DYNAMICAL SYSTEMS AND ERGODIC THEORY
BANACH CENTER PUBLICATIONS, VOLUME 23
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1989

TOWARDS A PROOF OF RECURRENCE FOR THE LORENTZ PROCESS

NANDOR SIMANYI

Mathematical Institute of the Hungarian Academy of Sciences
Budapest, Hungary

The content of this paper is an attempt of proving the recurrence property for the planar Lorentz process with periodic configuration of scatteres. This model describes the motion of a point – like particle in the plane among periodically placed scattering bodies. Trying to give an indirect proof of recurrence we supposed the contrary and gave an upper bound for the measure of wandering sets of the phase space. Moreover, a weak form of recurrence is also proved.

0. Introduction

The content of this paper is a modest attempt of proving the recurrence property for the planar Lorentz process with periodic configuration of scatteres. This model describes the motion of a point-like particle on the plane R^2 among twice periodically placed scatterers with smooth boundary. The motion is linear at unit speed and the reflections at the boundaries of the scatterers are elastic, i.e. the angle of incidence equals the angle of reflection. It is a natural requirement to investigate the ergodic properties of this process with infinite invariant Liouville measure. In this Kyoto lecture [Sin (1981)] Sinai expressed that the study of the Lorentz process may be important in the investigations of statistical physics of solid bodies and in the understanding of certain phenomena concerning heat conduction. Bunimovich and Sinai gave an excellent detailed study of the Lorentz process in [B–S(1981)]. They proved exponential correlation decay for the velocity of a particle in the Lorentz gas (this model is the many – particle variant of the Lorentz process without interaction between the particles) and they also proved Donsker's invariance principle for the Lorentz process, i.e. its weak convergence to a planar Wiener process. Nevertheless, these results do not answer the question whether the planar Lorentz process is recurrent or not. This problem seems to be quite hard. One hopes that the answer is affirmative because of the analogy with the usual symmetric random walk on the lattice Z^2, which walk is, by Pólya's famous theorem, recurrent. In [K–Sz(1985)] Krámli and Szász proved 'quasi recurrence' for the Lorentz process, that is, they proved that for any fixed constant $c > 3/2$ the process returns almost surely to the square with edge $(\log(n))^c$ in the n-th step for infinitely many natural numbers n.

Let us turn to the exact definitions and notations. We denote the configuration space by Q, which is the plane R^2 from which the union of a system of disjoint convex compact figures is removed. We suppose that these figures (the so called scatterers) have smooth (say C^∞) boundaries. We also suppose that this system is twice periodic, i.e. there are two linearly independent vectors such that the configuration of scatterers is invariant under the plane translations by these vectors, and we also suppose that the scatterers do not accumulate in finite regions. The phase space of the planar Lorentz process is $M = Q \times S^1$, where S^1 is the space of unit velocity vectors. The time evolution S^t on the space M corresponds to the linear motion at unit speed and to elastic reflections at the boundary δQ of Q. (The angle of incidence equals the angle of reflection.) Finally, the infinite invariant Liouville measure μ on M can be given by the formula $d\mu = dq^{(1)}\, dq^{(2)}\, d\omega$, where $q^{(1)}$ and $q^{(2)}$ are the two place coordinates of the point $x = (q, v) \in Q \times S^1 = M$ and ω is the angle of the velocity vector v.

In Section 1 we prove that the recurrence and the ergodicity of the dynamical system (M, S^t, μ) are equivalent. The proofs of this section are not detailed because they are not central to the topic of this paper.

In Section 2 we suppose that the planar Lorentz process is not recurrent (ergodic). Using this assumption we prove several types of 'pathological' behavior of this process. Moreover, a weak version of recurrence is proved. The main result of this paper is Theorem 2.1, where an upper bound is given for the measure of a wandering set W of the discrete time version (N, T, ν) of the process (M, S^t, μ). (For definitions cf. Section 1.) The optimal upper bound zero (non-existence of wandering sets) could give us the wanted recurrence. In the proof of Theorem 2.1 Donsker's invariance principle is heavily used. The Kolmogorov property of the partition for Sinai's billard system into local stable transversal fibres is also used. (For this property cf. Section 4 of the fundamental paper [Sin (1970)].) These two basic results seem to hold, when the dimension of the phase space is arbitrary, but planarity is very important in our proofs from the point of view of recurrence, because the analogous symmetric random walk on the lattice Z^d is transient for $d \geqslant 3$.

In my opinion, a proof of recurrence must be less elementary than my approach and it needs a better understanding of the analytic behavior of Sinai's billiard system. (For example, quantitative properties of its Markov partition seem to be very important.)

The numbering of definitions, propositions, theorems, corollaries are continuous in each section. The first number denotes the number of the section. (Thus, Proposition 1.3 follows Definition 1.2.)

1. Simple observations: Recurrence and ergodicity of the Lorentz process are equivalent

Let us consider the planar Lorentz process with periodic configuration of scatterers (for its definition cf. Section 0). For simplicity we suppose that the scatterers are discs with radius r ($0 < r < 1/2$) and their centres are the points of the lattice $\boldsymbol{Z}^2$. Analogous methods can be applied in the general case. We denote the configuration space by Q and the phase space $Q \times S^1$ by M. Of course, $Q = \{q \in \boldsymbol{R}^2 \colon \varrho(q, \boldsymbol{Z}^2) \geqslant r\}$ and S^1 is the space of unit velocity vectors (ϱ is the usual Euclidean metric on the plane $\boldsymbol{R}^2$). So we get a dynamical system (M, S^t, μ) endowed with the usual (infinite) Liouville measure μ for which $d\mu = dq^{(1)} dq^{(2)} d\omega$, where ω is the angle of the unit velocity vector v and $q^{(1)}$, $q^{(2)}$ are the coordinates of q. (S^t is the time-evolution of the system, expressing the motion of a point-particle in Q, moving at unit speed and colliding elastically at $\delta Q = \{q \in \boldsymbol{R}^2 \colon \varrho(q, \boldsymbol{Z}^2) = r\}$.) We consider, as usual, the Poincaré mapping of the original dynamical system (M, S^t, μ) in the following way: The phase space N of the Poincaré mapping is the boundary of M, i.e. $N = \delta Q \times S^1$. For any element $q \in \delta Q$ we identify the elements (q, v_1) and (q, v_2) of $\delta Q \times S^1$ iff v_2 is the reflected pair of the vector v_1 with respect to the tangent line of δQ at the point q. In our notation the element $(q, v) \in N$ always means that the vector v is directed inwards the billiard table Q. We can introduce the coordinates s, α in the space N, where s is the arc-length parameter of the point $q \in \delta Q$, measured counterclockwise around a scatterer, and α is the signed angle between the vectors $n(q)$ and v, where $n(q)$ is the normal vector of δQ at the point q, which normal vector is directed inwards the billiard table. ($0 \leqslant s < 2\pi r$, $-\pi/2 \leqslant \alpha \leqslant \pi/2$.) For $x = (q, v) \in N$ let $T(x)$ denote the point of N that represents the next reflection after the reflection represented by x. (T is the Poincaré mapping or the section mapping.) So we get the derived dynamical system (N, T, ν), where the infinite invariant measure ν satisfies the following equation:

$$(1) \qquad d\nu = ds\, d\alpha \cos\alpha .$$

The measure ν can be derived from μ in a natural way, because for every measurable function $f \colon N \to [0, 1/2 - r]$ and for every measurable subset $A \subset N$ we have

$$(2) \qquad \mu(A_f) = \int_A f\, d\nu ,$$

where $A_f = \{S^t(x)\colon x \in A \text{ and } 0 \leqslant t \leqslant f(x)\}$. Now we introduce the following subsets of N:

(3) $$N_{a,b} = \{(q, v) \in N\colon (q^{(1)} - a)^2 + (q^{(2)} - b)^2 = r^2\} \quad (a, b \in \boldsymbol{Z}).$$

Of course, the measure ν is invariant not only under the mapping T but under the plane-translations by vectors $(a, b) \in \boldsymbol{Z}^2$ as well. Moreover, these plane-translations commute with the mapping T. Let us normalize the measure ν in the following way: $\nu(N_{0,0}) = 1$. Throughout this paper we shall not distinguish between sets differing from each other by a set of ν-measure zero.

Definition 1.1. The dynamical system (N, T, ν) (or equivalently, the Lorentz process) is called *weakly recurrent* iff the set

(4) $$E = \{(q, v) \in N\colon \|q_k\| \not\to \infty \ (k \to \infty), \text{ where } T^k(q, v) := (q_k, v_k)\}$$

has positive ν-measure.

Definition 1.2. The system (N, T, ν) is called (as in the case of finite measures) *ergodic* iff for every T-invariant set $A \subset N$ we have either $\nu(A) = 0$ or $\nu(N - A) = 0$.

If the Lorentz process is not weakly recurrent, i.e. $\nu(E) = 0$, then for almost every trajectory of the dynamics T there exists a unique element of this trajectory for which the place component q has the smallest distance from the point $(1/2; 0)$. Consequently, the set

(5) $$W = \{(q, v) \in N\colon (q^{(1)} - 1/2)^2 + (q^{(2)})^2 < (q_k^{(1)} - 1/2)^2 + (q_k^{(2)})^2 \quad \text{for every } k \neq 0\}$$

has the wandering property, i.e.

(6) $$\nu(T^k(W) \cap T^l(W)) = 0 \ (k \neq l) \quad \text{and} \quad \nu\Big(N - \bigcup_{k=-\infty}^{+\infty} T^k(W)\Big) = 0.$$

It is clear that in this case the system (N, T, ν) is not ergodic. Consequently, ergodicity is a stronger property than the weak recurrence property.

Proposition 1.3. *Ergodicity and the weak recurrence property of the dynamical system (N, T, ν) are equivalent.*

Proof. Suppose that the system (N, T, ν) is weakly recurrent. In this case we have $\nu(E) > 0$ and E is a $\boldsymbol{Z}^2$-translation invariant measurable set. Factorizing the process by $\boldsymbol{Z}^2$-translations, we get a compact Sinai-billiard system, which is ergodic, as it was proved in [Sin (1970)]. Consequently, we get the relation $\nu(N - E) = 0$. Using this fact, we can prove the usual recurrence property for the system (N, T, ν) as it is stated in Poincaré's recurrence theorem for finite measure spaces. To this end, suppose the contrary of this property, that is, the existence of a measurable set $A \subset N$ such that for some measurable subset $B \subset A$ we have $\nu(B) > 0$ and $\nu(T^k(B) \cap A) = 0$ for every positive integer k. Thus a fortiori

$$\nu(T^k(B) \cap T^l(B)) = 0 \tag{7}$$

for every pair of integers $k \neq l$. Using the equation $\nu(N-E) = 0$ and choosing a subset of B instead of B we can assume that, iterating the mapping T, every point of B hits the set $N_{a,b}$ infinitely many times. (Here a and b are appropriate fixed integers.) Choosing again an appropriate subset of B and taking a suitable image of it under an iteration of T, we can assume that $B \subset N_{a,b}$ and $\nu(B) > 0$. Now let $C = \{x \in N_{a,b}$: x returns to $N_{a,b}$ infinitely many times$\}$. If F: $C \to C$ denotes the 'first return mapping' to the set $N_{a,b}$, then F is an invertible ν-preserving transformation of the set C onto itself, and, in virtue of (7), we have $\nu(F^k(B) \cap F^l(B)) = 0$ for every pair of integers $k \neq l$, which contradicts to the fact $\nu(B) > 0$, so it proves the usual recurrence for (N, T, ν).

We want to prove the ergodicity of the system (N, T, ν). To that end, it is enough to prove that the dynamical system $(N_{0,0}; F; \nu \restriction N_{0,0})$ is ergodic, where the letter F denotes again the 'first return mapping' to the set $N_{0,0}$. Let f: $N_{0,0} \to \mathbf{R}$ be any continuous function. By the pointwise ergodic theorem, the functions

$$\tilde{f}_+(x) := \lim_{n \to +\infty} (1/n) \Sigma \{f(F^k(x)): \ 0 \leqslant k < n\},$$

$$\tilde{f}_-(x) := \lim_{n \to +\infty} (1/n) \Sigma \{f(F^k(x)): \ -n < k \leqslant 0\}$$

exist and they are equal for almost every $x \in N_{0,0}$. From the paper [Sin (1970)] or from [B–S (1973)] we know that for almost every $x \in N_{0,0}$ there are local stable and local unstable transversal fibres of the compact billiard system (which can be obtained from the Lorentz process by factorizing it by $\mathbf{Z}^2$-translations), containing the point x. The images of a local stable transversal fibre γ under the iterated mappings F^k are shrinking to zero, i.e. $d(F^k(\gamma)) \to 0$, where $d(\xi)$ denotes the diameter of the set ξ in the phase space. This fact and the uniform continuity of the function f imply that either the function $\tilde{f}_+$ is equal to a constant on γ or it is not defined in any point of γ. An analogous statement holds for local unstable transversal fibres and for the function $\tilde{f}_-$. Thus the so called fundamental theorem for dispersing billiards (cf. [B–S (1973)]) and the absolute continuity of the partitions of the phase space into transversal fibres imply that $\tilde{f}_+$ is constant almost everywhere on the space $N_{0,0}$, that is, the dynamical system $(N_{0,0}; F; \nu \restriction N_{0,0})$ is ergodic. Proposition 1.3 is proved.

2. On the recurrence of the Lorentz process

In this section we shall modify slightly the model studied in the previous section. We assume that the moving particle cannot take an arbitrarily long walk on the billiard table without reflection at the boundary. This property is usually called the 'finite horizon property'. To establish this property we can consider the following, modified configuration of scatterers: The scatterers are the discs of radius r_1 with centres in the lattice $\mathbf{Z}^2$ and the discs of radius r_2 with centres in the translated lattice $\mathbf{Z}^2+(1/2;\ 1/2)$, such that

(8) $$\sqrt{2}/4 < r_1 < 1/2 \quad \text{and} \quad 1/2 < r_1 + r_2 < \sqrt{2}/2.$$

(Of course, we could investigate any twice periodic system of convex, compact scatterers with smooth boundaries and finite horizon.) Let h be the maximum of all lengths of paths without collision. We keep the notation (N, T, ν) for this modified dynamical system. The new definition of the cells $N_{a,b}$ $(a, b \in \mathbf{Z})$ will be the following:

(9) $$N_{a,b} = \{(q, v) \in N\colon (q^{(1)}-a)^2 + (q^{(2)}-b)^2 = r_1^2 \text{ or } (q^{(1)}-a-1/2)^2 + (q^{(2)}-b-1/2)^2 = r_2^2\}.$$

The standard normalization of ν is again $\nu(N_{a,b}) = 1$. The conditions in (8) easily imply that

(10) $$T(N_{a,b}) \subset \bigcup \{N_{k,l}\colon |k-a| \leqslant 2,\ |l-b| \leqslant 2\}.$$

In the remaining part of this paper we assume, as an indirect assumption, that

(AS) *the system (N, T, ν) is not weakly recurrent.*

Using this condition some 'pathological' properties of the process (N, T, ν) and some weak versions of recurrence will be proven, approximating a hoped-for contradiction.

Let us consider the wandering set W defined by (5) in Section 1. Our previous indirect assumption guarantees the existence of such a set. Thus the measure-preserving mapping T translates the set $T^k(W)$ simply by one step, transforming it onto the set $T^{k+1}(W)$. Of course, the system (N, T, ν) determines the measure of all such wandering sets uniquely (this is a very easy exercise). We can ask whether this measure is finite or not and, in the first case, what upper bound can be given for this measure. The following theorem gives an answer:

THEOREM 2.1. *$\nu(W) < \infty$, moreover, $\nu(W) < 8.376\sigma^2 h$, where σ^2 $(0 < \sigma^2 < \infty)$ is the diffusion coefficient of the Lorentz process, the existence of which was proven by Bunimovich and Sinai in* [B–S(1981)], *and h is the free path length defined at the beginning of this section.*

Proof. We shall use Donsker's invariance principle for the planar Lorentz process with periodic configuration of scatterers and with finite horizon. This result has also been proved in the paper [B–S(1981)]. It states that the sequence of the processes $q_{nt}/\sqrt{n}$ $(0 \leqslant t \leqslant 1)$ converges weakly to the two dimensional Wiener process $W_\sigma(t)$ $(0 \leqslant t \leqslant 1)$ with variance σ^2 at time 1, where $0 < \sigma^2 < \infty$. Here the process $q_{nt}/\sqrt{n}$ is started from a random (with respect to the measure $\nu \restriction N_{0,0}$) point $(q_0, v_0) \in N_{0,0}$, and q_{nt} denotes the place component of the point $S^{nt}(q_0, v_0) \in M$.

Six positive parameters will be used in the proof: p (< 1), K, α, ε_j $(j = 1, 2, 3)$. The actual values of them will be chosen optimally at the end of this proof. The parameters p and K are chosen in such a way that

$$(11) \qquad P\big(\|W_\sigma(t)\| < K \text{ for every } t, 0 \leqslant t \leqslant 1\big) > p.$$

(Throughout this paper the norm $\|(q^{(1)}, q^{(2)})\|$ of a vector from $\mathbf{R}^2$ is defined by $\max\{|q^{(1)}|, |q^{(2)}|\}$.) For a given $k, l \in \mathbf{Z}$ and $(q, v) \in N_{k,l}$ there is a unique $\mathbf{Z}^2$-translation carrying $T(q, v)$ into $N_{k,l}$. Denote by $F_{k,l}(q, v)$ this translated copy of $T(q, v)$. Thus $F_{k,l}$ is a ν-preserving mapping of $N_{k,l}$ onto itself, and for every pair $(k, l) \in \mathbf{Z}^2$ we get a compact Sinai-billiard system $(N_{k,l};\ F_{k,l};\ \nu \restriction N_{k,l})$ with finite horizon. Let us introduce the following definitions:

$$(12a) \qquad W_{k,l} := W \cap N_{k,l};$$

$$(12b) \qquad A_{k,l}^{(n)} := \{(q, v) \in N_{k,l}\colon \|q_m\| < (1+\varepsilon_1) K \sqrt{nh} \text{ for every integer } m, 0 \leqslant m \leqslant n\};$$

$$(12c) \qquad B_{k,l}^{(n)} := \{(q, v) \in N_{k,l}\colon \varrho\big(T^m(F_{k,l}^{r(n)}(q, v))\big) < (1+\varepsilon_1+\varepsilon_2).K \sqrt{nh} \text{ for every integer } m,\ 0 \leqslant m \leqslant n\} \quad (k, l \in \mathbf{Z};\ n \in \mathbf{N}).$$

Here $r(n) := [\sqrt[4]{n}]$ (the integer part of $\sqrt[4]{n}$), for any element $(q, v) \in N_{a,b}$ the symbol $\varrho(q, v)$ denotes the number $\max\{|a|, |b|\}$, and, finally, q_m is the place component of the element $T^m(q, v) \in N$. We want to prove that $\nu(W) \leqslant \alpha K^2$. Let us suppose the contrary: $\nu(W) > \alpha K^2$. In this case there is a positive integer L such that

$$(13) \qquad \Sigma\{\nu(W_{k,l})\colon |k| \leqslant L,\ |l| \leqslant L\} \geqslant \alpha K^2.$$

The invariance principle and (11) easily imply that

$$(14) \qquad \liminf_{n \to +\infty} \nu(A_{k,l}^{(n)}) > p.$$

Although the process starts from the cell $N_{k,l}$ instead of the cell $N_{0,0}$, the estimate in (14) remains valid, because in the definition of $A_{k,l}^{(n)}$ we have written $(1+\varepsilon_1)K\sqrt{nh}$ instead of $K\sqrt{nh}$. It follows directly from (12) that

(15) $$F_{k,l}^{-r(n)}(A_{k,l}^{(n)}) \subset B_{k,l}^{(n)} \quad \text{for } n \geqslant n_0(k, l).$$

Now (14) and (15) give us

(16) $$\liminf_{n\to+\infty} \nu(B_{k,l}^{(n)}) > p.$$

We want to prove the approximative independence of the sets $B_{k,l}^{(n)}$ and $W_{k,l}$ for large n. To this end, let us take a closer look at the dynamical system $(N_{k,l}; F_{k,l}; \nu \upharpoonright N_{k,l})$. We know from [Sin(1970)] that for almost every point $x \in N_{k,l}$ there are local stable and local unstable transversal fibres containing the point x. Let $\mathscr{K}_{k,l}$ be the following σ-algebra of measurable sets in the fundamental set $N_{k,l}$:

$$\mathscr{K}_{k,l} := \{A \subset N_{k,l}\text{: there is a measurable set } A' \subset N_{k,l} \text{ such that}$$
$$\nu((A-A')\cup(A'-A)) = 0 \text{ and } A' \text{ consists of } \textit{entire} \text{ local stable fibres}\}.$$

At the end of Section 4 in [Sin(1970)] it is proved that the σ-algebra $\mathscr{K}_{k,l}$ is a Kolmogorov σ-algebra of the system $(N_{k,l}; F_{k,l}; \nu \upharpoonright N_{k,l})$, i.e. it satisfies the following three conditions:

(i) $F_{k,l}(\mathscr{K}_{k,l}) \supset \mathscr{K}_{k,l}$;

(ii) the σ-algebra $\bigcap_n F_{k,l}^n(\mathscr{K}_{k,l})$ consists of the sets of measure 0 or 1;

(iii) the σ-algebra generated by the algebra $\bigcup_n F_{k,l}^n(\mathscr{K}_{k,l})$ contains all measurable subsets of $N_{k,l}$.

It is important for us that the set $F_{k,l}^{r(n)}(B_{k,l}^{(n)})$ is the union of a family of *entire* local stable transversal fibres. Indeed, if the points $(q_1, v_1) \in N_{k,l}$ and $(q_2, v_2) \in N_{k,l}$ are contained in the same local stable transversal fibre then for every non-negative integer m the elements $T^m(q_1, v_1)$ and $T^m(q_2, v_2)$ belong to the same scatterer, so $\varrho(T^m(q_1, v_1)) = \varrho(T^m(q_2, v_2))$. Thus the set $B_{k,l}^{(n)}$ belongs to the σ-algebra $F_{k,l}^{-r(n)}(\mathscr{K}_{k,l})$ and this gives us (using properties (i)–(iii))

(17) $$\lim [\nu(B_{k,l}^{(n)} \cap W_{k,l}) - \nu(B_{k,l}^{(n)})\,\nu(W_{k,l})] = 0.$$

From the relations (16) and (17) it follows that

(18) $$\nu(B_{k,l}^{(n)} \cap W_{k,l}) \geqslant p\nu(W_{k,l}) \quad \text{for } n \geqslant n_1(k, l).$$

Set $n_1 := \max\{n_1(k, l)\text{: } |k| \leqslant L\ |l| \leqslant L\}$. From (13) and (18) we get

(19) $$\nu\big(\bigcup\{B_{k,l}^{(n)} \cap W_{k,l}\text{: } |k| \leqslant L, |l| \leqslant L\}\big) \geqslant p\alpha K^2 \quad \text{for } n \geqslant n_1.$$

Let us observe that for $n \geq n_2(k, l)$ the set-inequality $B_{k,l}^{(n)} \subset C_{k,l}^{(n)}$ holds, where

(20) $$C_{k,l}^{(n)} := \{(q, v) \in N_{k,l}\colon \|q_m\| < (1+\varepsilon_1+\varepsilon_2+\varepsilon_3) K \sqrt{nh} \text{ for every integer } m,\ 0 \leq m \leq n\}.$$

Here we have used the finite horizon property (10). From the last set inequality and from (19) we get

(21) $$\nu\big(\bigcup \{C_{k,l}^{(n)} \cap W_{k,l}\colon |k| \leq L, |l| \leq L\}\big) \geq p\alpha K^2 \quad \text{for } n \geq n_2,$$

where $n_2 := \max\{n_2(k, l)\colon |k| \leq L, |l| \leq L\}$. The wandering property of the set W and (21) give us

(22) $$\nu\Big[\bigcup_{m=1}^{n} \bigcap_{\substack{k,l \\ |k| \leq L \\ |l| \leq L}} T^m(C_{k,l}^{(n)} \cap W_{k,l})\Big] \geq p\alpha K^2 n$$

for every integer $n \geq n_2$. The definition of the sets $C_{k,l}^{(n)}$ and (22) give us

(23) $$4h(1+\varepsilon_1+\varepsilon_2+\varepsilon_3)^2 \geq p\alpha.$$

Remember that the parameter α occurs in the indirect assumption $\nu(W) > \alpha K^2$ before the formula (13). From (23) we get contradiction if $p\alpha > 4h$; we must choose the parameters ε_j small enough, depending on $p\alpha$. Thus

(24) $$\nu(W) \leq 4(K_0(p))^2 h/p$$

where $K_0(p)$ is the infimum of the numbers K for which (11) is true, i.e.

(25) $$P(\|W_\sigma(t)\| < K_0(p) \text{ for every } t, 0 \leq t \leq 1) = p.$$

Since the two-dimensional Wiener process $W_\sigma(t)$ consists of two independent coordinate Wiener processes with variance $\sigma^2/2$, we get from (25) that

(26) $$L(\sqrt{2}K_0(p)/\sigma) = \sqrt{p},$$

where $L(x)$ is the distribution function of the maximal absolute value of the standard one-dimensional Wiener process running from time 0 till time 1. Now (24) and (26) imply that

(27) $$\nu(W) \leq 2(\sqrt{2}K_0(p)/\sigma)^2 \big(L(\sqrt{2}K_0(p)/\sigma)\big)^{-2} \sigma^2 h.$$

The parameter p can be chosen arbitrarily from the interval (0, 1), so, in virtue of (26), the parameter $K_0(p)$ can also be chosen arbitrarily from the interval $(0, \infty)$. The function $L(x)/x$ takes its maximal value near $x = 1.5076$ (this can be obtained with the help of tables for the function $L(x)$), and taking $\sqrt{2}K_0(p)/\sigma = 1.5076$ we can write into (27) the minus second power of the number $L(1.5076)/1.5076$, and we get the required upper bound for $\nu(W)$, so Theorem 2.1 is proved.

In the remaining part of this section we prove that the planar Lorentz process returns to special sets of cells even if this process is not recurrent (ergodic). To this end, we need a technical modification in the definition of the cells $N_{a,b}$. Namely, set

$$N^*_{a,b} := \{(q, v) \in N\colon\ a \leqslant q^{(1)} < a+1 \text{ and } b \leqslant q^{(2)} < b+1\}. \tag{28}$$

THEOREM 2.2. *For almost all points $x \in N$ and for every integer m there are two strictly increasing sequences $k_1 < k_2 < k_3 < \ldots$ and $l_1 < l_2 < l_3 < \ldots$ of natural numbers such that*

(i) $T^{k_j}(x) \in \bigcup \{N^*_{m,n}\colon\ n \in \mathbf{Z}\}$ $(j = 1, 2, \ldots)$,

(ii) $T^{l_j}(x) \in \bigcup \{N^*_{m,n}\colon\ n \in \mathbf{Z}\}$ $(j = 1, 2, \ldots)$,

(iii) $q^{(2)}(T^{k_j}(x)) \to +\infty$ $(j \to \infty)$,

(iv) $q^{(2)}(T^{l_j}(x)) \to -\infty$ $(j \to \infty)$.

Of course, an analogous statement is true for negative iterates of T and for horizontal strips of cells $N^*_{m,n}$.

Proof. Suppose that the set A of the points $x \in N$ for which the statement of the theorem is not true has positive ν-measure. In virtue of (8) and the definition of the sets $N^*_{a,b}$ we have

$$T(N^*_{a,b}) \subset \bigcup \{N^*_{k,l}\colon\ |k-a| \leqslant 1, |l-b| \leqslant 1\}, \tag{29}$$

and our assumption (AS) means that for almost every $(q, v) \in N$

$$\|q_k\| \to \infty \qquad (k \to \infty), \tag{30}$$

where $(q_k, v_k) = T^k(q, v)$. Obviously, $A = A_1 \cup A_2$, where the set A_1 consists of those points $x \in N$ for which there exists an integer m such that there is no appropriate sequence (k_j) and the set A_2 contains those points $x \in N$ for which there exists an integer m such that there is no appropriate sequence (l_j). Since A_1 and A_2 are T-invariant and $\mathbf{Z}^2$-translation invariant sets, using again the factorization by $\mathbf{Z}^2$-translations we get that either $\nu(A_1) = 0$ or $\nu(N - A_1) = 0$, and the analogous statement is true for A_2. In virtue of the relation $\nu(A) > 0$ and of the symmetry we can assume that $\nu(A_1) > 0$, i.e. $\nu(N - A_1) = 0$. Since our system is symmetric with respect to the x-axis, we have the relation $\nu(N - A_2) = 0$ as well. Consequently, for almost every point $x \in N$ there are integers m_1 and m_2 such that the set

$$B_{m_1}(x) := \{n \in \mathbf{Z}\colon\ \exists\, k \in N \text{ such that } T^k(x) \in N^*_{m_1,n}\}$$

is bounded from above, and the analogous set $B_{m_2}(x)$ is bounded from below. We claim that the sequence $(q_k^{(1)}(x))$ is either bounded from below or it is bounded from above. (Here $q_k^{(1)}(x)$ is the first coordinate of the place component of the point $(q_k, v_k) = T^k(x)$.) Clearly, we can assume that $m_1 \leqslant m_2$. The semi-boundedness properties of the sets $B_{m_1}(x)$ and $B_{m_2}(x)$ and (29) imply that from a large threshold $K(x)$ the trajectory $\{T^k(x): k \geqslant K(x)\}$ cannot go from the domain $\bigcup \{N^*_{m,n}: m < m_1, n \in \mathbf{Z}\}$ to the domain $\bigcup \{N^*_{m,n}: m > m_2, n \in \mathbf{Z}\}$ (or vice versa) without hitting the finite set of cells $\bigcup \{N^*_{m,0}: m_1 < m < m_2\}$. Thus the sequence $(q_k^{(1)}(x))$ must be either bounded from below or bounded from above by (30). Set

$$B_1 = \{x \in N: (q_k^{(1)}(x)) \text{ is bounded from below}\}$$

and

$$B_2 = \{x \in N: (q_k^{(1)}(x)) \text{ is bounded from above}\}.$$

We have just seen that $\nu(N-(B_1 \cup B_2)) = 0$, so one of the numbers $\nu(B_1)$ and $\nu(B_2)$ is positive. By symmetry we can assume that $\nu(B_1) > 0$. Because of the T invariance and the $\mathbf{Z}^2$-translation invariance of the set B_1 we have $\nu(N-B_1) = 0$, and by the symmetry with respect to the y-axis we have $\nu(N-B_2) = 0$. The symmetry with respect to the line $y = x$ gives us that the trajectory $\{T^k(x): k \in N\}$ of almost all points $x \in N$ is bounded, which contradicts to the original assumption (AS). Consequently, the indirect assumption $\nu(A) > 0$ leads to contradiction, so Theorem 2.2 is proved.

COROLLARY 2.3. *Let $a_0 \in N$ be a fixed natural number. Let us factorize the Lorentz process in such a way that $(q_1, v_1) \sim (q_2, v_2)$ iff $q_1 - q_2 = (ka_0; 0)$ for an appropriate integer k and, moreover, $v_1 = v_2$. We claim that this new billiard system on cylinder surface is recurrent.* (*It must also be ergodic, cf. Section* 1.)

Proof. Immediate consequence of Theorem 2.2.

Note. The generic translator vector $(a_0, 0)$ in Corollary 2.3 can be replaced by any non-zero vector of $\mathbf{Z}^2$. We have to make minor modifications only in the proof of Theorem 2.2.

In the remaining part of this paper we shall briefly discuss some connections between the dynamics T and the $\mathbf{Z}^2$-translations. Let $E_{m,n}$ denote the plane-translation on the phase space N defined by the vector $(m, n) \in \mathbf{Z}^2$. Of course, $E_{(\cdot,\cdot)}$ is a ν-preserving action of the group $\mathbf{Z}^2$ on the space N, which commutes with the dynamics T. For $(m, n) \in \mathbf{Z}^2$ let $E'_{m,n}: W \to W$ be the following, ν-preserving mapping of the wandering set W onto itself:

$$E'_{m,n}(w) := T^k(E_{m,n}(w)) \in W \quad (w \in W), \tag{31}$$

where the integer exponent k is chosen in such a unique way that $T^k(E_{m,n}(w))$ lies in the set W again (for the definition of the set W see (5) in Section 1). Using the commutativity between $E_{(\cdot,\cdot)}$ and T, it is easy to see that $E'_{m,n}$ is a ν-preserving action of the group $\mathbf{Z}^2$ on the finite measure space (W, ν).

COROLLARY 2.4. *For every non-zero vector* $(m, n) \in \mathbf{Z}^2$ *the mapping* $E'_{m,n}$: $W \to W$ *is ergodic.*

Proof. Let $A \subset W$ be an $E'_{m,n}$-invariant measurable set. The T-invariant set $\bigcup \{T^k(A): k \in \mathbf{Z}\}$ must be $E_{m,n}$-invariant. Factorizing the process (N, T, ν) by the plane translation with the vector (m, n), Corollary 2.3 (generalized along the lines of the Note after it) gives us that either $\nu(\bigcup \{T^k(A): k \in \mathbf{Z}\}) = 0$ or $\nu(N - \bigcup \{T^k(A): k \in \mathbf{Z}\}) = 0$. In the first case we obtain that $\nu(A) = 0$ and in the second case $\nu(W - A) = 0$.

Acknowledgement. Thanks are due to A. Krámli and D. Szász for the initiation of the problem of recurrence.

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