UNIVERSAL EMBEDDINGS OF $l^1(\alpha)$ INTO THE SPACE
OF CONTINUOUS FUNCTIONS ON A PRODUCT SPACE

BY

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Preliminaries. The ordinals are defined in such a way that an \textit{ordinal} is the set of smaller ordinals. A \textit{cardinal} is an ordinal not in one-to-one correspondence with any smaller ordinal. The \textit{cofinality} of a cardinal $\alpha$, denoted by $\text{cf}(\alpha)$, is the least cardinal $\beta$ such that $\alpha$ is a cardinal sum of $\beta$-many cardinals, each smaller than $\alpha$.

A cardinal $\alpha$ is \textit{regular} if $\alpha = \text{cf}(\alpha)$, and \textit{singular} otherwise. We denote by $\alpha^+$ the least cardinal which is strictly greater than $\alpha$, and by $\omega$ the first infinite cardinal.

If $\alpha$ and $\beta$ are cardinals, we denote by $\alpha \Delta$ the cardinal sum $\sum_{\beta \in \alpha} \alpha$.

The cardinality of a set $A$ is denoted by $|A|$. We denote by $\mathcal{P}_\alpha(A)$ the set of those subsets of $A$ that have cardinality less than $\alpha$, and by $\mathcal{P}(A)$ the set of all subsets of $A$.

Let $\alpha$ and $\kappa$ be cardinals. We say that $\alpha$ is \textit{strongly $\kappa$-inaccessible} if $\beta^\lambda < \alpha$ for every $\beta < \alpha$ and $\lambda < \kappa$. If, in addition, $\alpha > \kappa$, we write $\alpha \gg \kappa$.

For set-theoretic background we refer the reader to [4].

0.1. Theorem (Erdős–Rado). Let $\alpha > \omega$ be a regular cardinal and
\[
\{N_\xi: \xi < \alpha\} \subset \mathcal{P}_\omega(\alpha).
\]
Then there are $A \subset \alpha$, with $|A| = \alpha$, and $N \subset \alpha$ such that
\[
N_{\xi_1} \cap N_{\xi_2} = N \quad \text{for } \xi_1, \xi_2 \in A, \xi_1 \neq \xi_2.
\]

0.2. Proposition. Let $\alpha, \beta$ be cardinals, $\alpha > \beta \gg \omega^+$, let
\[
\{J_\xi: \xi < \alpha\} \subset \mathcal{P}_\beta(\alpha) \quad \text{and} \quad \{N_\xi: \xi < \alpha\} \subset \mathcal{P}(\alpha)
\]
with $N_\xi \cap N_\eta = \emptyset$ for every $\xi < \eta < \alpha$. Then there exists $A \subset \alpha$, with $|A| = \alpha$, such that
\[
N_\xi \cap J_\eta = \emptyset \quad \text{for every } \xi, \eta \in A, \xi \neq \eta.
\]

The above proposition in the case $\beta = \omega^+$ is contained in [1]. To prove it we use the Hajnal free set theorem.
All topological spaces in this paper are assumed to be infinite and Hausdorff. Let \( X \) be a topological space and \( \alpha \) an infinite cardinal. We say that \( X \) has precaliber \( \alpha \) if every family \( \{ U_\xi : \xi < \alpha \} \) of non-empty open subsets of \( X \) contains a subfamily with the same cardinality and the finite intersection property.

If \( X \) has precaliber \( \alpha \), then \( \text{cf}(\alpha) > \omega \) (Corollary 2.25 in [5]). \( X \) is said to have caliber \( \alpha \) if every family \( \{ U_\xi : \xi < \alpha \} \) of non-empty open subsets of \( X \) contains a subfamily of the same cardinality with non-empty intersection.

The Souslin number \( S(X) \) of \( X \) is defined to be the smallest cardinal \( \alpha \) such that there is no family of \( \alpha \)-many pairwise disjoint non-empty open subsets of \( X \). By the Erdös–Tarski theorem, \( S(X) \) is an uncountable regular cardinal.

\( X \) is called pseudo-\( \alpha \)-compact if for every family of \( \alpha \)-many non-empty open subsets of \( X \) there exists \( x \in X \) such that every neighbourhood of \( x \) meets infinitely many sets from the family.

We set

\[
  r(X) = \min \{ \alpha : X \text{ is pseudo-} \alpha \text{-compact} \},
\]

\[
  \text{ca}(X) = \min \{ \alpha : X \text{ has caliber } \alpha \}.
\]

Then it is clear that

\[
  r(X) \leq \text{ca}(X) \leq d(X)^+,
\]

where \( d(X) \) is the density character of \( X \).

All Banach spaces in this paper are assumed to be real. Let \( X \) be a topological space. By \( C^*(X) \) we denote the Banach space of all real-valued bounded continuous functions on \( X \) with the supremum norm. If \( Y \) is a Banach space and \( \alpha \) a cardinal, we say that \( l_1(\alpha) \) embeds universally in \( Y \) if for any closed subspace \( Z \) of \( Y \), with \( \dim Z = \alpha \), there exists an isomorphic embedding of \( l_1(\alpha) \) into \( Z \). It is clear that an isomorphic embedding of \( l_1(\alpha) \) into \( Z \) exists iff there exist a uniformly bounded family \( \{ z_\xi : \xi < \alpha \} \subset Z \) and a constant \( M > 0 \) such that

\[
  \left\| \sum_{i=1}^n c_i z_\xi_i \right\| \geq M \sum_{i=1}^n |c_i|
\]

for all \( c_1, \ldots, c_n \in \mathbb{R} \), \( \xi_1, \ldots, \xi_n < \alpha \) pairwise different and for each \( n \in \mathbb{N} \). Such a family is said to be equivalent to the usual basis \( \{ e_\xi : \xi < \alpha \} \) of \( l_1(\alpha) \), where \( e_\xi(\eta) = 0 \) for \( \xi \neq \eta \) and \( e_\xi(\xi) = 1 \).

Let \( X \) be a set and \( \{ (A_i, B_i) : i \in I \} \) be a family of subsets of \( X \) with the property \( A_i \cap B_i = \emptyset \) for \( i \in I \). The above family is called independent if for every pair of finite disjoint subsets \( K, F \) of \( I \) we have

\[
  \left( \bigcap_{i \in K} A_i \right) \cap \left( \bigcap_{i \in F} B_i \right) \neq \emptyset.
\]

The connection between independent families of sets and the isomorphic embedding of \( l_1(\alpha) \) into subspaces of spaces of the form \( C^*(X) \) is described by the following lemma, due to Rosenthal [9].
0.3. Lemma. Let $X$ be a set and $\{f_i : i \in I\}$ a family of uniformly bounded real functions on $X$. Let also $r, \delta$ be real numbers with $\delta > 0$ such that if

$$A_i = \{x \in X : f_i(x) > r + \delta\}, \quad B_i = \{x \in X : f_i(x) < r\},$$

then the family $\{(A_i, B_i) : i \in I\}$ is independent. Then the following inequality is valid:

$$\left\| \sum_{j=1}^{n} c_j f_j \right\| \geq \frac{\delta}{2} \sum_{j=1}^{n} |c_j|$$

for every $c_1, \ldots, c_n \in \mathbb{R}$, $i_1, \ldots, i_n \in I$ pairwise different and each $n \in \mathbb{N}$.

If $(X_i)_{i \in I}$ is a family of topological spaces and $f \in C^*(\prod_{i \in I} X_i)$, we say that $f$ depends only on $J \subset I$ if for each $x, y \in \prod_{i \in J} X_i$ with $pr_j(x) = pr_j(y)$ we have $f(x) = f(y)$.

If $J \subset I$, it is clear that $C^*(\prod_{i \in J} X_i)$ embeds isometrically in $C^*(\prod_{i \in I} X_i)$.

Let $(Z_i)_{i \in I}$ be a family of Banach spaces. By $(\sum_{i \in I} \oplus Z_i)_{\infty}$ we denote the Banach space

$$\{z = (z_i)_{i \in I} : z_i \in Z_i \text{ and } \sup_{i \in I} \|z_i\| < +\infty\}$$

with the norm

$$\|z\| = \sup_{i \in I} \|z_i\|.$$

1.1. Definition. Let $(X_i)_{i \in I}$ be a family of topological spaces and

$$X = \prod_{i \in I} X_i.$$

For every cardinal $\alpha$ let

$$C^*_\alpha(X) = \{f \in C^*(X) : \text{there exists } J \in \mathcal{P}_\alpha(I) \text{ such that } f \text{ depends only on } J\}.$$

We set

$$l(X) = \min \{\alpha : C^*_\alpha(X) = C^*(X)\}, \quad k(X) = \min \{\alpha : C^*_\alpha(X) = C^*(X)\}.$$

1.2. Remarks. (i) It is clear that

$$l(X) \leq k(X) \leq l(X)^{+},$$

and if $\operatorname{cf}(l(X)) > \omega$, then $k(X) = l(X)$.

(ii) If $X_i$ is a compact topological space for every $i \in I$, then, by the Stone–Weierstrass theorem, $l(X) = \omega$.

(iii) From [3] we have $k(X) \leq r(X)^{+}$. 

5 – Colloquium Mathematicum LVII. 2
1.3. Theorem. Let \( \alpha \) be a cardinal with \( \alpha > \omega^+ \). Let \((X_i)_{i \in I}\) be a family of topological spaces and
\[
X = \prod_{i \in I} X_i.
\]

We suppose that \( X \) has precaliber \( \alpha \) and \( \alpha > k(X) \). If \( Z \) is a closed subspace of \( C^*(X) \) and if, for every \( J \in P_\alpha(I) \), \( Z \) is not contained in \( C^*(\prod_{i \in J} X_i) \), then \( Z \) contains isomorphically a copy of \( l_1(\alpha) \).

Proof. For every \( f \in Z \) there is an \( J_f \in P_{k(X)}(I) \) such that \( f \) depends only on \( J_f \). Without loss of generality we may assume that
\[
I = \bigcup_{f \in Z} J_f.
\]

By transfinite induction, we may construct a family
\[
\{ f_\xi: \xi < \alpha \} \subset Z,
\]
with \( \| f_\xi \| = 1 \), such that if we set
\[
J_f_\xi = J_\xi \quad \text{and} \quad T_\eta = \bigcup_{\zeta < \eta} J_\zeta,
\]
then \( f_\eta \) does not depend only on \( T_\eta \), hence \( J_\eta \not\subset T_\eta \). We distinguish two cases:

Case I. There exists an \( A \subset \alpha \), with \( |A| = \alpha \), such that
\[
T_\eta \cap J_\xi \neq \emptyset \quad \text{for all} \quad \eta \in A.
\]
Since \( X \) has precaliber \( \alpha \), we have \( \text{cf}(\alpha) > \omega \). Hence there exist \( A_1 \subset A \), with \( |A_1| = \alpha \), and rational numbers \( r, \delta \) with \( \delta > 0 \) such that the set
\[
\Lambda_\eta = \left\{ x \in \prod_{i \in J_\eta} X_i: \sup_{z \in \{x\} \times \prod_{i \in I \setminus J_\eta} X_i} f_\eta(z) > r + \delta > r > \inf_{z \in \{x\} \times \prod_{i \in I \setminus J_\eta} X_i} f_\eta(z) \right\}
\]
is non-empty for every \( \eta \in A_1 \), where \( J_\eta = T_\eta \cap J_\eta \). It is clear that \( \Lambda_\eta \) is an open subset of \( \prod_{i \in J_\eta} X_i \). We set \( N_\eta = J_\eta \setminus T_\eta \). Clearly, \( N_\eta \cap N_\xi = \emptyset \) for every \( \eta, \xi \in A_1 \).

By Proposition 0.2 there exists \( A_2 \subset A_1 \), with \( |A_2| = \alpha \), such that \( J_\xi \cap N_\eta = \emptyset \) for every \( \xi, \eta \in A_2 \) with \( \xi \neq \eta \). We set
\[
M_\eta = \Lambda_\eta \times \prod_{i \in I \setminus J_\eta} X_i.
\]
Since \( X \) has precaliber \( \alpha \), there exists \( A_3 \subset A_2 \), with \( |A_3| = \alpha \), such that the family \( \{ M_\eta: \eta \in A_3 \} \) has the finite intersection property. Let
\[
B_\eta = \{ x \in X: f_\eta(x) > r + \delta \}, \quad C_\eta = \{ x \in X: f_\eta(x) < r \}
\]
for every \( \eta \in A_3 \). Clearly, \( B_\eta \) and \( C_\eta \) are not empty. We will see that the above family is independent.
Let $F$ and $G$ be finite disjoint subsets of $A_3$. Then there is
\[ z \in \bigcap_{\eta \in F \cup G} M_\eta. \]
We set $z_\eta = \text{pr}_{J_\eta}(z_\eta)$. There exists
\[ y_\eta \in \prod_{\iota \in I \setminus J_\eta} X_\iota \]
such that $f_\eta(z_\eta, y_\eta) > r + \delta$ if $\eta \in F$ and $f_\eta(z_\eta, y_\eta) < r$ if $\eta \in G$. We consider $x \in \prod_{\iota} X_\iota$ such that
\[ \text{pr}_{J_\eta}(x) = z_\eta \quad \text{and} \quad \text{pr}_{N_\eta}(x) = \text{pr}_{N_\eta}(y_\eta) \]
for every $\eta \in F \cup G$ (such an $x$ exists, since $J_\eta \cap N_\xi = \emptyset$ for every $\xi, \eta \in A_3$ with $\xi \neq \eta$). Since $f_\eta(x) = f_\eta(z_\eta, y_\eta)$ for $\eta \in F \cup G$, we have
\[ x \in \left( \bigcap_{\eta \in F} B_\eta \right) \cap \left( \bigcap_{\eta \in G} C_\eta \right). \]
Thus by Lemma 0.3 the family $\{f_\eta, \eta \in A_3\}$ is equivalent to the usual basis of $l_1(\alpha)$.

Case II. Case I does not hold. Then there exists $A \subset \alpha$, $|A| = \alpha$, such that $J_\eta \cap T_\eta = \emptyset$ for all $\eta \in A$. Then, as in case I, we can find an $A_1 \subset A$, with $|A_1| = \alpha$, such that $\{f_\eta, \eta \in A_1\}$ is equivalent to the usual basis of $l_1(\alpha)$.

1.4. Theorem. Let $(X_\iota)_{\iota \in I}$ be a family of topological spaces, let
\[ X = \prod_{\iota \in I} X_\iota \quad \text{and} \quad l(X) = \omega. \]
We suppose that, for each $\iota \in I$, $X_\iota$ has precaliber $\omega^+$ and $C^*(\prod_{\iota \in I} X_\iota)$ is separable for every $F \in \mathcal{P}_\omega(I)$. Then, if $C^*(X)$ is non-separable, $l_1(\omega^+)$ embeds universally in $C^*(X)$.

Proof. Let $Z$ be a closed subspace of $C^*(X)$ with $\dim Z = \omega^+$, and $0 < \delta < 1$. There exists a family $\{f_\xi : \xi < \omega^+\} \subset Z$ with $\|f_\xi\| = 1$ and $\|f_\xi - f_\eta\| > \delta$ for every $\xi < \eta < \omega^+$.

For every $\xi < \omega^+$ there is a $g_\xi \in C^*(X)$ such that
\[ \|g_\xi - f_\xi\| < \delta/20 \]
and $g_\xi$ depends only on a set $J_\xi$ with $|J_\xi| < \omega$. It is clear that
\[ \|g_\xi - g_\eta\| > \delta/20 \quad \text{for every} \quad \xi < \eta < \omega^+. \]
By Theorem 0.1 there is an $A \subset \omega^+$, with $|A| = \omega^+$, and a $J \subset I$ such that
\[ J_\xi \cap J_\eta = J \quad \text{for every} \quad \xi, \eta \in A \quad \text{with} \quad \xi \neq \eta. \]
We have that $C^*(\prod_{\iota \in J} X_\iota)$ is separable. Hence we may suppose that $J \neq J_\xi$ for every $\xi \in A$. We distinguish two cases.
Case I. Let \( J \neq \emptyset \). Then there exist \( A_1 \subset A \), with \( |A_1| = \omega^+ \), and \( r \in R \) such that \( A_\xi \neq \emptyset \) for every \( \xi \in A_1 \), where
\[
A_\xi = \{ x \in \prod_{\xi \in J} X_\xi : \sup_{\xi \in J} \{ g_\xi(z) : z \in \{ x \} \times \prod_{\xi \notin J} X_\xi \} > r + \frac{9}{8} > r > \inf_{\xi \in J} \{ g_\xi(z) : z \in \{ x \} \times \prod_{\xi \notin J} X_\xi \} \}.
\]
It is clear that \( A_\xi \) is an open subset of \( \prod_{\xi \in J} X_\xi \). Since \( X_\xi \) has precaliber \( \omega^+ \), it is easy to see that \( \prod_{\xi \in J} X_\xi \) has precaliber \( \omega^+ \), hence there is an \( \xi \in A_1 \), with \( |A_2| = \omega^+ \), such that the family \( \{ A_\xi : \xi \in A_2 \} \) has the finite intersection property. We set
\[
B_\xi = \{ x \in X : g_\xi(x) > r + \frac{9}{8} \}, \quad C_\xi = \{ x \in X : g_\xi(x) < r \}
\]
for every \( \xi \in A \). It is easy to see that the family \( \{ (B_\xi, C_\xi) : \xi \in A_2 \} \) is independent, and so by Lemma 0.3 we have
\[
\left\| \sum_{i=1}^{n} c_i g_{\xi_i} \right\| \geq \left( \frac{9}{16} \right) \sum_{i=1}^{n} |c_i|
\]
for every \( \xi_1, \ldots, \xi_n \in A_2 \) pairwise different, \( c_1, \ldots, c_n \in R \) and \( n \in N \). Thus we have
\[
\left\| \sum_{i=1}^{n} c_i f_{\xi_i} \right\| \geq \left( \frac{9}{20} \right) \sum_{i=1}^{n} |c_i|
\]
for every \( \xi_1, \ldots, \xi_n \in A_2 \) pairwise different, \( c_1, \ldots, c_n \in R \) and \( n \in N \). Thus the family \( \{ f_{\xi_i} : \xi \in A_2 \} \) is equivalent to the usual basis of \( l_1(\alpha) \).

Case II. Let \( J = \emptyset \). Then, as in case I, we find \( A_1 \subset A \), with \( |A_1| = \alpha \), such that the family \( \{ f_{\xi_i} : \xi \in A_1 \} \) is equivalent to the usual basis of \( l_1(\alpha) \).

1.5. Lemma. Let \((X_i)_{i \in I}\) be a family of topological spaces and
\[
X = \prod_{i \in I} X_i.
\]
Then \( C^*(X) \) can be embedded isometrically in
\[
\left( \sum_{\alpha \in \mathcal{A}(I)} \oplus C^*(\prod_{i \in \alpha} X_i) \right)_{\infty}.
\]

Proof. Let \( x^* = (x_i)_{i \in I} \) be an element of \( X \) and take the function
\[
T : C^*(X) \rightarrow \left( \sum_{\alpha \in \mathcal{A}(I)} \oplus C^*(\prod_{i \in \alpha} X_i) \right)_{\infty}
\]
such that \( T(f) = (f_\alpha)_{\alpha \in \mathcal{A}(I)} \), where
\[
f_\alpha : \prod_{i \in A} X_i \rightarrow R \quad \text{and} \quad f_\alpha(x) = f(y), \quad \text{where} \quad y_i = x_i \quad \text{for} \quad i \in A,
\]
and \( y_i = x_i \) for \( i \in I \setminus A \). It is easy to see that \( T \) is a well-defined linear isometry. Combining Theorems 1.3 and 1.4 yields the following general corollary:
1.6. COROLLARY. Let $\alpha$ be a cardinal. Let $(X_i)_{i \in I}$ be a family of topological spaces and

$$X = \prod_{i \in I} X_i.$$  

We suppose that $X$ has precaliber $\alpha$, $\alpha > l(X)$, and $\beta^{l(X)} < \alpha$ for every $\beta < \alpha$. We also suppose that

$$\alpha > \sup \{ \dim C^*(\prod_{i \in I} X_i) : F \in \mathcal{P}_\omega(I) \}.$$  

If $\dim C^*(X) \geq \alpha$, then $l_1(\alpha)$ embeds universally in $C^*(X)$.

Proof. Since $X$ has precaliber $\alpha$, we have $\text{cf}(\alpha) > \omega$.

If $\alpha = \omega^+$, then the corollary reduces to Theorem 1.4. Let $\alpha > \omega^+$ and $Z$ be a closed linear subspace of $C^*(X)$ with $\dim Z \geq \alpha$. We will prove that the conditions of Theorem 1.3 are valid.

If $l(X) = \omega$, then $k(X) \leq \omega^+ < \alpha$.

If $l(X) > \omega$ and $\text{cf}(l(X)) > \omega$, then $k(X) = l(X) < \alpha$.

If $l(X) > \omega$ and $\text{cf}(l(X)) = \omega$, then

$$\alpha > \alpha = \alpha \geq l(X)^{l(X)} \geq l(X)^{\text{cf}(l(X))} > l(X),$$

so $k(X) < \alpha$.

We suppose that there exists $J \in \mathcal{P}_\alpha(I)$ such that

$$\dim C^*(\prod_{i \in J} X_i) \geq \alpha.$$  

Then, if $0 < \delta < 1$, there exists a family

$$\{ f_\xi : \xi < \alpha \} \subset C^*(\prod_{i \in J} X_i)$$

with $\|f_\xi\| = 1$ and $\|f_\xi - f_\eta\| \geq \delta$ for every $\xi < \eta < \alpha$. Without loss of generality we may assume that $f_\xi$ depends only on $J_\xi \subset J$ with $|J_\xi| < l(X)$ for all $\xi < \alpha$.

There exists a regular cardinal $\beta$ such that

$$\alpha \geq \beta > |J|^{l(X)},$$

$$\beta > \dim C^*(\prod_{i \in F} X_i) \quad \text{for every} \quad F \in \mathcal{P}_\omega(J) \text{ and } \beta > l(X).$$

Then there are a $J' \subset J$ and an $A \subset \alpha$, with $|A| = \beta$, such that $J_\xi = J'$ for every $\xi \in A$. Consequently,

$$\dim C^*(\prod_{i \in J'} X_i) \geq \beta.$$  

From Lemma 1.5 we have

$$\beta \leq \dim C^*(\prod_{i \in J'} X_i) \leq \sup \{ \dim C^*(\prod_{i \in J'} X_i)^{J_1} : F \in \mathcal{P}_\omega(J') \} < \beta,$$

which is a contradiction. Thus, for $Z$, the conditions of Theorem 1.3 are valid. Therefore $l_1(\alpha)$ embeds isomorphically into $Z$. 

1.7. Remark. It is clear from the proof of Corollary 1.6 that when \( \alpha \) is a regular cardinal, then condition (\( \ast \)) may be weakened to:

\[
\alpha > \dim C^*(\prod_{i \in F} X_i) \quad \text{for every } F \in \mathcal{P}_\alpha(I).
\]

1.8. Corollary. Let \( \alpha \) be a cardinal. Let also \( (X_i)_{i \in I} \) be a family of compact topological spaces and

\[
X = \prod_{i \in I} X_i.
\]

We suppose that \( X_i \) has caliber \( \alpha \) for every \( i \in I \) and

\[
\alpha > \sup \{w(X_i): i \in I\},
\]

where \( w(X_i) \) is the topological weight of \( X_i \). If \( \dim C^*(X) \geq \alpha \), then \( l_1(\alpha) \) embeds universally in \( C^*(X) \).

1.9. Remarks. (i) In the case of a regular cardinal \( \alpha \) we need only the condition \( \alpha > w(X_i) \) for every \( i \in I \), as in Corollary 1.6.

(ii) It is obvious that if \( \alpha \) is an infinite cardinal and \( Y \) a topological space with \( w(Y) \geq \alpha \), which is a continuous image of the product of a family \( (X_i)_{i \in I} \) of compact topological spaces with the properties of Corollary 1.8, then \( l_1(\alpha) \) embeds universally in \( C(Y) \).

(iii) Corollary 1.8 extends a result of Argyros and Negrepontis [2] and also contains the result of Hagler [6] for dyadic spaces.

Since \( k(X) \leq r(X)^+ \), Corollary 1.6 gives easily the following

1.10. Corollary. Let \( \alpha \) be an uncountable cardinal, \( (X_i)_{i \in I} \) be a family of topological spaces and

\[
X = \prod_{i \in I} X_i.
\]

We suppose that \( X \) has precaliber \( \alpha \) and \( \beta^{(X)} < \alpha \) for every \( \beta < \alpha \). We also suppose that

\[
\alpha > \sup \{\dim C^*(\prod_{i \in F} X_i): F \in \mathcal{P}_\alpha(I)\}.
\]

If \( \dim C^*(X) \geq \alpha \), then \( l_1(\alpha) \) embeds universally in \( C^*(X) \).

Finally, from Theorem 1.3 we obtain easily the following corollary, which is contained in [7].

1.11. Corollary. Let \( \alpha \) be a regular cardinal with \( \alpha \geq \omega^+ \), and \( (X_i)_{i \in I} \) a family of topological spaces with \( \alpha \geq k(X) \), where

\[
X = \prod_{i \in I} X_i.
\]
We suppose that \( l_1(\alpha) \) embeds universally into \( C^* \left( \prod_{i \in F} X_i \right) \) for every \( F \in \mathcal{P}_\omega(I) \) such that

\[
\dim C^* \left( \prod_{i \in F} X_i \right) \geq \alpha.
\]

Then \( l_1(\alpha) \) embeds universally into \( C^*(X) \) if \( \dim C^*(X) \geq \alpha \).

**Proof.** Let \( Z \) be a subspace of \( C^*(X) \) with \( \dim Z = \alpha \). If, for every \( J \in \mathcal{P}_\omega(I) \),

\[
Z \not\subseteq C^* \left( \prod_{i \in J} X_i \right),
\]

then the result follows from Theorem 1.3. We suppose that there is \( J \in \mathcal{P}_\omega(I) \) with

\[
Z \subseteq C^* \left( \prod_{i \in J} X_i \right).
\]

If \( 0 < \theta < 1 \), there is \( \{ f_\xi : \xi < \alpha \} \subseteq Z \) with \( \| f_\xi \| = 1 \) and \( \| f_\xi - f_\eta \| > \theta \) for every \( \xi < \eta < \alpha \). For every \( \xi < \alpha \) there are \( J_\xi \subseteq J \), with \( |J_\xi| < k(X) \), and \( f_\xi \) depending only on \( J_\xi \). Since \( \alpha \) is a regular cardinal, \( \alpha \gg k(X) \) and \( |J| < \alpha \), we see that there exist \( A \subset \alpha \), with \( |A| = \alpha \), and \( L \subset J \) such that \( J_\xi = L \) for every \( \xi \in A \). Let \( Y \) be the closed subspace generated by \( \{ f_\xi : \xi \in A \} \). Since

\[
Y \subseteq C^* \left( \prod_{i \in L} X_i \right),
\]

by Lemma 1.5, \( Y \) embeds isometrically in \( \left( \sum_{A \in \mathcal{P}_\omega(L)} \bigoplus Y_A \right)_\infty \), where \( Y_A \) is the closed image of \( Y \) in \( C^* \left( \prod_{i \in A} X_i \right) \) through the canonical projection of

\[
\left( \sum_{A \in \mathcal{P}_\omega(L)} \bigoplus C^* \left( \prod_{i \in A} X_i \right) \right)_\infty
\]

onto

\[
C^* \left( \prod_{i \in A} X_i \right).
\]

If \( \dim Y_A < \alpha \) for every \( A \in \mathcal{P}_\omega(L) \), then

\[
\alpha = \dim Y \leq \sup \{(\dim Y_A)^{|A|} : A \in \mathcal{P}_\omega(L)\} < \alpha.
\]

So there is a \( A \in \mathcal{P}_\omega(L) \) such that \( \dim Y_A = \alpha \) and, consequently, \( l_1(\alpha) \) embeds isomorphically in \( Y_A \), hence in \( Y \), since \( \text{cf}(\alpha) > \omega \) (see [8]). Thus \( l_1(\alpha) \) embeds isomorphically in \( Z \).

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