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Linear functional equations with variable coefficients

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Abstract. Sufficient conditions are found for the existence and uniqueness of local C^∞-solutions of multidimensional linear functional equations of the form $\varphi(\Lambda x)-Q(x)\varphi(x) = \gamma(x)$.

1. Introduction and statement of the results. The present paper is devoted to the study of local C^∞-solvability (for any right-hand side) of the multidimensional linear functional equation

$$\varphi(\Lambda x)-Q(x)\varphi(x) = \gamma(x), \tag{1}$$

where Λ: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ is a linear nonsingular operator, Q: $\boldsymbol{R}^n \to \boldsymbol{C}^{m^2}$, γ: $\boldsymbol{R}^n \to \boldsymbol{C}^m$ are C^∞-maps in a neighbourhood of the origin, φ: $\boldsymbol{R}^n \to \boldsymbol{C}^m$ is a map to be found. The conditions of local nontrivial C^∞-solvability of the corresponding homogeneous equation

$$\varphi(\Lambda x)-Q(x)\varphi(x) = 0 \tag{2}$$

are also elucidated.

Abel and Schröder equations pertain to this type as well as their various generalizations detailedly studied in a major series of works of the Kraków school of functional equations, summed up in the monograph by Kuczma [7]. In these and in a number of later works the writers investigated existence and uniqueness of continuous finitely smooth and analytic solutions of equations in a single variable ($x \in \boldsymbol{R}^1$). Multidimensional functional equations were considered in papers [3], [4], [8], where sufficient conditions for existence and uniqueness of finitely smooth solutions were obtained; works [10] and [11] investigated the analytic case.

Equations of type (1) were dealt with in [5], where for a constant nonsingular matrix $Q(x) \equiv Q$ necessary and sufficient conditions for existence of an infinitely differentiable local solution of (1) for any right-hand side $\gamma(x)$ were found, together with necessary and sufficient conditions for existence of locally nontrivial C^∞-solution of the corresponding homogeneous equation. The case where the argument transformation is nonlinear but hyperbolic (the Jacobi matrix at zero has no spectral points on the unit circle) was considered in [2], [6]. The present paper completes investigation of equation (1) with variable coefficients (with an arbitrary operator function $Q(x)$).

To be locally C^∞-solvable, equation (1) must be formally solvable, i.e., there must exist a formal map $\hat{\varphi}$: $\boldsymbol{R}^n \to \boldsymbol{C}^m$ such that $\hat{\varphi}(\Lambda x)-\hat{Q}(x)\hat{\varphi}(x) = \hat{\gamma}(x)$. (Here, $\hat{Q}$, $\hat{\gamma}$ are formal Taylor series at zero.) Let $\hat{\varphi}$ be any formal solution of (1) while φ_0 is a local C^∞-map whose Taylor series at zero is equal to $\hat{\varphi}$. We shall seek the solution of (1) in the form $\varphi = \varphi_0+\psi$, where ψ is to be found and is flat at zero(1). For ψ we obtain the equation

$$\psi(\Lambda x)-Q(x)\psi(x) = \tau(x),$$

where a map $\tau(x) = -\varphi_0(\Lambda x)+Q(x)\varphi_0(x)+\gamma(x)$ is flat at zero. Thus, provided that formal solvability condition holds, local C^∞-solvability of equation (1) for any right-hand side is reduced to existence of flat-at-zero local C^∞-solution of equation (1) for any right-hand side that is flat at zero.

Let $\{\lambda_i\}_{i=1}^n$ and $\{q_i\}_{j=1}^m$ be eigenvalues of operators Λ and $Q(0)$, respectively. It is well known that absence of the so-called resonance relations, i.e., validity of the inequalities

$$(3) \qquad q_j \neq \prod_{i=1}^{n} \lambda_i^{p_i}, \quad j = 1, 2, \ldots, m, \ p_i \geqslant 0 \ - \ \text{integers},$$

is a necessary condition even for a formal solvability of (1) for any right-hand side γ. For a constant nonsingular matrix $Q(x) \equiv Q$ it was shown in [5] that absence of the resonance relations between moduli of eigenvalues of Λ and $Q(0)$, i.e., validity of the inequalities

$$(4) \qquad |q_j| \neq \prod_{i=1}^{n} |\lambda_i|^{p_i}, \quad j = 1, 2, \ldots, m, \ p_i \geqslant 0 \ - \ \text{integers},$$

is a necessary and sufficient condition of local C^∞-solvability of (1) for any right-hand side, provided that there exists at least one spectral point of Λ lying on a unit circle and distinct from 1 (in all other cases validity of inequalities (3) is a sufficient condition for local C^∞-solvability of (1) for any right-hand side).

Denote by $\mathscr{L}_+$ (respectively, $\mathscr{L}_-$, $\mathscr{L}_1$) the Λ-invariant subspace of the space $\boldsymbol{R}^n$ that corresponds to the part of the Λ-spectrum lying inside the unit disk (respectively, out-side the unit disk, on the unit circle); denote by $\mathscr{M}_0$ the $Q(0)$-invariant subspace of the space $\boldsymbol{C}^m$ that corresponds to the zero-spectrum of the operator $Q(0)$.

THEOREM 1. *Let* $\mathscr{L}_1 = 0$. *Then the sufficient condition for equation* (1) *to be locally* C^∞*-solvable for any right-hand side is the validity of inequalities* (3).

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(1) C^∞-map f is said to be *flat on a set* $\mathscr{M}$ if $f^{(s)}(x) = 0$ for all $x \in \mathscr{M}$ and $s = 0, 1, 2, \ldots$

Now, let $\mathscr{L}_1 \neq 0$. If $\mathscr{L}_+ = 0$, then the sufficient condition for equation (1) to be locally C^∞-solvable for any right-hand side is the validity of inequalities (4). But if $\mathscr{L}_+ \neq 0$, then the sufficient condition for equation (1) to be locally C^∞-solvable for any right-hand side is the validity of (4) plus the following two conditions:

(a) the subspace $\mathscr{M}_0$ is $Q(x)$-invariant for every x from some neighbourhood of the origin;

(b) $Q_0(x) = \text{const} = Q_0(0)$, $x \in \mathscr{L}_1$, where $Q_0(x)$ is a restriction of the operator $Q(x)$ to the subspace $\mathscr{M}_0$.

Remark 1. Condition (b) is essential for the validity of Theorem 1, with $\mathscr{L}_1 \neq 0$ and $\mathscr{L}_+ \neq 0$, as is shown in the following

EXAMPLE. Consider a one-dimensional ($m = 1$) functional equation

$$\varphi(\lambda\xi, \eta) - \eta\varphi(\xi, \eta) = \gamma(\xi, \eta), \qquad x = (\xi, \eta) \in \mathbf{R}^2, \tag{5}$$

where $0 < \lambda < 1$. Here condition (b) does not hold. We indicate a function $\gamma(x)$ for which equation (5) is not locally C^∞-solvable. We put $\theta_k = (k!)^{-1}$ and $\xi_k = \lambda^k \xi_0$, $k = 0, 1, 2, \ldots$, where $\xi_0 > 0$ is sufficiently small. Let $\tau_1(\xi)$ be a C^∞-function that is flat at zero and satisfies the condition $\tau_1(\xi_k) = \theta_k$, $k = 0, 1, 2, \ldots$, and

$$\tau_2(\eta) = \begin{cases} e^{-(2\eta)^{-1}}, & \eta > 0, \\ 0, & \eta \leqslant 0. \end{cases}$$

We put $\gamma(\xi, \eta) = \tau_1(\xi)\tau_2(\eta)$ in (5). If $\varphi(\xi, \eta)$ is a flat-at-zero C^∞-solution of (5) for $\|(\xi, \eta)\| \leqslant \delta$, then $\varphi(\xi, \eta)$ is flat on subspaces $(\xi, 0)$ and $(0, \eta)$. From (5) we obtain

$$\varphi(\xi, \eta) = \eta^{-k}\varphi(\lambda^k\xi, \eta) - \sum_{i=0}^{k-1} \eta^{-i-1}\gamma(\lambda^i\xi, \eta), \qquad \|(\xi, \eta)\| \leqslant \delta. \tag{6}$$

Since the function $\varphi(\xi, \eta)$ is flat on the subspace $(0, \eta)$, the first term in the right-hand side of (6) converges to zero as $k \to \infty$ (for every fixed $\eta \neq 0$). It means that the solution of (5) must be presented by the series

$$\varphi(\xi, \eta) = -\sum_{i=0}^{\infty} \eta^{-i-1}\gamma(\lambda^i\xi, \eta), \qquad \|(\xi, \eta)\| \leqslant \delta.$$

Hence, by virtue of the choice of γ,

$$\varphi(\lambda^p\xi_0, \eta) = -\eta^{p-1}e^{(2\eta)^{-1}} + e^{-(2\eta)^{-1}} \sum_{i=0}^{p-1} \frac{\eta^{p-1}}{i!}, \qquad \eta > 0,\ p = 1, 2, \ldots,$$

which contradicts to smoothness (and even continuity) of the function φ at the origin.

Now, let us consider the homogeneous equation (2). In the case when $\det Q(0) = 0$ we assume that the origin is an isolated zero of $\det Q(x)$. Note that if condition (3) holds, then equation (2) has no nontrivial formal solutions, hence, it can have only locally nontrivial C^∞-solution that is flat at zero.

We put $\sigma_k(A) = \min(\|A^k\|, \|A^{-k}\|)$, $k = 0, 1, 2, \ldots$, and treat hyperbolic and nonhyperbolic cases separately.

THEOREM 2. *Let A be a hyberbolic operator, i.e., $\mathscr{L}_1 = 0$. Equation* (2) *has a locally nontrivial C^∞-solution that is flat at zero if and only if one of the following conditions*:

(7) *the sequence $\{\sigma_k(A)\}$ is not bounded;*
(8) *the sequence $\{\sigma_k(A)\}$ is bounded, $\mathscr{L}_+ \neq 0$ and* $\det Q(0) = 0$;
is valid.

THEOREM 3. *Let A be a nonhyperbolic linear operator, i.e., $\mathscr{L}_1 \neq 0$.*

If $\mathscr{L}_+ \neq 0$ and $\mathscr{L}_- \neq 0$, then equation (2) *has a locally nontrivial C^∞-solution that is flat at zero.*

If $\mathscr{L}_+ = 0$ and inequalities (4) *hold, then equation* (2) *has a locally nontrivial C^∞-solution that is flat at zero if and only if condition* (7) *is valid.*

If $\mathscr{L}_+ \neq 0$, $\mathscr{L}_- = 0$ and inequalities (4) *with conditions* (a), (b) *of Theorem* 1 *hold, then equation* (2) *has a locally nontrivial C^∞-solution that is flat at zero if and only if one of conditions* (7), (8) *is valid.*

Remark 2. The condition that the sequence $\sigma_k(A)$ is bounded means that either $|\lambda_i| \leqslant 1$, $i = 1, 2, \ldots, n$, or $|\lambda_i| \geqslant 1$, $i = 1, 2, \ldots, n$, and that Jordan blocks of dimension 1 correspond to the eigenvalues of A of modulus 1 (if there are any). That the sequence $\sigma_k(A)$ is not bounded means that either A has some eigenvalues with absolute values greater than, some less than, 1, or that there is a Jordan block of dimension greater than 1 for some eigenvalue of A of modulus 1.

2. Proof of the theorems.

Proof of Theorem 1. First we remind that if conditions (3) hold, then equation (1) is formally solvable for any right-hand side, hence, the problem is reduced to solvability of equation (1) for any right-hand side that is flat at zero.

(i) For the case when $\mathscr{L}_+ = 0$, $\mathscr{L}_- = 0$ and the operator $Q(0)$ has no eigenvalues on the unit circle the solvability of equation (1) for any right-hand side was proved in [5].

(ii) Suppose now that $\mathscr{L}_+ = 0$, $\mathscr{L}_1 \neq 0$ and conditions (4) are valid. Denote by $\mathscr{P}_-$ (respectively, $\mathscr{P}_1$) the projection on the subspace $\mathscr{L}_-$ (respectively, $\mathcal{L}_1$). Putting $x_- = \mathcal{P}_- x = 0$ in (1) we obtain:

$$\varphi(0, A_1 x_1) - Q(0, x_1)\varphi(0, x_1) = \gamma(0, x_1). \tag{9}$$

(Here $A_1 = \mathcal{P}_1 A$, $x_1 = \mathcal{P}_1 x$.) By virtue of (i) and conditions (4), equation (9) has a local C^∞-solution $\varphi(0, x_1)$. Differentiate equation (1) with respect to x_- and again put $x_- = 0$:

$$\varphi'_{x_-}(0, A_1 x_1) - Q(0, x_1)\varphi'_{x_-}(0, x_1)A_-^{-1} = Q'_{x_-}(0, x_1)\varphi(0, x_1)A_-^{-1} + \gamma'_{x_-}(0, x_1)A_-^{-1}. \tag{10}$$

(Here $A_- = \mathcal{P}_- A$.) By (4), equation (10) satisfies the conditions of (i) and so has a local C^∞-solution $\varphi'_{x_-}(0, x_1)$. On going on with differentiating with respect to x_- and restricting the equation obtained to the subspace $\mathcal{L}_1$, we find a C^∞-jet $(\varphi(0, x_1), \varphi'(0, x_1), \ldots)$. Let $\varphi_0(x)$ be the Whitney [9] C^∞-extention of this C^∞-jet to the δ-neighbourhood of the origin. We seek a solution of equation (1) in the form $\varphi_0 + \varphi$, where ψ is to be found and is flat on $\mathcal{L}_1$. We obtain

$$\psi(Ax) - Q(x)\psi(x) = \tau(x) \tag{11}$$

for ψ, where $\tau(x) = -\varphi_0(Ax) + Q(x)\varphi_0(x) + \gamma(x)$ is flat on $\mathcal{L}_1$. We rewrite (11) in the form

$$\psi(x) = \tilde{Q}(x)\psi(A^{-1}x) + \tilde{\tau}(x), \tag{12}$$

where $\tilde{Q}(x) = Q(A^{-1}x)$, $\tilde{\tau}(x) = \tau(A^{-1}x)$. Let $\mathcal{U}(\delta)$ be the closed δ-neighbourhood of the origin and $\varkappa_\delta$: $\boldsymbol{R}^n \to \boldsymbol{R}^1$ a C^∞-function taking the value 1 on some small neighbourhood of the origin and the value 0 outside $\mathcal{U}(\delta)$. We consider the linear operator

$$(T\psi)(x) = \varkappa_\delta(x)\varkappa_\delta(A^{-1}x)[\tilde{Q}(x)\psi(A^{-1}x) + \tilde{\tau}(x)]$$

acting on the space of C^∞-maps ψ: $\mathcal{U}(\delta) \to \boldsymbol{C}^m$ that are flat on $\mathcal{L}_1$. To prove local C^∞-solvability of (12) it is enough to prove local C^∞-solvability of the functional equation

$$\psi(x) = (T\psi)(x). \tag{13}$$

Let $\{\mathcal{C}_{sv}\}$, $\{v_s\}$ be sets of positive numbers. We write $\mathcal{K} = \mathcal{K}(\{\mathcal{C}_{sv}\}, \{v_s\}, \delta)$ for the compact convex set of C^∞-maps ψ: $\mathcal{U}(\delta) \to \boldsymbol{C}^m$ that are flat on $\mathcal{L}_1$ for which

$$\|\psi^{(s)}(x)\| \leqslant \mathcal{C}_{sv}\|x_-\|^v, \quad v \geqslant v_s,\ s = 0, 1, 2, \ldots \tag{14}$$

By the general fixed-point principle, to show that equation (13) is solvable it is enough to find constants $\mathcal{C}_{sv}$, v_s such that $\mathcal{K}$ is T-invariant.

We choose a vector norm on $\boldsymbol{R}^n$ such that $\|A_-^{-1}\| = \lambda < 1$. Let $\|A^{-1}\|$

$= \mu$ and $\sup\limits_{\|x\| \leqslant \delta} \|\tilde{Q}(x)\| = \mathcal{M}$. We choose a nondecreasing sequence of numbers ν_s such that $\mathcal{M}\lambda^\nu \mu^s \leqslant \tilde{\lambda} < 1$ for all $\nu \geqslant \nu_s$ $(s = 0, 1, 2, \ldots)$. Let the map ψ belong to the compact set $\mathcal{K}$ defined by inequalities (14). On differentiating the right-hand side of (13) s times $(s = 0, 1, 2, \ldots)$ we obtain

$$(T\psi)^{(s)}(x) = \varkappa_\delta(x)\varkappa_\delta(\Lambda^{-1}x)\tilde{Q}(x)\psi^{(s)}(\Lambda^{-1}x)(\Lambda^{-1})^{\otimes s} + \\ + \Gamma_s(x, \psi(x), \psi'(x), \ldots, \psi^{(s-1)}(x)),$$

where Γ_s stand for terms that do not contain the s-th derivative of the map ψ $(\Gamma_0(x) = \varkappa_\delta(x)\varkappa_\delta(\Lambda^{-1}x)\tilde{\tau}(x))$. Hence, on considering the definition of compact set (14) we have

$$\|(T\psi)^{(s)}(x)\| \leqslant \mathcal{M}\mathcal{C}_{s\nu}\lambda^\nu \|x_-\|^\nu \mu^s + \gamma_{s\nu}(\mathcal{C}_{0\nu}, \ldots, \mathcal{C}_{s-1,\nu})\|x_-\|^\nu$$
$$\leqslant \mathcal{C}_{s\nu}\|x_-\|^\nu \left(\tilde{\lambda} + \frac{\gamma_{s\nu}(\mathcal{C}_{0\nu}, \ldots, \mathcal{C}_{s-1,\nu})}{\mathcal{C}_{s\nu}}\right),$$
$$\nu \geqslant \nu_s, \quad s = 0, 1, 2, \ldots,$$

where $\gamma_{s\nu}$ is a certain linear function with nonnegative coefficients ($\gamma_{0\nu} = \text{const}$). We choose consecutively the numbers $\mathcal{C}_{s\nu}$ such that

$$\tilde{\lambda} + \frac{\gamma_{s\nu}(\mathcal{C}_{0\nu}, \ldots, \mathcal{C}_{s-1,\nu})}{\mathcal{C}_{s\nu}} < 1.$$

Then, if $\psi \in \mathcal{K}$, we have $\|(T\psi)^{(s)}(x)\| \leqslant \mathcal{C}_{s\nu}\|x_-\|^\nu$, $\nu \geqslant \nu_s$, $s = 0, 1, 2, \ldots$; that is, T maps $\mathcal{K}$ into itself. Hence, equation (13) has a local C^∞-solution $\psi \in \mathcal{K}$.

We note that the above reasoning implies that formal solvability, i.e., validity of (3), is sufficient for local solvability of the equation (1) in the case that $\mathcal{L}_+ = \mathcal{L}_1 = 0$.

(iii) We suppose that $\mathcal{L}_+ \neq 0$, $\mathcal{L}_1 = 0$ and conditions (3) hold. By (3), the map γ may be assumed to be flat at zero. The map γ can be written in the form $\gamma = \gamma_+ + \gamma_-$, where γ_+ (respectively, γ_-) is a map that is flat on $\mathcal{L}_+$ (respectively, $\mathcal{L}_-$). To prove solvability of equation (1) it suffices to show that each of the equations

$$\varphi_+(\Lambda x) - Q(x)\varphi_+(x) = \gamma_+(x), \tag{15}$$

$$\varphi_-(\Lambda x) - Q(x)\varphi_-(x) = \gamma_-(x) \tag{16}$$

is solvable, where φ_+ (respectively, φ_-) is a map to be found and is flat on $\mathcal{L}_+$ (respectively, $\mathcal{L}_-$). (If, in particular, $\mathcal{L}_- = 0$, then the problem is reduced to solvability of (16).) The proof of existence of local C^∞-solutions of (15) is carried out in exactly the same way as the proof that (11) is solvable.

We consider now equation (16). First we suppose that the matrix $Q(0)$ is nonsingular. Then equation (16) is reduced to an equation of the type (15) by multiplying the both parts of (16) by $Q^{-1}(x)$ and substitution $x \mapsto \Lambda^{-1}x$.

Suppose now that the matrix $Q(0)$ in equation (16) is nilpotent. We rewrite (16) in the form

(17) $$\varphi_-(x_+, x_-) = (T\varphi_-)(x_+, x_-),$$

where

$$(T\varphi_-)(x_+, x_-) = Q(\Lambda_+^{-1} x_+, \Lambda_-^{-1} x_-)\varphi_-(\Lambda_+^{-1} x_+, \Lambda_-^{-1} x_-) + \tilde{\gamma}_-(x_+, x_-),$$

$x_+ = \mathscr{P}_+ x$, $\Lambda_+ = \mathscr{P}_+ \Lambda$, $\mathscr{P}_+$ is the projection on the subspace $\mathscr{L}_+$, $\tilde{\gamma}_-(x) = \gamma_-(\Lambda^{-1} x)$. First suppose that $Q(0) = 0$. We choose vector norms on $\mathscr{L}_+$ and $\mathscr{L}_-$ such that for some μ_1, $\mu_2 > 1$ and some $0 < \lambda_1, \lambda_2 < 1$ the inequalities

(18) $$\mu_1 \|x_+\| \leqslant \|\Lambda_+^{-1} x_+\| \leqslant \mu_2 \|x_+\|, \qquad \lambda_1 \|x_-\| \leqslant \|\Lambda_-^{-1} x_-\| \leqslant \lambda_2 \|x_-\|$$

would be satisfied, and we put $\|x\| = \max\{\|x_+\|, \|x_-\|\}$ for $x = (x_+, x_-) \in \boldsymbol{R}^n$. Let $\mathscr{M} = \sup_{\|x\| \leqslant \delta} \|Q(x)\|$, $\mu = \|\Lambda^{-1}\|$. For each pair of nonnegative integers s, ν we choose numbers $\varepsilon_{s\nu} > 0$ such that

(19) $$\varepsilon_{s\nu} \mu_2^{\nu} \mu^{s} \leqslant \lambda < 1.$$

Since, by assumption, $Q(0) = 0$, for any $\varepsilon_{s\nu}$ we can find $\delta_{s\nu} > 0$ such that

(20) $$\|Q(x)\| \leqslant \varepsilon_{s\nu} \quad \text{for } \|x\| \leqslant \delta_{s\nu}.$$

For each pair of nonnegative integers s, ν we choose numbers $\alpha_{s\nu} > 0$ such that

(21) $$\alpha_{s\nu} < \lambda \delta_{s\nu}^{2\nu} \mu_1^{\nu} \mu_2^{-2\nu} \mathscr{M}^{-1} \mu^{-s}.$$

We note that, since we are interested in a local C^{∞}-solution of (17), the maps $Q(x)$ and $\tilde{\gamma}_-(x)$ may be assumed to be equal to zero outside the set $S(\delta) = \{x \in \boldsymbol{R}^n \mid \|\mathscr{P}_+ x\| \leqslant \delta\}$. We consider the compact convex set $\mathscr{K} = \mathscr{K}(\{\mathscr{C}_{s\nu}\}, \{\nu_s\}, \{\delta_{s\nu}\}, \{\alpha_{s\nu}\}, \delta)$ of C^{∞}-maps $\varphi_-: S(\delta) \to \boldsymbol{C}^m$ that are flat on $\mathscr{L}_-$ and satisfy inequalities

(22) $$\|\varphi_-^{(s)}(x)\| \leqslant \mathscr{C}_{s\nu} \xi_{s\nu}(x), \qquad \nu \geqslant \nu_s,\ s = 0, 1, 2, \ldots,$$

where

$$\xi_{s\nu}(x) = \begin{cases} \|x\|^{\nu}, & \|x_+\| \leqslant \delta_{s\nu}, \\ \alpha_{s\nu} \|x_+\|^{-\nu}, & \delta_{s\nu} < \|x_+\| \leqslant \delta, \\ 0, & \delta < \|x_+\|. \end{cases}$$

Let us show that we can choose numbers $\mathscr{C}_{s\nu}$ and a nondecreasing sequence ν_s such that the operator T maps the compact set (22) into itself. As in (ii), the proof of possibility of such a choice is carried out by induction on s.

We consider the following cases:

(α) Let $\|\Lambda_+^{-1} x_+\| \leqslant \delta_{s\nu}$ and, besides, $\|\Lambda_-^{-1} x_-\| \leqslant \delta_{s\nu}$. On differentiating the right-hand side of (17) s-times ($s = 0, 1, 2, \ldots$) and taking into account

(18)–(20) and compact set definition (22), we obtain that for all $\nu \geqslant \nu_{s-1}$

$$(23)\qquad \|(T\varphi_-)^{(s)}(x)\| \leqslant \varepsilon_{s\nu}\mathscr{C}_{s\nu}\mu_2^{\nu}\|x\|^{\nu}\mu^s+\gamma_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})\|x\|^{\nu}$$
$$\leqslant \mathscr{C}_{s\nu}\|x\|^{\nu}\left[\lambda+\frac{\gamma_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})}{\mathscr{C}_{s\nu}}\right],$$

where $\gamma_{s\nu}$ is a certain linear function with nonnegative coefficients ($\gamma_{0\nu} = \text{const}$, $\nu_{-1} = 0$).

(β) Let $\|\Lambda_+^{-1}x_+\| \leqslant \delta_{s\nu}$, though $\|\Lambda_1^{-1}x_-\| > \delta_{s\nu}$. We note that in this case $\|\Lambda^{-1}x\| = \|\Lambda_-^{-1}x_-\|$ and $\|x\| = \|x_-\|$. On taking into account this notation and also (18), (22), we obtain that for all $\nu \geqslant \nu_{s-1}$

$$\|(T\varphi_-)^{(s)}(x)\| \leqslant \mathscr{M}\mathscr{C}_{s\nu}\lambda_2^{\nu}\|x\|^{\nu}\mu^s+\tilde{\gamma}_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})\|x\|^{\nu}.$$

We choose numbers $\tilde{\nu}_s \geqslant \nu_{s-1}$ such that $\mathscr{M}\lambda_2^{\nu}\mu^s \leqslant \tilde{\lambda} < 1$ for all $\nu \geqslant \tilde{\nu}_s$. Then

$$(24)\qquad \|(T\varphi_-)^{(s)}(x)\| \leqslant \mathscr{C}_{s\nu}\|x\|^{\nu}\left[\tilde{\lambda}+\frac{\tilde{\gamma}_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})}{\mathscr{C}_{s\nu}}\right],\qquad \nu \geqslant \tilde{\nu}_s.$$

(γ) Now, let $\|x_+\| \leqslant \delta_{s\nu} < \|\Lambda_+^{-1}x_+\|$. Then it follows from (18) that $\|x_+\| > \delta_{s\nu}\mu_2^{-1}$. On taking into account this inequality and also (18), (21), (22), we obtain that for all $\nu \geqslant \nu_{s-1}$

$$(25)\qquad \|(T\varphi_-)^{(s)}(x)\| \leqslant \mathscr{M}\mathscr{C}_{s\nu}\alpha_{s\nu}\mu_1^{-\nu}\|x_+\|^{-\nu}\mu^s+\tilde{\tilde{\gamma}}_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})\|x\|^{\nu}$$
$$\leqslant \mathscr{C}_{s\nu}\|x\|^{\nu}\left[\lambda+\frac{\tilde{\tilde{\gamma}}_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})}{\mathscr{C}_{s\nu}}\right].$$

(δ) Finally, let $\|x_+\| > \delta_{s\nu}$. On taking into account (18) and (22), we obtain

$$\|(T\varphi_-)^{(s)}(x)\| \leqslant \mathscr{M}\mathscr{C}_{s\nu}\alpha_{s\nu}\mu_1^{-\nu}\|x_+\|^{-\nu}\mu^s+\tilde{\tilde{\tilde{\gamma}}}_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})\|x_+\|^{-\nu},$$
$$\nu \geqslant \nu_{s-1}.$$

We choose $\tilde{\tilde{\nu}}_s \geqslant \nu_{s-1}$ such that $\mathscr{M}\mu_1^{-\nu}\mu^s \leqslant \tilde{\tilde{\lambda}} < 1$ for all $\nu \geqslant \tilde{\tilde{\nu}}_s$. Then

$$(26)\qquad \|(T\varphi_-)^{(s)}(x)\| \leqslant \mathscr{C}_{s\nu}\alpha_{s\nu}\|x_+\|^{-\nu}\left[\tilde{\tilde{\lambda}}+\frac{\tilde{\tilde{\tilde{\gamma}}}_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})}{\alpha_{s\nu}\mathscr{C}_{s\nu}}\right],\qquad \nu \geqslant \tilde{\tilde{\nu}}_s.$$

We put $\nu_s = \max\{\tilde{\nu}_s, \tilde{\tilde{\nu}}_s\}$. Inequalities (23)–(26) imply that

$$\|(T\varphi_-)^{(s)}(x)\| \leqslant \mathscr{C}_{s\nu}\xi_{s\nu}(x)\left[\tilde{\tilde{\tilde{\lambda}}}+\frac{\varkappa_{s\nu}(\mathscr{C}_{0\nu},\ldots,\mathscr{C}_{s-1,\nu})}{\alpha_{s\nu}\mathscr{C}_{s\nu}}\right],\qquad \nu \geqslant \nu_s,$$

where $\tilde{\tilde{\tilde{\lambda}}} = \max\{\lambda, \tilde{\lambda}\} < 1$ and $\varkappa_{s\nu}$ is a certain linear function with nonnegative coefficients. We choose consecutively numbers $\mathscr{C}_{sv}$ such that

$$\tilde{\tilde{\lambda}} + \frac{\varkappa_{sv}(\mathscr{C}_{0v}, \ldots, \mathscr{C}_{s-1,v})}{\alpha_{sv}\mathscr{C}_{sv}} < 1.$$

Then the operator T maps compact set (22) into itself. Consequently, equation (17) has a local C^{∞}-solution.

Suppose now that the matrix $Q(0)$ is nilpotent, but nonzero. Without loss of generality, $Q(0)$ may be assumed to have a normal Jordan form. We carry out a transformation of the map to be found in equation (16): $\varphi_-(x) = A(x)\tilde{\varphi}_-(x)$, where $A(x) = \operatorname{diag}\{\|x\|^{\alpha_1}, \ldots, \|x\|^{\alpha_m}\}$ and $0 < |\alpha_i - \alpha_j| < 1$, $i \neq j$. This transformation is an isomorphism of the space of C^{∞}-maps that are flat at zero. Equation (16) will be transformed to a similar equation with the operator function $\tilde{Q}(x) = A^{-1}(\Lambda x)Q(x)A(x)$. For a mentioned choice of parameters α_i we obtain that $\tilde{Q}(0) = 0$ which reduces the case to the one already considered.

The complete the proof of solvability of (16) it is enough to prove the following lemma on normalization:

LEMMA. *Let Λ be a hyperbolic linear operator, i.e., $\mathscr{L}_1 = 0$. Then there exists a local C^{∞}-transformation $A(x)$ of the matrix function $Q(x)$ to a block-diagonal form $\tilde{Q}(x) = A^{-1}(\Lambda x)Q(x)A(x) = \operatorname{diag}\{\tilde{Q}_1(x), \tilde{Q}_0(x)\}$ in which the matrix $\tilde{Q}_1(0)$ is nonsingular and the matrix $\tilde{Q}_0(0)$ is nilpotent.*

Proof of the lemma. There exists a formal transformation $\hat{A}(x)$ that reduces the matrix $\hat{Q}(x)$ to the matrix $\hat{\tilde{Q}}(x) = \hat{A}^{-1}(\Lambda x)\hat{Q}(x)\hat{A}(x)$ the latter satisfying the condition

$$\hat{\tilde{Q}}(\Lambda_s x)Q_s = Q_s\hat{\tilde{Q}}(x), \tag{27}$$

where Λ_s (respectively, Q_s) is a semi-simple part of the matrix Λ (respectively, $Q(0)$). This statement is, in essence, a variant of the Poincaré–Dulac theorem (see [1]) for a formal map $\hat{\Phi}$: $\boldsymbol{R}^n \times \boldsymbol{C}^m \to \boldsymbol{R}^n \times \boldsymbol{C}^m$ defined by the formula $\hat{\Phi}(x, y) = (\Lambda x, \hat{Q}(x)y)$ and it is proved after the same scheme.

It follows from (27) that $\hat{\tilde{Q}}(x) = \operatorname{diag}\{\hat{\tilde{Q}}_1(x), \hat{\tilde{Q}}_0(x)\}$, where $\hat{\tilde{Q}}_1(0)$ is nonsingular and $\hat{\tilde{Q}}_0(0)$ is nilpotent. Hence, we may at once assume that the matrix function $Q(x)$ has the form: $Q(x) = \operatorname{diag}\{Q_1(x), Q_0(x)\} + \tau(x)$, where $Q_1(0)$ is nonsingular, $Q_0(0)$ is nilpotent, and $\tau(x)$ is a matrix function that is flat at zero. We seek the transformation $A(x)$ of the matrix $Q(x)$ to an upper triangular block matrix $\tilde{Q}(x) = A^{-1}(\Lambda x)Q(x)A(x)$ in the form:

$$A(x) = \begin{pmatrix} E & 0 \\ a_{21}(x) & E \end{pmatrix}, \quad \hat{a}_{21} = 0,$$

where the dimensions of unit blocks correspond to those of $Q_1(x)$, $Q_0(x)$.

Thus we arrive at a system for $\tilde{Q}(x)$ and $a_{21}(x)$:

$$\begin{aligned} \tilde{Q}_1(x) &= Q_1(x)+\tau_{12}(x)a_{21}(x)+\tau_{11}(x), \\ \tilde{Q}_{12}(x) &= \tau_{12}(x), \\ a_{21}(\Lambda x)\tilde{Q}_1(x) &= \tau_{21}(x)+Q_0(x)a_{21}(x)+\tau_{22}(x)a_{21}(x), \\ \tilde{Q}_0(x)+a_{21}(\Lambda x)\tilde{Q}_{12}(x) &= Q_0(x)+\tau_{22}(x), \end{aligned}$$

where $\tilde{Q}_1(x)$, $\tilde{Q}_0(x)$, $\tilde{Q}_{12}(x)$ are the blocks of the upper triangular matrix $\tilde{Q}(x)$ while $\tau_{ij}(x)$ are the respective blocks of the matrix $\tau(x)$. On substituting the expression for $\tilde{Q}_1(x)$ from the first equation to the third one, we obtain

$$a_{21}(\Lambda x) = \mathscr{H}(x, a_{21}(x)), \tag{28}$$

where $\mathscr{H}(x, y) = [(Q_0(x)+\tau_{22}(x))y+\tau_{21}(x)][Q_1(x)+\tau_{12}(x)y+\tau_{11}(x)]^{-1}$; note that $\mathscr{H}(x, 0)$ is a C^∞-map that is flat at zero while the matrix $\mathscr{H}'_y(0, 0)$ is nilpotent. On restricting equation (28) to the subspace $\mathscr{L}_-$ and following the line of reasoning employed in the proof of solvability of (11), we prove existence of a local C^∞-solution $a_{21}(0, x_-)$ for the equation obtained. Similarly, on differentiating (28) and restricting the equation obtained to the subspace $\mathscr{L}_-$, we find a C^∞-jet $(a_{21}(0, x_-), a'_{21}(0, x_-), \ldots)$. Let $a_0(x)$ be a C^∞-Whitney extension of this jet to the δ-neighbourhood of the origin. We seek the solution of (28) in the form $a_{21} = a_0+\psi$, where ψ is a matrix function to be found and is flat on $\mathscr{L}_-$. For ψ we obtain the equation

$$\psi(\Lambda x) = \mathscr{G}(x, \psi(x)), \tag{29}$$

where $\mathscr{G}(x, y) = \mathscr{H}(x, a_0(x)+y)-a_0(\Lambda x)$; note that $\mathscr{G}(x, 0)$ is flat on $\mathscr{L}_-$ while the matrix $\mathscr{G}'_y(0, 0)$ is nilpotent. On following the line of reasoning employed in the proof of solvability of (17), we prove local C^∞-solvability of equation (29).

Thus, equation (28) has a C^∞-solution that is flat at zero. It means that the matrix $Q(x)$ may be assumed to be upper triangular block matrix with needed blocks on the diagonal. To transform $Q(x)$ to a block diagonal form one must apply subsequent transformation

$$A(x) = \begin{pmatrix} E & a_{12}(x) \\ 0 & E \end{pmatrix}, \quad \hat{a}_{12} = 0.$$

As a result, we obtain an equation for $a_{12}(x)$ which is solvable by previous reasoning. The lemma is proved.

(iv) To complete the proof of Theorem 1 we must consider the case when $\mathscr{L}_+ \neq 0$ and $\mathscr{L}_1 \neq 0$. On restricting equation (1) and its derivatives to the subspace $\mathscr{L}_1$ and taking into account inequalities (4), we demonstrate, in the same fashion as in (ii), that it is enough to consider equation (1) with an arbitrary right-hand side γ that is flat on $\mathscr{L}_1$. Such a map γ can be written in the form $\gamma = \gamma_+ + \gamma_-$, where γ_+ (respectively, γ_-) is flat on $\mathscr{L}_1 + \mathscr{L}_+$ (respectively, on $\mathscr{L}_1 + \mathscr{L}_-$). This reduces the problem to considering two equations of the form

$$\varphi_+(\Lambda x) - Q(x)\varphi_+(x) = \gamma_+(x), \tag{30}$$

$$\varphi_-(\Lambda x) - Q(x)\varphi_-(x) = \gamma_-(x), \tag{31}$$

where φ_+ (respectively, φ_-) is a map to be found and is flat on $\mathscr{L}_1 + \mathscr{L}_+$ (respectively, $\mathscr{L}_1 + \mathscr{L}_-$). (If, in particular, $\mathscr{L}_- = 0$, then the problem is reduced to solvability of (31).) Solvability of equation (30) is proved exactly as that of equation (15).

By condition (α), $Q(x)$ may be assumed to be a lower triangular block matrix with nonsingular at zero and nilpotent at zero diagonal blocks. So, to prove solvability of equation (31) it is enough to prove solvability of two equations of type (31) with a matrix $Q(x)$ that is nonsingular or nilpotent at zero. When $\det Q(0) \neq 0$ equation (31) is reduced to (30). But if the matrix $Q(0)$ is nilpotent, then the proof of solvability of equation (31) is similar to that of equation (17). Theorem 1 is proved.

Remark 3. Suppose that conditions of Theorem 1 hold, while $\gamma(x)$ in equation (1) is a C^∞-map equal to zero on a closed set $\mathscr{S}$ which is Λ- and Λ^{-1}-invariant. From the proof of Theorem 1 one can see that there exists a local C^∞-solution of (1) that is equal to zero on the set $\mathscr{S}$.

Proof of Theorem 3. (i) First we suppose that $\mathscr{L}_+ = \mathscr{L}_- = 0$ while the operator $\Lambda_1 = \mathscr{P}_1 \Lambda$ is orthogonal. If condition (4) holds, then it means that the operator $Q(0)$ has no spectral points on the unit circle. Let us show that in this case equation (2) has no nontrivial solution that is flat at zero. Let $\mathscr{M}_+$ (respectively, $\mathscr{M}_-$) be a $Q(0)$-invariant subspace of the space $\mathbf{C}^m$ that corresponds to the part of $Q(0)$-spectrum lying inside the open unit disk (respectively, outside the closed unit disk). We denote by $\tilde{\mathscr{P}}_+$ (respectively, $\tilde{\mathscr{P}}_-$) the projection on the subspace $\mathscr{M}_+$ (respectively, $\mathscr{M}_-$). We put $Q(x) = Q(0) + q(x)$, $q(0) = 0$. Equation (2) is equivalent to the following system

$$\begin{aligned} \varphi_+(x) &= Q_+(0)\varphi_+(\Lambda_1^{-1}x) + \tilde{\mathscr{P}}_+\big(q(\Lambda_1^{-1}x)\varphi(\Lambda_1^{-1}x)\big), \\ \varphi_-(x) &= Q_-^{-1}(0)\varphi_-(\Lambda_1 x) - Q_-^{-1}(0)\tilde{\mathscr{P}}_-\big(q(x)\varphi(x)\big), \end{aligned} \tag{32}$$

where $Q_\pm(0) = \tilde{\mathscr{P}}_\pm Q(0)$. We write $\tilde{q}$ for the greatest of the moduli of the eigenvalues of $Q_+(0)$ and $Q_-^{-1}(0)$. Let $\varepsilon_1, \varepsilon_2 > 0$ be such that $\tilde{\tilde{q}} = |\tilde{q}| + \varepsilon_1 + \varepsilon_2 < 1$. We take a δ-neighbourhood of the origin such that $\|q(x)\| \leqslant \varepsilon_2$ for $\|x\| \leqslant \delta$. Suppose that $\varphi(x) = (\varphi_+(x), \varphi_-(x))$ is a nontrivial C^∞-solution of (32) for $\|x\| \leqslant \delta$ and is flat at zero. On choosing a suitable norm of the matrix $Q(0)$ and taking into account orthogonality of the operator Λ_1, from

(32) we obtain

$$\|\varphi(x)\| \leqslant (|\tilde{q}|+\varepsilon_1)\|\varphi(x)\|+\varepsilon_2\|\varphi(x)\| = \tilde{\tilde{q}}\|\varphi(x)\|, \qquad \|x\| \leqslant \delta,$$

whence it follows that $\varphi(x) \equiv 0$.

(ii) Next suppose that $\mathscr{L}_+ = 0$ and conditions (4) hold.

(iiA) Let, moreover, the sequence $\sigma_k(A)$ be bounded. If $\mathscr{L}_- = 0$, then the case is reduced to the one considered in (i). Let $\mathscr{L}_- \neq 0$. We show that equation (2) has no local nontrivial C^∞-solution that is flat at zero. We put $x_- = 0$ in (2):

$$\varphi(A_1 x_1, 0) - Q(x_1, 0)\varphi(x_1, 0) = 0.$$

By (i), $\varphi(x_1, 0) \equiv 0$. We differentiate equation (2) with respect to the variable x_- and put $x_- = 0$ again:

$$\varphi'_{x_-}(A_1 x_1, 0) - Q(x_1, 0)\varphi'_{x_-}(x_1, 0)A_-^{-1} = 0.$$

By (4) the equation obtained satisfies the conditions of (i), hence, $\varphi'_{x_-}(x_1, 0) \equiv 0$. In exactly the same way one shows that all other derivatives of any solution φ of (2) vanish on $\mathscr{L}_1$.

Now let the map φ that is flat at zero satisfy (2) for $\|x\| \leqslant \delta$. The above reasoning shows that φ must be flat on $\mathscr{L}_1$. Therefore, for any $\nu \geqslant 0$ there exists such a constant $\mathscr{C}_\nu > 0$ that $\|\varphi(x)\| \leqslant \mathscr{C}_\nu \|x_-\|^\nu$ for $\|x\| \leqslant \delta$. From (2) we obtain

$$\varphi(x) = \Big(\prod_{i=1}^{k} Q(A^{-i}x)\Big)\varphi(A^{-k}x), \qquad \|x\| \leqslant \delta. \tag{33}$$

We choose a vector norm on $\boldsymbol{R}^n$ so that $\|A_-^{-1}\| = \lambda < 1$. We put $\mathscr{M} = \sup\limits_{\|x\| \leqslant \delta} \|Q(x)\|$ and choose ν so that $\mathscr{M}\lambda^\nu \leqslant \tilde{\lambda} < 1$. From (33) we have

$$\|\varphi(x)\| \leqslant \mathscr{M}^k \mathscr{C}_\nu \lambda^{k\nu}\|x_-\|^\nu \leqslant \mathscr{C}_\nu \tilde{\lambda}^k.$$

The right-hand side of the inequality obtained tends to zero as $k \to \infty$; that is, $\varphi(x) \equiv 0$.

(iiB) Now let the sequence $\sigma_k(A)$ be not bounded. In the case under consideration it means that there is a Jordan block of dimension greater than 1 for some eigenvalue of A of modulus 1.

First we consider several particular cases. Let equation (2) have the form

$$\varphi(\zeta+\eta, \eta) - Q(\zeta, \eta)\varphi(\zeta, \eta) = 0 \tag{34}$$

that is $n = 2$ and a two-dimensional Jordan block corresponds to the eigenvalue $\lambda = 1$ of the operator A. Let us show that equation (34) has a local nontrivial C^∞-solution that is flat at zero. We put $\varphi(0, \eta) \equiv g(\eta)$, where $g(\eta)$ is a C^∞-map that, together with its derivatives, tends to zero sufficiently quickly as $\eta \to 0$. In accordance with equation (34) we put

$$(35) \qquad \varphi\left(\zeta, \frac{1}{k}\zeta\right) = \left(\prod_{i=1}^{k} Q\left(\frac{k-i}{k}\zeta, \frac{1}{k}\zeta\right)\right) g\left(\frac{1}{k}\zeta\right), \quad k = 1, 2, \ldots,$$

$$(36) \qquad \varphi\left(\zeta, \frac{1}{k}\zeta\right) = \left(\prod_{i=0}^{-k-1} Q^{-1}\left(\frac{k+i}{k}\zeta, \frac{1}{k}\zeta\right)\right) g\left(\frac{1}{k}\zeta\right), \quad k = -1, -2, \ldots$$

By means of (35) φ is determined as a C^∞-map on each line $\eta = \zeta/k$, $k = 1, 2, \ldots$ Since the origin is an isolated zero of the function $\det Q(x)$, equalities (36) determine φ as a C^∞-map on each line $\eta = \zeta/k$, $k = -1, -2, \ldots$ (for a suitable choice of the map g). As $|k| \to \infty$ the lines $\eta = \zeta/k$, $k = \pm 1, \pm 2, \ldots$, converge to the subspace $(\zeta, 0)$ and the value of φ on these lines tends to zero for a suitable choice of the map $g(\eta)$.

We write $\mathcal{S}$ for the bundle of lines: $\zeta = 0$, $\eta = 0$, $\eta = \zeta/k$, $k = \pm 1, \pm 2, \ldots$ We note that $\mathcal{S}$ is closed and Λ- and Λ^{-1}-invariant. Let $\varphi_0(\zeta, \eta)$ be a C^∞-map of $\boldsymbol{R}^2$ into $\boldsymbol{C}^m$ which is equal to φ (determined by (35), (36)) on $\mathcal{S}$. We seek a solution of equation (34) in the form $\varphi = \varphi_0 + \psi$, where ψ is a C^∞-map to be found that is flat at zero and is equal to zero on $\mathcal{S}$. We obtain the equation

$$(37) \qquad \psi(\zeta+\eta, \eta) - Q(\zeta, \eta)\psi(\zeta, \eta) = \gamma(\zeta, \eta)$$

for ψ, where $\gamma(\zeta, \eta) = -\varphi_0(\zeta+\eta, \eta) + Q(\zeta, \eta)\varphi_0(\zeta, \eta)$ is a C^∞-map that is equal to zero on $\mathcal{S}$. By Theorem 1 and Remark 3, equation (37) has a local C^∞-solution ψ that is equal to zero on $\mathcal{S}$. Then $\varphi_0 + \psi$ is a local nontrivial C^∞-solution of equation (34).

The existence of local nontrivial C^∞-solution of the equation

$$\varphi(-\zeta+\eta, -\eta) - Q(\zeta, \eta)\varphi(\zeta, \eta) = 0$$

(in the particular case when $n = 2$ and a two-dimensional Jordan block corresponds to the eigenvalue $\lambda = -1$ of the operator Λ) can be proved similarly.

We consider one more particular case. Let $n = 4$ and two-dimensional Jordan blocks correspond to the eigenvalues $e^{i\beta}$ and $e^{-i\beta}$ ($\beta \neq 0 \pmod{\pi}$) of the operator Λ. We can assume that the matrix Λ is in real normal form. We consider the equation

$$(38) \qquad \varphi(\xi\cos\beta - \eta\sin\beta + \zeta, \xi\sin\beta + \eta\cos\beta + \theta, \zeta\cos\beta - \theta\sin\beta, \zeta\sin\beta + \theta\cos\beta) - Q(\xi, \eta, \zeta, \theta)\varphi(\xi, \eta, \zeta, \theta) = 0.$$

We put $\varphi(0, 0, \zeta, \theta) = g(\zeta, \theta)$, where g is a C^∞-map that, together with its derivatives, tends to zero sufficiently quickly as $(\zeta, \theta) \to 0$. In accordance with

(38) we put

$$\varphi\left(u, \frac{1}{k}\Lambda_1 u\right) = \left(\prod_{i=1}^{k} Q\left(\frac{k-i}{k}\Lambda_1^{-i} u, \frac{1}{k}\Lambda_1^{-i+1} u\right)\right) g\left(\frac{1}{k}\Lambda_1^{1-k} u\right),$$

$$k = 1, 2, \ldots,$$

$$\varphi\left(u, \frac{1}{k}\Lambda_1 u\right) = \left(\prod_{i=0}^{-k-1} Q^{-1}\left(\frac{k+i}{k}\Lambda_1^{i} u, \frac{1}{k}\Lambda_1^{i+1} u\right)\right) g\left(\frac{1}{k}\Lambda_1^{1-k} u\right),$$

$$k = -1, 2, \ldots,$$

where

$$u = (\zeta, \eta), \qquad \Lambda_1 = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

By means of these equalities φ is determined as a C^∞-map (for a suitable choice of the map $g(\zeta, \theta)$) on each plane $v = 1/k\,\Lambda_1 u$, $v = (\zeta, \theta)$, $k = \pm 1, \pm 2, \ldots$ As $|k| \to \infty$ these planes converge to the plane $v = 0$ and the value of φ on these planes tends to zero for a suitable choice of the map $g(v)$. Further reasoning is similar to that in the two-dimensional case.

Next we consider the general case. Let the matrix Λ contain a Jordan block of dimension greater than 1 corresponding to the eigenvalue $\lambda = 1$. Let ζ, η be the variables that correspond to the first two rows of this block. We write x' for the set of remaining variables, i.e., $x = (\zeta, \eta, x')$. We restrict equation (2) to the subspace $(\zeta, \eta, 0)$:

$$\varphi(\zeta+\eta, \eta, 0) - Q(\zeta, \eta, 0)\,\varphi(\zeta, \eta, 0) = 0. \tag{39}$$

As we have shown above, equation (39) has a local nontrivial C^∞-solution $\varphi_0(\zeta, \eta)$. We seek the solution of (2) in the form $\varphi(x) = \varphi_0(\zeta, \eta) + \psi(x)$, where $\psi(x)$ is a C^∞-map to be found that is flat at zero and is equal to zero on the subspace $(\zeta, \eta, 0)$. For ψ we obtain an equation of type (1) with right-hand side $\gamma(x) = -\varphi_0(\zeta+\eta, \eta) + Q(x)\,\varphi_0(\zeta, \eta)$ that is flat at zero and is equal to zero on the subspace $(\zeta, \eta, 0)$. By Theorem 1 and Remark 3, the equation obtained has a local C^∞-solution ψ that is flat at zero and is equal to zero on the subspace $(\zeta, \eta, 0)$. Then the map $\varphi(x) = \varphi_0(\zeta, \eta) + \psi(x)$ is a local nontrivial C^∞-solution of equation (2). In the similar way one shows that equation (2) has a nontrivial solution in the case when the matrix Λ contains a Jordan block of dimension greater than 1 corresponding to the eigenvalue $e^{i\beta}$, $\beta \neq 0 \pmod{2\pi}$.

(iii) Suppose now that $\mathcal{S}_+ \neq 0$ but $\mathcal{S}_- = 0$ and that conditions (4) plus conditions (a), (b) of Theorem 1 hold.

(iiiA) Let the sequence $\sigma_k(\Lambda)$ be bounded. If the matrix $Q(0)$ is nonsingular, then, on multiplying the both sides of (2) by $Q^{-1}(x)$ and substituting $\Lambda^{-1}x$ for x, we reduce equation (2) to the one considered in (iiA).

On taking into account condition (a), we, finally, must consider equation

(2) with a matrix $Q(x)$ that is nilpotent at zero. Like in the nonhomogeneous case, the problem is reduced to the case when $Q(0) = 0$. Let a point $x_0 \in \mathcal{S}_+$ be such that $\det Q(x_0) \neq 0$. Then, to construct a nontrivial solution of equation (2) we consider a sequence of points $x_k = A_+^k x_0$, $k = 0, \pm 1, \pm 2, \ldots$ Let $y_0 \neq 0$ be a vector from the space $\boldsymbol{C}^m$. In accordance with (2) we put

$$\varphi(x_k) = \Big(\prod_{i=1}^{k} Q(x_{k-i})\Big) y_0, \qquad k = 1, 2, \ldots,$$

$$\varphi(x_k) = \Big(\prod_{i=0}^{-k-1} Q^{-1}(x_{k+i})\Big) y_0, \qquad k = -1, -2, \ldots, \tag{40}$$

$$\varphi(x_0) = y_0.$$

By the choice of the point x_0, $\varphi(x_k) \neq 0$, $k = 1, 2, \ldots$ Let $\varphi_0\colon \boldsymbol{R}^n \to \boldsymbol{C}^m$ be a C^∞-map that is flat at zero and assumes the values determined by (40) on the sequence $\{x_k\}$. We seek a solution of (2) in the form $\varphi = \varphi_0 + \psi$. For ψ we obtain an nonhomogeneous equation whose right-hand side is flat at zero and is equal to zero on the sequence $\{x_k\}$. By Theorem 1 and Remark 3, this equation has a local C^∞-solution ψ that is flat at zero and is equal to zero on the sequence $\{x_k\}$. Then, $\varphi = \varphi_0 + \psi$ is a local nontrivial ($\varphi(x_k) \neq 0$, $k = 0, 1, 2, \ldots$) C^∞-solution of (2) that is flat at zero.

(iiiB) If the sequence $\sigma_k(A)$ is not bounded, then the proof of local nontrivial C^∞-solvability of equation (2) is carried out just as the proof of (iiB).

(iv) Finally, suppose that $\mathcal{S}_+ \neq 0$ and $\mathcal{S}_- \neq 0$. Since we are interested in a local solution of equation (2), we may assume that $Q(x) = E$ outside a certain δ-neighbourhood of the origin. Let λ_1, λ_2 be the eigenvalues of the operator A such that $|\lambda_1| < 1 < |\lambda_2|$. We write $h_i(x)$ for linear functionals satisfying $h_i(Ax) = \lambda_i h_i(x)$, $i = 1, 2$. We take numbers $\alpha, \beta > 0$ such that $|\lambda_1|^\alpha |\lambda_2|^\beta = 1$. We put $\sigma(x) = |h_1(x)|^\alpha |h_2(x)|^\beta$. Consider the map

$$\varphi(x) = \begin{cases} e^{-(\sigma(x))^{-1}} \Big(\prod_{k=1}^{\infty} Q(A^{-k} x)\Big) y_0, & x_+ \neq 0, \\ 0, & x_+ = 0, \end{cases} \tag{41}$$

where $y_0 \neq 0$ is an arbitrary vector from $\boldsymbol{C}^m$. It is easy to check that map (41) belongs to C^∞ and is a nontrivial solution of (2). Theorem 3 is completely proved.

Proof of Theorem 2 (by using the lemma) is carried out similarly.

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