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Positive solution of asymptotically linear elliptic eigenvalue problems for certain differential equations of the fourth order

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Abstract. In this paper we are concerned with the existence of positive solutions of the non-linear elliptic eigenvalue problems. The results of this paper generalize the results of paper [1]. Namely, we transfer the results of paper [1] concerning differential equation of the second order, to a certain class of differential equations of the fourth order. The method used in this paper is based, in principle on the results of papers [1], [3] and [4].

Introduction. Let Ω be a bounded Jordan-measurable domain in the space R^m ($m \geq 1$), which can be approximated by an increasing sequence of domains Ω_n with regular boundaries (i.e. it is assumed that the boundary $\partial\Omega_n$ of each Ω_n is a surface of class C_σ^1; for the definition of a surface of class C_σ^1 see [5], p. 149). We do not require any regularity properties of the boundary of Ω.

We shall consider a differential equation of the form

$$\mathscr{L}u = \lambda f(u) \quad \text{in } \Omega, \tag{1}$$

where $\mathscr{L}$ is a differential expression of the form $\mathscr{L} = \mathscr{L}_1 \mathscr{L}_0$ and the expressions $\mathscr{L}_i$ ($i = 0, 1$) are symmetric strongly uniformly elliptic linear differential expressions of the second order, i.e.,

$$\mathscr{L}_k \varphi := -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}^k \frac{\partial \varphi}{\partial x_j}\right) + a^k \varphi, \quad k = 0, 1, \tag{2}$$

for $\varphi \in C^2(\Omega)$; λ is a real parameter. We make the following assumptions: $a_{ij}^k = a_{ji}^k$ ($i, j = 1, \ldots, m$) and a^k are real functions of class C^{3-2k} and C^{2-2k} in Ω ($k = 0, 1$), respectively. Further, $f\colon R^+ \to R^+$ is a continuously differentiable function, asymptotically linear in the sense that there exist $m_\infty > 0$, a function g and a constant C such that

$$\forall s \in R^+ f(s) = m_\infty s + g(s), \quad |g(s)| \leq C, \quad f(0) \geq 0. \tag{3}$$

We shall also consider the generalized boundary condition (cf. [2] and

[3]) which, in the case where the boundary $\partial\Omega$ is regular, can be written in the form

$$(4)\qquad R_i\varphi^i = 0 \quad \text{on } \partial\Omega, \quad i = 0, 1; \quad \varphi^0 := u, \quad \varphi^1 := \mathscr{L}_0 u,$$

where $R_i u = 0$ on $\partial\Omega$ means that

$$(5)\qquad \frac{du}{d\nu_i} - h^i u = 0 \quad \text{on } \partial\Omega - \Gamma_i, \quad u = 0 \quad \text{on } \Gamma_i, \quad i = 0, 1$$

and Γ_i denotes the $(m-1)$-dimensional part of $\partial\Omega$ (Γ_i being connected or not); in the extreme cases Γ_i may be whole boundary of Ω or the empty set. Here h^i $(i = 0, 1)$ are non-negative continuous functions in $\bar{\Omega}$ and $du/d\nu_i$ $(i = 0, 1)$ are the transversal derivatives of u with respect to the expressions $\mathscr{L}_i$ $(i = 0, 1)$, respectively, i.e.,

$$\frac{du}{d\nu_i} = \sum_{k,j=1}^{m} a^i_{kj} \frac{\partial u}{\partial x_k} \cos(n, x_j), \qquad i = 0, 1,$$

n being the interior normal to $\partial\Omega$.

The object subsequent considerations are positive solutions of problem (1), (4). By a *positive solution* of problem (1), (4) we mean a pair (λ, u), where $\lambda > 0$ and u is a function belonging to $C^4(\Omega) \cap L_2(\Omega)$ and satisfying equation (1) and condition (4) (in generalized sense) such that $u > 0$ (i.e., $u \geqslant 0$ in Ω, and $u \neq 0$). Let us denote by $E := C(\bar{\Omega})$ the space of continuous functions in $\bar{\Omega}$. The norm of the space E is denoted by $\|\cdot\|$ and defined by

$$(6)\qquad \|\varphi\| := \sup_{x \in \bar{\Omega}} |\varphi(x)|.$$

Let $\Sigma \subset R^+ \times E$ be the closure of the set of positive solutions of problem (1), (4). It is our aim to investigate the existence and global behavior of the components of Σ. According to the behavior of the function f near 0, we will distinguish three cases (see [1]).

The linear eigenvalue problem

$$(6)\qquad \mathscr{L}u = \lambda u \quad \text{in } \Omega,$$

$$(7)\qquad R_i u = 0 \quad \text{on } \partial\Omega, \quad i = 0, 1,$$

corresponds to problem (1), (4).

We shall need the following assumptions:

Hypothesis Z_1. If Γ_i is the empty set, then $h^i(x) > 0$ for each $x \in \bar{\Omega}$, $i = 0, 1$.

Hypothesis Z_2. Given (6) and (7), there exist a principal eigenvalue λ_1 and a corresponding first eigenfunction $u_1 \in C^4(\Omega)$.

1. Notation and preliminary results. Let $H := L_2(\Omega)$, with inner product

$(\cdot, \cdot)$ and norm $\|\cdot\|_{L_2}$ and let $H^p(\Omega)$ be the Sobolev space of order p. Let us write

(8) $$H_i^2(\text{b.c.}) := \{u \in H^2(\Omega)\colon\ R_i u = 0 \text{ on } \partial\Omega\}, \quad i = 0, 1,$$

and

(9) $$H^4(\text{b.c.}) := \{u \in H^4(\Omega)\colon\ R_i u = 0 \text{ on } \partial\Omega,\ i = 0, 1\}.$$

Let L^0 and L_i^0 $(i = 0, 1)$ be the linear operators induced by $\mathscr{L}$ and $\mathscr{L}_i$ $(i = 0, 1)$ in H with domains $D(L^0) = H^4(\text{b.c.})$ and $D(L_i^0) = H_i^2(\text{b.c.})$ $(i = 0, 1)$, respectively. Then L^0 and L_i^0 $(i = 0, 1)$ are closed operators with compact resolvents.

By the regularity theory for linear elliptic boundary value problems, the restrictions $(L^0)^{-1}|E$ and $(L_i^0)^{-1}|E$ $(i = 0, 1)$ map E into itself are compact (viewed as operators in E). Let L and L_i $(i = 0, 1)$ be the restrictions of L^0 and L_i^0 $(i = 0, 1)$ to E, respectively, defined by

$$D(L) := \{u \in E\colon\ u \in D(L^0),\ L^0 u \in E\}, \quad Lu = L^0 u, \quad u \in D(L),$$

and analogously

$$D(L_i) := \{u \in E\colon\ u \in D(L_i^0),\ L_i^0 u \in E\}, \quad L_i u = L_i^0 u, \quad u \in D(L_i), \quad i = 0, 1.$$

Then L and L_i $(i = 0, 1)$ are closed operators in E with compact inverses, and $L = L_1 L_0$.

2. Statement of the results. Let us denote by $f'_+(0)$ the right-sided derivative of f at 0, and write $\lambda_\infty := (m_\infty)^{-1}\lambda_1$ and $\lambda_0 := [f'_+(0)]^{-1}\lambda_1$ if $f'_+(0) \neq 0$. Further, let Σ_0 and Σ_∞ be the components of Σ which meet $(\lambda_0, 0)$ and (λ_∞, ∞), respectively.

We have the following theorems (see [1]):

THEOREM 1. *Suppose $f(0) = 0$ and $f'_+(0) > 0$. Then*

1° *λ_∞ is a bifurcation point from infinity and it is unique. More precisely, the component Σ_∞ is not the empty set. If*

(10) $$\liminf_{s \to +\infty} g(s) > 0$$

or

(11) $$\limsup_{s \to +\infty} g(s) < 0,$$

respectively, then Σ_∞ bifurcates to the left (right) of λ_∞.

2° *λ_0 is a bifurcation point from the trivial solution, and it is unique for positive solutions. The component Σ_0 is unbounded and it emanates from $(\lambda_0, 0)$.*

3° *If $f(s) > 0$ for all $s > 0$, there is a number λ^* such that problem* (1), (4) *admits no positive solution with $\lambda > \lambda^*$. In this case $\Sigma_0 = \Sigma_\infty$.*

4° *If* $f(s_0) = 0$ *for some* $s_0 > 0$, *there exists no positive solution* (λ, u) *with* $\|u\| = s_0$.

Hence $\Sigma_0 \cap \Sigma_\infty = \emptyset$, *and problem* (1), (4) *admits at least two positive solutions for all* $\lambda > \max(\lambda_\infty, \lambda_0)$.

THEOREM 2. *Suppose* $f(0) = 0$, *but* $f'_+(0) = 0$. *Then*

1° *Assertion* 1° *of Theorem 1 holds.*

2° *There is no bifurcation of positive solutions from the line of trivial solutions* $R^+ \times \{0\}$.

THEOREM 3. *Suppose* $f(0) > 0$. *Then*

1° *Assertion* 1° *of Theorem* 1 *holds.*

2° *The component* Σ_0 *is unbounded and it meets* (0, 0). *If* $f(s) > 0$ *for all* $s \geqslant 0$, *then* $\Sigma_0 = \Sigma_\infty$.

3° *If* $f(s_0) = 0$ *for some* $s_0 > 0$, *there exists no positive solution* (λ, u) *with* $\|u\| = s_0$.

Hence $\Sigma_0 \cap \Sigma_\infty = \emptyset$, *and problem* (1), (4) *has at least two positive solutions for* $\lambda > \lambda_\infty$.

3. Remarks on the proofs of Theorems 1.2.3. The proofs of the theorems from Section 2 are based on the results of papers [3] and [4]. Now, these results can be generalized. In [3] we consider the linear eigenvalue problem (6), (7). Under Assumption Z_1 (see Introduction), the operator L_1^0 defined in Section 1 has an inverse. Let us denote it by $K := (L_1^0)^{-1}$. The operator K satisfies the following conditions (see [4]):

1° K: $L_2(\Omega) \supset E \to H_1^2(\text{b.c.})$ is a linear bounded operator on $H = L_2(\Omega)$;

2° K is symmetric, i.e., $(u, Kv) = (Ku, v)$ for $u, v \in H$;

3° K is positive, i.e., $(u, Ku) > 0$ for $u \neq 0$;

4° if $u \geqslant 0$ in Ω, then $Ku \geqslant 0$ in Ω.

From this it follows that problem (6), (7) in the space H may be replaced by the following equivalent problem

$$L_0^0 u = \lambda K u, \quad u \in H_0^2(\text{b.c}). \tag{12}$$

Problem (12) was considered in [3]. Under additional assumption, formulated in [4] as:

"HYPOTHESIS A. No eigenfunction of problem (6), (7) can vanish identically in any subdomain of domain Ω",

we proved in [4] the following theorems:

THEOREM 4. *The first eigenfunction* u_1 *of problem* (6), (7) *does not vanish at any point of the domain* Ω.

THEOREM 5. *If there exists a function* v *of class* C^4 *in* Ω *such that* $v(x) \neq 0$ *for each* $x \in \Omega$, *satisfying the boundary condition* (7) *and equation* (6) *with* $\lambda = t$, *then* $t = \lambda_1$ *and* $v = cu_1$, *where* $c = \text{const} \neq 0$.

THEOREM 6. *The first eigenvalue of problem* (6), (7) *is a single eigenvalue, i.e., each function* $\varphi \not\equiv 0$ *of class* C^4 *in* Ω *satisfying the boundary condition* (7) *and equation* (6) *with* $\lambda = \lambda_1$ *is equal to the first eigenfunction of* (6), (7) *multiplied by a constant* $c \neq 0$.

Now we prove that Theorems 4, 5 and 6 hold without Hypothesis A. Indeed, Hypothesis A is used in [3] only once, in the proof of the following

LEMMA 1. *Every function* $u \in H_0^2(\text{b.c.})$ *satisfying equation* (12) *with* $\lambda = \lambda_1$ *preserves sign in* Ω.

Proof. Let u be a function satisfying the assumptions of Lemma 1, and $\lambda_1 = \lambda_2 = \ldots = \lambda_s < \lambda_{s+1}$ (i.e., λ_1 is an s-fold eigenvalue of (12)). Suppose that $u(x) \geqslant 0$ for $x \in \bar{\Omega}_1 \subset \Omega$. Let us write $\Omega_2 := \Omega - \bar{\Omega}_1$. We have $u(x) < 0$ for $x \in \Omega_2$. If Ω_2 is non-emty, then it is open. Put

$$U(x) := \begin{cases} u(x) & \text{for } x \in \bar{\Omega}_1, \\ 0 & \text{for } x \in \Omega_2. \end{cases}$$

If Hypothesis A is satisfied, then clearly the functions $U, u_1, \ldots, u_s$, are linearly independent in Ω. We shall prove that the functions $U, u_1, \ldots, u_s$ are linearly independent in Ω without using Hypothesis A. Indeed, let us suppose that there exist $c_1, \ldots, c_s \in R$ such that $c_1^2 + \ldots + c_s^2 > 0$ and such that $U + c_1 u_1 + \ldots + c_s u_s = 0$ in Ω. It follows that the function $\varphi = c_1 u_1 + \ldots + c_s u_s \in H_0^2(\text{b.c.})$, satisfies equation (12) with $\lambda = \lambda_1$, $\varphi(x) = 0$ for each $x \in \Omega_2$ and $\varphi(x) = -U(x) \leqslant 0$ for each $x \in \Omega$, a contradiction to Theorem 7 from [3]. The continuation of the proof of Lemma 1 is the same as in [3].

These considerations show that, under Assumptions Z_1 and Z_2, the operator L^0 has a principal eigenvalue $\lambda_1 > 0$; the corresponding eigenspace is one-dimensional and is spanned by a function $\varphi \in E := C(\bar{\Omega})$, which can be chosen to be positive in Ω. The adjoint operator $(L^0)^*$ has also the eigenvalue λ_1 and the corresponding eigenspace is also one-dimensional.

We now turn to problem (1), (4). Let $\tilde{f}$ be an extension of the function f to R such that $\tilde{f}$ is continuous on R and $\tilde{f}(s) > 0$ for all $s < 0$. Let F denote the Nemytskii operator associated with $\tilde{f}$, i.e. $F(u)(x) := \tilde{f}(u(x))$ for any function u defined on $\bar{\Omega}$. Then F is a bounded and continuous operator of E into itself.

Let us observe that problem (1), (4) for $u \in D(L)$ may be written in the form

$$L_0 u = \lambda K F(u) \quad \text{in } \Omega. \tag{13}$$

Since $\tilde{f}(s) \geqslant 0$ for $s \in R$, the following result is an immediate consequence of the maximum principle for the operator L_0 and the properties of the operators K and F.

LEMMA 2. *Let* $u \in D(L)$ *be a function such that* $L_0 u \geqslant \lambda K F(u)$ *in* Ω, $\lambda \geqslant 0$. *Then* $u \geqslant 0$ *in* Ω.

From Lemma 2 it follows that the closure of the set of non-trivial solutions (λ, u) of equation (13) in $R^+ \times E$ is exactly Σ.

Problem (1), (4) with $\lambda \geqslant 0$ is now equivalent to the functional equation

$$u = \lambda L^{-1} F(u) \tag{14}$$

in the Banach space E, where $L^{-1} = L_0^{-1} K = L_0^{-1} L_1^{-1} = (L_1 L_0)^{-1}$. Since L^{-1} is a compact operator in E, equation (14) is identical with the equation considered in paper [1]. Therefore, applying the method of paper [1] almost without modification and using the above results we obtain the proofs of Theorems 1, 2 and 3.

References

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