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NOT EVERY EQUATIONAL CLASS OF INFINITARY ALGEBRAS CONTAINS A SIMPLE ALGEBRA

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An algebra is *simple* if it has exactly two congruence relations.

Magari [2] proves that, for every non-trivial algebra A (with only finitary operations), either A has a simple quotient or the subalgebra of A^A generated by the constant functions and the identity function has a simple quotient; in the latter case, the simple quotient which Magari constructs actually has a trivial (i.e. one-element) subalgebra. It follows from this that every non-trivial equational class of finitary algebras contains a simple algebra, and, in an equational class of finitary algebras in which no non-trivial algebra contains a trivial subalgebra, every non-trivial algebra has a simple quotient.

These results are no longer true if one permits infinitary operations. This note * consists of two examples. In the first we describe an equational class of infinitary algebras which contains no simple algebras, and in the second — an equational class of infinitary algebras which contains a simple algebra and in which no non-trivial algebra has a trivial subalgebra, but not every non-trivial algebra has a simple quotient.

1. Let $\mathscr{A}$ be the class of algebras $\big(L, \wedge, \vee, 0, 1, (c_i)_{i\in N}, g\big)$, where $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice with smallest element 0 and largest element 1, c_i is a nullary operation on L for each $i \in N$, g is an $\aleph_0$-ary operation on L, and

(1) $c_0 = 0$;

(2) if $i < j$, then $c_i \wedge c_j = c_i$;

(3) $g\big((c_i)_{i\in N}\big) = 0$;

(4) for all countable sequences $(b_i)_{i\in N}$ of elements of L and for all $j \in N$, $g\big((b_i \vee c_j)_{i\in N}\big) = g\big((b_i)_{i\in N}\big) \vee c_j$;

(5) if $\{b_i \mid i \in N\}$ is finite, then $g\big((b_i)_{i\in N}\big) = 1$.

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Note that an algebra satisfies (5) iff it satisfies all identities of the form $g\big((x_i)_{i\in N}\big) = 1$, where the sequences $(x_i)_{i\in N}$ run through all countable sequences from a given infinite set, each with only finitely many different entries, and thus $\mathscr{A}$ is an equational class.

Note that if $\big(L, \wedge, \vee, 0, 1, (c_i)_{i\in N}, g\big) \in \mathscr{A}$, then either L has only one element or $0 \neq 1$ and thus, by (3) and (5), $\{c_i \mid i \in N\}$ is an infinite subset of L. The following proposition shows that $\mathscr{A}$ is non-trivial:

PROPOSITION 1. *If* $(L, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice which contains a countably infinite ascending chain* $c_0 = 0 < c_1 < c_2 < c_3 < \ldots < c_n < \ldots$, *then there exists a mapping* $g\colon L^N \to L$ *such that* $\boldsymbol{L} = \big(L, \wedge, \vee, 0, 1, (c_i)_{i\in N}, g\big) \in \mathscr{A}$.

Proof. Define $g\colon L^N \to L$ as follows: For $(b_i)_{i\in N} \in L^N$, if there exists a $j \in N$ such that

$$(b_i \vee c_j)_{i\in N} = (\underbrace{c_j, c_j, \ldots, c_j}_{j+1\ \text{terms}}, c_{j+1}, c_{j+2}, \ldots),$$

then $g\big((b)_{i\in N}\big) = c_k$, where k is the smallest such j; otherwise, $g\big((b_i)_{i\in N}\big) = 1$.

Then $\boldsymbol{L} = \big(L, \wedge, \vee, 0, 1, (c_i)_{i\in N}, g\big)$ obviously satisfies (1) and (2).

For notational convenience, let

$$\bar{c}_j = (\underbrace{c_j, c_j, \ldots, c_j}_{j+1\ \text{terms}}, c_{j+1}, c_{j+2}, \ldots).$$

Since $(c_i \vee c_0)_{i\in N} = (c_i \vee 0)_{i\in N} = (c_i)_{i\in N} = \bar{c}_0$, it follows that $\boldsymbol{L}$ satisfies (3). Similarly, for each $j \in N$, since $i < j$ implies $c_i < c_j$, $g(\bar{c}_j) = c_j$.

If $(b_i)_{i\in N}$ is a countable sequence of elements of L such that $\{b_i \mid i \in N\}$ is finite, then, for each $j \in N$, $\{b_i \vee c_j \mid i \in N\}$ is also finite, and, consequently, $(b_i \vee c_j)_{i\in N} \neq \bar{c}_j$; thus $g\big((b_i)_{i\in N}\big) = 1$. It follows that $\boldsymbol{L}$ satisfies (5).

To prove that $\boldsymbol{L} \in \mathscr{A}$ it remains only to show that, for each sequence $(b_i)_{i\in N}$ of elements of L and for each $j \in N$,

$$g\big((b_i \vee c_j)_{i\in N}\big) = g\big((b_i)_{i\in N}\big) \vee c_j.$$

We consider two cases.

Case 1. There exists a k which is the smallest natural number with the property that $(b_i \vee c_k)_{i\in N} = \bar{c}_k$ (and then $g\big((b_i)_{i\in N}\big) = c_k$). Then, if $j \geqslant k$,

$$(b_i \vee c_j)_{i\in N} = (b_i \vee c_k \vee c_j)_{i\in N} = \bar{c}_j;$$

thus

$$g\big((b_i \vee c_j)_{j\in N}\big) = c_j = c_k \vee c_j = g\big((b_i)_{i\in N}\big) \vee c_j.$$

If $j < k$, then

$$(b_i \vee c_j \vee c_k)_{i\in N} = (b_i \vee c_k)_{i\in N} = \bar{c}_k,$$

and k is the smallest natural number with this property; thus

$$g\big((b_i \vee c_j)_{i\in N}\big) = c_k = c_k \vee c_j = g\big((b_i)_{i\in N}\big) \vee c_j.$$

Case 2. There is no $k \in N$ such that $(b_i \vee c_k)_{i \in N} = \bar{c}_k$. Then $g\big((b_i)_{i \in N}\big) = 1$. If there exists a $k \in N$ with $(b_i \vee c_j \vee c_k)_{i \in N} = \bar{c}_k$, then $b_0 \vee c_j \vee c_k = c_k$, thus $c_j \leqslant c_k$; consequently,

$$(b_i \vee c_k)_{i \in N} = (b_i \vee c_j \vee c_k)_{i \in N} = \bar{c}_k,$$

and this is a contradiction. Thus

$$g\big((b_i \vee c_j)_{i \in N}\big) = 1 = 1 \vee c_j = g\big((b_i)_{i \in N}\big) \vee c_j.$$

This completes the proof.

COROLLARY. *$\mathscr{A}$ is non-trivial.*

Now, for each algebra $\boldsymbol{L} = \big(L, \wedge, \vee, 0, 1, (c_i)_{i \in N}, g\big) \in \mathscr{A}$, and for each $j \in N$, $L_j = [c_j, 1] = \{a \in L \mid c_j \leqslant a\}$ is a sublattice of L, and, moreover, (4) implies that L_j is closed under the operation g.

PROPOSITION 2. *For each algebra $\boldsymbol{L} = \big(L, \wedge, \vee, 0, 1, (c_i)_{i \in N}, g\big) \in \mathscr{A}$ and for each $j \in N$,*

$$\boldsymbol{L}_j = \big(L_j, \wedge | L_j, \vee | L_j, c_j, 1, (c_j \vee c_i)_{i \in N}, g | L_j\big) \in \mathscr{A},$$

and the mapping φ_j: $L \to L_j$ defined by $\varphi(a) = a \vee c_j$ is an $\mathscr{A}$-algebra homomorphism.

Proof. To see that $\boldsymbol{L}_j \in \mathscr{A}$ it is only necessary to show that it satisfies (1)-(5). Conditions (1), (2), (4) and (5) follow immediately from the corresponding properties of $\boldsymbol{L}$, since all the operations in $\boldsymbol{L}_j$ except the constants are just the restrictions of the corresponding operations on L. We have

$$g\big((c_i \vee c_j)_{i \in N}\big) = g\big((c_i)_{i \in N}\big) \vee c_j = 0 \vee c_j = c_j$$

since $\boldsymbol{L}$ satisfies (3) and (4), and thus $\boldsymbol{L}_j$ satisfies (3).

Since $(L, \wedge, \vee)$ is a *distributive* lattice, it follows that φ_j is a lattice homomorphism. Moreover, since $\boldsymbol{L}$ satisfies (4), it follows that, for all countable sequences $(b_i)_{i \in N}$ of elements of L, $g\Big(\varphi_j\big((b_i)_{i \in N}\big)\Big) = \varphi_j\Big(g\big((b_i)_{i \in N}\big)\Big)$. The homomorphism φ_j obviously maps the nullary operations of $\boldsymbol{L}$ to the corresponding nullary operations of $\boldsymbol{L}_j$, and thus φ_j is an $\mathscr{A}$-algebra homomorphism.

COROLLARY. *$\mathscr{A}$ contains no simple algebras.*

Proof. For any $\boldsymbol{L} = \big(L, \wedge, \vee, 0, 1, (c_i)_{i \in N}, g\big) \in \mathscr{A}$ with $0 \neq 1$, conditions (3) and (5) imply that there exists a j with $c_j \neq 0$ and $c_j \neq 1$. But then, if φ_j is the homomorphism described in Proposition 2, the kernel of φ_j is a proper non-trivial congruence on $\boldsymbol{L}$, and hence $\boldsymbol{L}$ is not simple.

It should be pointed out that one could obtain an example of an equational class with only *finitely many operations* which does not contain a simple algebra by discarding from $\mathscr{A}$ the constants c_i, adding a new

$\aleph_0$-ary operation, say h, and a condition on h analogous to (4), and then "re-introducing" the constants c_i as derived operations, by defining, for instance, c_i to be the image under h of the sequence with 1 in the i-th place and zeroes elsewhere.

2. Let $\mathscr{B}$ be the equational class of all $\aleph_0$-complete Boolean algebras, that is, algebras $\left(B, \wedge, \vee, ', 0, 1, \bigwedge, \bigvee\right)$, where $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra, and $\bigwedge$, $\bigvee$ are $\aleph_0$-ary operations satisfying the obvious identities. The two-element Boolean algebra $\mathit{2}$ is a simple $\aleph_0$-complete Boolean algebra. It was proved in Banaschewski and Nelson [1] that $\mathit{2}$ is the only subdirectly irreducible $\aleph_0$-complete Boolean algebra, and hence $\mathit{2}$ is the only simple algebra in $\mathscr{B}$.

Clearly, no non-trivial algebra in $\mathscr{B}$ has a trivial subalgebra, since the class of Boolean algebras already has this property.

It is well known that the lattice $\boldsymbol{B}$ of all regular-open subsets of the real line is an $\aleph_0$-complete Boolean algebra [3]; however, if $\varphi: \boldsymbol{B} \to \mathit{2}$ were an $\aleph_0$-complete homomorphism, then $\varphi^{-1}(\{0\})$ would be an $\aleph_0$-complete proper prime (and hence maximal) ideal in $\boldsymbol{B}$. Since every regular-open subset of the real line is the join (in $\boldsymbol{B}$) of all the open intervals with rational end points contained in it, it follows that every $\aleph_0$-complete ideal in $\boldsymbol{B}$ is complete, and hence principal. But there are no proper principal ideals in $\boldsymbol{B}$ which are maximal, and hence there is no $\aleph_0$-complete homomorphism from $\boldsymbol{B}$ to $\mathit{2}$.

Thus $\mathscr{B}$ provides the desired example of an equational class which contains a simple algebra, and in which no non-trivial algebra contains a trivial subalgebra, but not every non-trivial algebra has a simple quotient.

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