}$$

is invariant with respect to all transformations σ_β, $|\beta| < 1$, defined by Definition 2.

(ii) *If $u \in D'(\Lambda)$ is rotation invariant and satisfies* (16), *then u is G-invariant.*

Proof. Let $u \in D'(\Lambda)$ satisfy (16). Then from (12) it follows that

$$\frac{d}{d\beta}u[\varphi\circ\sigma_\beta]|_{\beta=0} = -\left(t\frac{\partial u}{\partial x_1} + x_1\frac{\partial u}{\partial t}\right)[\varphi] = 0 \quad \text{for } \varphi \in C_0^\infty(\Lambda). \tag{17}$$

Let us fix arbitrary $\varphi \in C_0^\infty(\Lambda)$ and β_0, $|\beta_0| < 1$. To prove (i) it is sufficient to show:

$$\frac{d}{d\beta}u[\varphi\circ\sigma_\beta]|_{\beta=\beta_0} = 0. \tag{18}$$

Set $h = \varphi\circ\sigma_{\beta_0}$, $h \in C_0^\infty(\Lambda)$. Then by Proposition 1

$$h\circ\sigma_\alpha = \varphi\circ\sigma_{\gamma(\alpha)}, \qquad \text{where } \gamma(\alpha) = \tanh(\tanh^{-1}\alpha + \tanh^{-1}\beta_0).$$

Hence we obtain: $\gamma(0) = \beta_0$, $\gamma'(\alpha) \neq 0$,

$$\left.\frac{d(h\circ\sigma_\alpha)}{d\alpha}\right|_{\alpha=0} = \left.\frac{d(\varphi\circ\sigma_{\gamma(\alpha)})}{d\alpha}\right|_{\alpha=0} = \left.\frac{d(\varphi\circ\sigma_\gamma)}{d\gamma}\right|_{\gamma=\beta_0}\cdot\gamma'(0).$$

therefore

$$\left.\frac{d(\varphi\circ\sigma_\gamma)}{d\gamma}\right|_{\gamma=\beta_0} = \frac{1}{\gamma'(0)}\cdot\left.\frac{d(h\circ\sigma_\alpha)}{d\alpha}\right|_{\alpha=0}.$$

From this, applying (17), we obtain

$$\left.\frac{d}{d\gamma}u[\varphi\circ\sigma_\gamma]\right|_{\gamma=\beta_0} = \frac{1}{\gamma'(0)}u\left[\left.\frac{d(h\circ\sigma_\alpha)}{d\alpha}\right|_{\alpha=0}\right] = 0,$$

which gives (18).

The second part of Theorem 3 follows from (i) by Proposition 2.

2. Now we take up the case where instead of $t^2-x_1^2-\ldots-x_n^2$ we have the form $t_1^2+\ldots+t_m^2-x_1^2-\ldots-x_n^2$, $m\geqslant 2$, $n\geqslant 2$, $(t_1,\ldots,t_m,x_1,\ldots,x_n)\in E^{m+n}$.

Write $t=(t_1,\ldots,t_m)$, $x=(x_1,\ldots,x_n)$.

We begin with

DEFINITION 5. Let H denote the identity component[19] of the group of linear transformations which leave the quadratic form $|t|^2-|x|^2$ invariant.

In particular, the following transformations belong to H:

$$\sigma_\beta^i(t,x)=\left(\frac{t_1+\beta x_i}{\sqrt{1-\beta^2}},t_2,\ldots,t_m,x_1,\ldots,x_{i-1},\frac{x_i+\beta t_1}{\sqrt{1-\beta^2}},x_{i+1},\ldots,x_n\right),$$
$$|\beta|<1,\ i=1,2,\ldots,n,$$

$$R_a^j(t,x)=(t_1\sqrt{1-a^2}+at_j,t_2,\ldots,t_{j-1},-at_1+\sqrt{1-a^2}t_j,t_{j+1},\ldots,t_m,x),$$
$$|a|<1,\ j=2,3,\ldots,m.$$

LEMMA 4. *Let $u\in D'(\Omega)$, $\Lambda\subset E^{m+n}$, Λ H-invariant. Suppose that u is H-invariant. Then u satisfies the following system of differential equations:*

$$\left(t_1\frac{\partial}{\partial x_i}+x_i\frac{\partial}{\partial t_1}\right)u=0,\quad i=1,\ldots,n,$$
$$\left(t_1\frac{\partial}{\partial t_j}-t_j\frac{\partial}{\partial t_1}\right)u=0,\quad j=2,\ldots,m,$$

on every open subset $\Lambda_1\subset\Lambda$.

Proof. Using the transformations σ_β^i, $i=1,\ldots,n$, and R_a^j, $j=2,\ldots,m$, we proceed as in the proof of Lemma 2.

Let $\tilde t=(t_2,\ldots,t_m)$, $\tilde h=(h_2,\ldots,h_m)$.

In our case we can also define the "rectifying" map

$$\tau:\ h_1=|t|^2-|x|^2,\quad h_i=t_i,\quad i=2,\ldots,m,\quad y_j=x_j,\quad j=1,\ldots,n.$$

The domain of τ is $E_+^{m+n}=\{(t,x):\ t_1>0\}$. Then the properties of τ are analogous to those of μ from Lemma 1.

An approach similar to that applied to G-invariant functions leads to the following continuous operation $\tilde K$: $D(E_+^{m+n})\to D(E^1)$

$$\big(\tilde K(\varphi)\big)(h_1)=\int\limits_{E^{m+n-1}}\frac{\varphi(\sqrt{h_1-|\tilde h|^2+|y|^2},\tilde h,y)}{2\sqrt{h_1-|\tilde h|^2+|y|^2}}\,d\tilde h\,dy,$$

where $\varphi\in C_0^\infty(E_+^{m+n})$.

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[19] It can be understood in the sense of the topology in the space of $(m+n)\times(m+n)$ matrices, i.e., in $E^{(m+n)^2}$.

It follows immediately from the definition of the group H that for every point $p \in E^{m+n}\setminus\{0\}$ there is $h \in H$ such that $h(p) \in E_+^{m+n}$. Replacing Ω_1 by $E^{m+n}\setminus\{0\}$, after similar considerations as in the proof of Proposition 7, we conclude that K can be extended in a unique way to a continuous H-invariant operation from $D(E^{m+n}\setminus\{0\})$ into $D(E^1)$. We shall denote this extension by $\tilde{K}$.

In the following considerations we need the property: if $a \in C_0^\infty(E^1)$, then there is $\varphi \in C_0^\infty(E_+^{m+n})$ such that $K(\varphi) = a$ which can be proved in the way indicated in Property 2.

Now we are in the position to prove the fundamental

THEOREM 4. *Let $u \in D'(E^{m+n}\setminus\{0\})$. If u is H-invariant, then there is a unique distribution $T \in D'(E^1)$ such that:*

$$u[\varphi] = T[K(\varphi)] \quad \textit{for } \varphi \in C_0^\infty(E^{m+n}\setminus\{0\}). \tag{19}$$

And conversely, every distribution $T \in D'(E^1)$ defines by means of formula (19) *a unique H-invariant distribution $u \in D'(E^{m+n}\setminus\{0\})$.*

Proof. After some obvious technical changes it is sufficient to repeat the proof of Theorem 1, using the preceding properties of K instead of Property 2.

Note that in spite of similarities with the case of G-invariant distributions, H-invariant distributions cannot be treated as a special case of the earlier and thus the assumptions $n \geqslant 2$, $m \geqslant 2$ are significant.

We note that the case $m = 1$, $n = 1$ can also be treated with the help of the above described methods. Namely, there are four operations $K^i\colon D(\Gamma_i) \to D(E^1)$, $i = 1, 2, 3, 4$, where $\Gamma_1 = \{(t, x)\colon t > x\}$, $\Gamma_2 = \{(t, x)\colon t > -x\}$, $\Gamma_3 = \{(t, x)\colon t < x\}$, $\Gamma_4 = \{(t, x)\colon t < -x\}$, such that a distribution $u \in D'(E^2\setminus\{0\})$ is invariant with respect to the identity component of the group of linear maps preserving $t^2 - x^2$ iff there are four distributions $T_1, \ldots, T_4 \in D'(E^1)$ such that:

$$u[\varphi] = T_i[K^i(\varphi)] \quad \text{for } \varphi \in C_0^\infty(\Gamma_i),\ i = 1, 2, 3, 4, \tag{20}$$

and $T_1 = T_2$ on E_+^1, $T_2 = T_3$ on E_-^1, $T_3 = T_4$ on E_+^1, $T_4 = T_1$ on E_-^1.

3. We introduce the following definition.

DEFINITION 6. Let A be a real symmetric non-singular $k \times k$ matrix ($k \geqslant 2$). By G_A we denote the identity component of the set of all linear transformations g such that $g^t A g = A$ [20]. Let $B = B_{(A)}$ stand for a non-

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[20] Notice that G_A with the operation of superposition of transformations forms a group.

singular matrix such that $C = C_{(A)} = (B^{-1})^t A B^{-1}$ has the canonical form

$$\begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \varepsilon_k \end{pmatrix}, \quad |\varepsilon_i| = 1, \ i = 1, \dots, k.$$

In the sequel we shall denote by the same symbols both matrices and the quadratic forms corresponding to them.

LEMMA 5. *Let $\Lambda \subset E^k$ be an open set and let $u \in D'(\Lambda)$. Suppose that A is a real symmetric non-degenerate matrix and B, C are the matrices corresponding to A as in Definition 6. If both the set Λ and the distribution u are G_A-invariant, then the set $B(\Lambda)$ is G_C-invariant and so is the distribution $v = u \circ B^{-1} \in D'(B(\Lambda))$*

Proof. Let $g \in G_A$; then $g^t A g = A$ and writing $A = B^t C B$ we have $(BgB^{-1})^t CBgB^{-1} = C$.

Hence $BgB^{-1} \in G_C$, because the continuous transformation $G_A \ni a \mapsto BaB^{-1}$ maps connected sets onto connected sets. Similarly, if $h \in G_C$, then $B^{-1}hB \in G_A$. Thus for every $h \in G_C$ there is $g \in G_A$ such that $h = BgB^{-1}$. Since Λ is G_A-invariant, it follows from the above that $B(\Lambda)$ is G_C-invariant.

In a similar way, u being G_A-invariant, we derive that $u \circ B^{-1}$ is G_C-invariant.

THEOREM 5. *We retain the assumptions of Lemma 5 with $\Lambda = E^k \setminus \{0\}$. Then we have $C = \sum_{i=1}^{k} \varepsilon_i x_i^2$. Let m be the number of the positive ε_i's, and let $n = k - m$.*

(i) *If $m = k$, then the distribution u is G_A-invariant iff there exists a unique distribution $V \in D'(E^1_+)$ such that*

$$u[\varphi] = V[T_B(\varphi)], \quad \varphi \in C_0^\infty(E^k \setminus \{0\}),$$

where

$$T_B(\varphi)(r) = \frac{1}{|\det B|} \int\limits_{|x|=r} \varphi \circ B^{-1}(x)\, d\sigma_x;$$

(ii) *if $m = 1$, $n \geqslant 2$, then denoting by t the variable corresponding to the $\varepsilon_i > 0$ and by $x_1 \dots x_n$ $(k = n+1)$ the remaining ones, we have:*

the distribution u is G_A-invariant iff there are exactly two distributions T, $T^* \in D'(E^1)$ such that

$$u[\varphi] = T[J_B(\varphi)] \quad \textit{on } B(\Omega_1),$$
$$u[\varphi] = T^*[J_B^*(\varphi)] \quad \textit{on } B(\Omega_2),$$

and $T = T^*$ *on* E^1_-, *where*

$$J_B(\varphi) = \frac{1}{|\det B|} J(\varphi \circ B^{-1}), \qquad \varphi \in C_0^\infty(B(\Omega_1)),$$

$$J_B^*(\varphi) = \frac{1}{|\det B|} J^*(\varphi \circ B^{-1}), \qquad \varphi \in C_0^\infty(B(\Omega_2));$$

(iii) *suppose* $m \geqslant 2$, $n \geqslant 2$. *Denote the variables corresponding to the positive* ε_i*—s by* $t_1, \ldots, t_m$ *and the remaining ones by* $x_1, \ldots, x_n$.

The distribution u *is* G_A*-invariant iff there is a unique distribution* $T \in D'(E^1)$ *such that*

$$u[\varphi] = T[K_B(\varphi)], \qquad \varphi \in C_0^\infty(E^k \setminus \{0\}),$$

where

$$K_B = \frac{1}{|\det B|} K(\varphi \circ B^{-1});$$

(iv) *let* $m = 1$, $n = 1$; *then denoting the variable corresponding to the positive* ε_i *by* t *and the other one by* x *we obtain: the distribution* $u \in D'(E^2 \setminus \{0\})$ *is* G_A*-invariant iff there are four distributions* $T_1, T_2, T_3, T_4 \in D'(E^1)$ *such that*

$$u[\varphi] = T_i[K_B^i(\varphi)] \quad \textit{for } \varphi \in C_0^\infty(B(\Gamma_i)),\ i = 1, 2, 3, 4,$$

and $T_1 = T_2$ *on* E^1_+, $T_2 = T_3$ *on* E^1_-, $T_3 = T_4$ *on* E^1_+, $T_4 = T_1$ *on* E^1_-, *where*

$$K_B^i(\varphi) = \frac{1}{|\det B|} K^i(\varphi \circ B^{-1}).$$

Proof. It is enough to apply Lemma 5 to Theorem 4 in [4], to Theorem 2 and to the remarks which follow the proof of Theorem 2.

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