ANNALES
POLONICI MATHEMATICI
XLIX (1988)

Differential subordination and meromorphic functions

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Abstract. The notion of Hadamard product is used to define certain classes of meromorphic functions in the unit disc which reduce in special cases to the classes of starlike meromorphic functions and of close-to-convex meromorphic functions. As the applications of differential subordination we prove inclusion theorems for these classes and investigate other properties.

Let $E = \{z \in C\colon |z| < 1\}$ and $H(E)$ be the set of all functions holomorphic in E. Let $g, G \in H(E)$. We say that $g(z)$ is *subordinate to* $G(z)$ (written $g(z) \prec G(z)$) if $G(z)$ is univalent, $g(0) = G(0)$ and $g(E) \subset G(E)$. Now onwards we assume everywhere $h \in H(E)$ is convex univalent in E and satisfies $h(0) = 1$ and $\mathrm{Re}(h(z)) > 0$ for $z \in E$.

Let M be the set of all meromorphic functions in E and having in E the Laurent series expansion

$$f(z) = \frac{1}{z}\Big(1 + \sum_{n=1}^{\infty} a_n z^n\Big).$$

The convolution or Hadamard product $(f*g)(z)$ of two functions

$$f(z) = \frac{1}{z}\Big(1 + \sum_{n=1}^{\infty} a_n z^n\Big) \quad \text{and} \quad g(z) = \frac{1}{z}\Big(1 + \sum_{n=1}^{\infty} b_n z^n\Big)$$

in M is defined as follows:

$$(f*g)(z) = \frac{1}{z}\Big(1 + \sum_{n=1}^{\infty} a_n b_n z^n\Big).$$

Let

$$L_a(z) = \frac{1}{z(1-z)^a},$$

a is any real number. For $f \in M$, we have the following easily verified results:

$$z(L_a * f)'(z) = a(L_{a+1} * f)(z) - (a+1)(L_a * f)(z) \tag{1}$$

and

(2) $$z(L_a * f)'(z) = (L_a * zf')(z).$$

In the sequel, we need the following three lemmas.

LEMMA 1 [1]. *Let $\beta, \gamma \in C$, let $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\mathrm{Re}(\beta h(z)+\gamma) > 0$, $z \in E$ and let $p \in H(z)$, $p(z) = 1+p_1 z+ \ldots$ Then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

LEMMA 2 [3]. *Let $\beta, \gamma \in C$, let $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\mathrm{Re}(\beta h(z)+\gamma) > 0$, $z \in E$ and let $q \in H(E)$ with $q(0) = 1$ and $q(z) \prec h(z)$, $z \in E$. If $p(z) = 1+p_1 z+ \ldots$ is analytic in E, then*

$$p(z)+\frac{zp'(z)}{\beta q(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

LEMMA 3 ([2], p. 12). *Suppose that $h(z) = \sum_{n=1}^{\infty} h_n z^n$ is convex univalent and maps $|z| < 1$ onto D. Let $\omega = g(z) = \sum_{n=1}^{\infty} g_n z^n$ be regular in $|z| < 1$ and assume there only values ω which lie in D. Then $|g_n| \leqslant |h_1|$ and in particular $|h_n| \leqslant |h_1|$ for $n \geqslant 1$.*

DEFINITION 1. Let $M_a^s(h)$ denote the class of functions $f \in M$ such that

$$-\frac{z(L_a * f)'(z)}{(L_a * f)(z)} \prec h(z),$$

where $(L_a * f)(z) \neq 0$ for $z \in E$.

Remark 1. If $a = 1$ and $h(z) = (1-z)/(1+z)$, then since $(L_a * f)(z) \equiv f(z)$, $M_a^s(h)$ reduces to the class of meromorphic starlike univalent functions.

THEOREM 1. *If $f \in M_{a+1}^s(h)$ and $\mathrm{Re}\, h$ is bounded in E, then $f \in M_a^s(h)$ holds for $a+1 > d = \operatorname{Max}_{z \in E} \mathrm{Re}\, h(z)$ provided $(L_a * f)(z) \neq 0$ for $z \in E$.*

Proof. Let

$$p(z) = \frac{z(L_a * f)'(z)}{(L_a * f)(z)}.$$

From (1), we get

$$p(z)+(a+1) = \frac{a(L_{a+1} * f)(z)}{(L_a * f)(z)}.$$

Taking logarithmic derivatives and multiplying by z, we get,

$$\frac{zp'(z)}{p(z)+(a+1)}+p(z)=\frac{z(L_{a+1}*f)'(z)}{(L_{a+1}*f)(z)}. \tag{3}$$

If $f\in M^s_{a+1}(h)$, then

$$\frac{z(-p(z))'}{-(-p(z))+(a+1)}-p(z)\prec h(z). \tag{4}$$

From Lemma 1, it follows that $-p(z)\prec h(z)$ for $\mathrm{Re}(-h(z)+(a+1))>0$, $z\in E$ which means $f\in M^s_a(h)$ for $a+1>d$.

DEFINITION 2. Let

$$f(z)=\frac{1}{z}\Big(1+\sum_{n=1}^{\infty}a_n z^n\Big).$$

Define

$$h_\gamma(z)=\sum_{j=1}^{\infty}\left(\frac{\gamma+1}{\gamma+j}\right)z^{j-2}\qquad \text{for } \mathrm{Re}\,\gamma>-1$$

and

$$F(z)=(f*h_\gamma)(z)=\sum_{j=1}^{\infty}\left(\frac{\gamma+1}{\gamma+j}\right)a_{j-1}z^{j-2}$$

with $a_0=1$. Then

$$F(z)=\frac{\gamma+1}{z^{\gamma+2}}\int_0^z t^{\gamma+1}f(t)\,dt.$$

DEFINITION 3. An infinite sequence $\{d_n\}_1^\infty$ of complex numbers is said to *preserve property* T if whenever $f(z)=\sum_{n=1}^{\infty}a_n z^{n-2}$ possesses property T the convolution $J(z)=f(z)*\sum_{n=1}^{\infty}d_n z^{n-2}$ also possesses property T.

THEOREM 2. *Suppose* $f\in M^s_a(h)$ *and* $\mathrm{Re}\,h$ *is bounded in E. Then* $F\in M^s_a(h)$ *holds for* $\mathrm{Re}(\gamma+2)>d$ *provided* $(L_a*F)(z)\neq 0$ *for* $z\in E$.

Proof. We have from Definition 2

$$zF'(z)+(\gamma+2)F(z)=(\gamma+1)f(z);$$

and so

$$(L_a*zF')(z)+(\gamma+2)(L_a*F)(z)=(\gamma+1)(L_a*f)(z).$$

By (2), we have

$$z(L_a*F)'(z)+(\gamma+2)(L_a*F)(z)=(\gamma+1)(L_a*f)(z). \tag{5}$$

Let $p(z) = z(L_a * F)'(z)/(L_a * F)(z)$. Then (5) gives

$$p(z)+(\gamma+2) = (\gamma+1)\frac{(L_a * f)(z)}{(L_a * F)(z)}.$$

Taking logarithmic derivatives and multiplying by z, we get

$$\frac{zp'(z)}{p(z)+(\gamma+2)}+p(z) = \frac{z(L_a * f)'(z)}{(L_a * f)(z)}. \tag{6}$$

Since $f \in M_a^s(h)$, it follows from (6) that

$$\frac{z(-p(z))'}{-(-p(z))+(\gamma+2)}-p(z) \prec h(z).$$

By Lemma 1, $-p(z) \prec h(z)$ for $\mathrm{Re}(-h(z)+\gamma+2) > 0$, $z \in E$; that is $F \in M_a^s(h)$ for $\mathrm{Re}(\gamma+2) > d$.

COROLLARY 2.1. *For each fixed γ with $\mathrm{Re}(\gamma+2) > d$, the sequence $\{(\gamma+1)/(\gamma+n)\}_{n=1}^{\infty}$ preserves the property $f \in M_a^s(h)$.*

Proof. Corollary follows from Definition 3 with $d_n = (\gamma+1)/(\gamma+n)$.

THEOREM 3. *Let $f \in M$ and let h be continuous in addition on the unit circle. Then $f \in M_a^s(h)$ if and only if $(L_a * f) \neq 0$, $z \in E$ and*

$$f(z) * \frac{(1-h(x))(1-z)-az}{z(1-z)^{a+1}} \neq 0, \quad |x| = 1, \ z \in E. \tag{7}$$

Proof. Let $f \in M$ and satisfy (7) and $(L_a * f) \neq 0$, $z \in E$. Put $g(z) = (L_a * f)(z)$. Then $g(z) \neq 0$, $z \in E$. We can rewrite (7) as

$$G(z) = \frac{(L_{a+1} * f)(z)}{(L_a * f)(z)} \neq \frac{a+1}{a} - \frac{h(x)}{a}, \quad |x| = 1, \ z \in E. \tag{8}$$

From (1) we get

$$G(z) = \frac{a+1}{a} + \frac{1}{a}\frac{zg'(z)}{g(z)}, \quad z \in E. \tag{9}$$

(8) and (9) imply

$$-\frac{zg'(z)}{g(z)} \neq h(x), \quad |x| = 1, \quad z \in E. \quad -\frac{zg'(z)}{g(z)}\bigg|_{z=0} = 1$$

is in $h(E)$. Also $-zg'(z)/g(z)$ is analytic in E and so maps E onto a region which contains 1 and is a subset of $h(E)$. Therefore

$$-\frac{zg'(z)}{g(z)} \prec h(z).$$

Hence $f \in M_a^s(h)$.

Conversely, $f \in M_a^s(h)$ implies

$$-\frac{zg'(z)}{g(z)} \prec h(z), \qquad z \in E$$

and so

$$-\frac{zg'(z)}{g(z)} \neq h(x), \qquad |x| = 1, \ z \in E.$$

By retracing the steps we obtain the converse.

THEOREM 4. *Let* $f \in M_a^s(h)$ *and*

$$f(z) = \frac{1}{z}\left(1 + \sum_{n=1}^{\infty} a_n z^n\right).$$

Then

$$|a_n| \leqslant \frac{|h_1|(1+|h_1|)(2+|h_1|)\dots(n-1+|h_1|)}{a(a+1)(a+2)\dots(a+n-1)}, \qquad n \geqslant 1,$$

where $h(z) = 1 + h_1 z + \dots$, $|z| < 1$.

Proof. Let

$$\frac{-z(L_a * f)'(z)}{(L_a * f)(z)} = p(z) = 1 + p_1 z + \dots + p_n z^n + \dots$$

We have $h(z) = 1 + h_1 z + \dots$, $|z| < 1$. Since $f \in M_a^s(h)$, $p(z) \prec h(z)$. Then, by Lemma 3, we have $|p_n| \leqslant |h_1|$, $n \geqslant 1$. Let

$$L_a(z) = \frac{1}{z(1-z)^a} = \frac{1}{z}\left(1 + \sum_{n=1}^{\infty} b_n z^n\right), \quad \text{where} \quad b_n = \frac{a(a+1)\dots(a+n-1)}{n!}.$$

Now

$$1/z - \sum_{n=1}^{\infty} (n-1) a_n b_n z^{n-1} = (1 + p_1 z + \dots + p_n z^n + \dots)\left(1/z + \sum_{n=1}^{\infty} a_n b_n z^{n-1}\right).$$

Comparing the coefficients on either side, we obtain

$$0 = a_1 b_1 + p_1; \qquad |a_1 b_1| = |p_1| \leqslant |h_1|,$$

$$|a_2 b_2| \leqslant \tfrac{1}{2}|h_1|(1+|h_1|), \qquad |a_3 b_3| \leqslant \tfrac{1}{3}|h_1|(1+|h_1|)(1+\tfrac{1}{2}|h_1|).$$

Then

$$|a_n b_n| \leqslant \frac{|h_1|}{n}(1+|h_1|)\left(1+\frac{|h_1|}{2}\right)\dots\left(1+\frac{|h_1|}{n-1}\right).$$

Therefore

$$|a_n| \leqslant \frac{|h_1|(1+|h_1|)(2+|h_1|)\dots(n-1+|h_1|)}{a(a+1)(a+2)\dots(a+n-1)}, \qquad n \geqslant 1.$$

DEFINITION 4. Let $M_a^c(h)$ be the class of functions $f \in M$ such that

$$\frac{-z(L_a * f)'(z)}{(L_a * \Phi)(z)} \prec h(z) \quad \text{for some } \Phi \in M_a^s(h).$$

Remark 2. If $a = 1$ and $h(z) = (1-z)/(1+z)$, then $M_a^c(h)$ reduces to the class of meromorphic close-to-convex functions.

THEOREM 5. *If $f \in M_{a+1}^c(h)$ and $\operatorname{Re} h$ is bounded in E, then $f \in M_a^c(h)$ holds for $a+1 > d$ provided $(L_a * \Phi)(z) \neq 0$ for $z \in E$.*

Proof. Let

$$p(z) = \frac{z(L_a * f)'(z)}{(L_a * \Phi)(z)}.$$

By (1) we have

$$(L_a * \Phi)(z)\, p(z) + (a+1)(L_a * f)(z) = a(L_{a+1} * f)(z).$$

This gives

(10) $$(L_a * \Phi)(z)\, zp'(z) + zp(z)(L_a * \Phi)'(z) + (a+1)z(L_a * f)'(z) = az(L_{a+1} * f)'(z).$$

Let

$$q(z) = \frac{z(L_a * \Phi)'(z)}{(L_a * \Phi)(z)}.$$

Then (1) implies

(11) $$q(z) + (a+1) = a\frac{(L_{a+1} * \Phi)(z)}{(L_a * \Phi)(z)}.$$

(10) can be written as

(12) $$zp'(z) + p(z)\big(q(z) + (a+1)\big) = a\frac{z(L_{a+1} * f)'(z)}{(L_a * \Phi)(z)}.$$

Dividing (12) by (11),

(13) $$\frac{zp'(z)}{q(z)+(a+1)} + p(z) = \frac{z(L_{a+1} * f)'(z)}{(L_{a+1} * \Phi)(z)}.$$

If $f \in M_{a+1}^c(h)$, then

$$\frac{z(-p(z))'}{-(-q(z)) + (a+1)} - p(z) \prec h(z).$$

By Lemma 2, we have $-p(z) \prec h(z)$ for $\operatorname{Re}(-h(z)+a+1) > 0$, $z \in E$. Since $\Phi \in M^s_{a+1}(h)$, Φ belongs to $M^s_a(h)$ for $a+1 > d$ by Theorem 1. Hence $f \in M^c_a(h)$ for $a+1 > d$.

THEOREM 6. *Suppose $f \in M^c_a(h)$ with respect to the function $\Phi \in M^s_a(h)$. Define Ψ by $\Psi(z) = (\Phi * h_\gamma)(z)$. Then $F \in M^c_a(h)$ with respect to the function Ψ for $\operatorname{Re}(\gamma+2) > d$ provided $(L_a * \Psi)(z) \neq 0$ for $z \in E$.*

Proof. Since $\Psi(z) = (\Phi * h_\gamma)(z)$, we have

$$\Psi(z) = \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} \Phi(t)\, dt.$$

By Theorem 2, $\Psi \in M^s_a(h)$ for $\operatorname{Re}(\gamma+2) > d$ provided $(L_a * \Psi)(z) \neq 0$, $z \in E$, since $\Phi \in M^s_a(h)$. Also

$$z(L_a * \Psi)'(z) + (\gamma+2)(L_a * \Psi)(z) = (\gamma+1)(L_a * \Phi)(z) \tag{14}$$

and

$$z(L_a * F)'(z) + (\gamma+2)(L_a * F)(z) = (\gamma+1)(L_a * f)(z). \tag{15}$$

Let

$$p(z) = \frac{z(L_a * F')(z)}{(L_a * \Psi)(z)}.$$

Then from (15)

$$p(z)(L_a * \Psi)(z) + (\gamma+2)(L_a * F)(z) = (\gamma+1)(L_a * f)(z).$$

Differentiating with respect to z, multiplying by z and dividing by $(L_a * \Psi)(z)$, we get

$$zp'(z) + p(z)\big(q(z)+\gamma+2\big) = (\gamma+1)\frac{z(L_a * f)'(z)}{(L_a * \Psi)(z)},$$

where

$$q(z) = \frac{z(L_a * \Psi)'(z)}{(L_a * \Psi)(z)} \prec -h(z).$$

Hence

$$\frac{zp'(z)}{q(z)+(\gamma+2)} + p(z) = (\gamma+1)\frac{z(L_a * f)'(z)}{z(L_a * \Psi)'(z)+(\gamma+2)(L_a * \Psi)(z)} = \frac{z(L_a * f)'(z)}{(L_a * \Phi)(z)},$$

by (14). Since $f \in M^c_a(h)$,

$$-\frac{z(-p(z))'}{(-q(z))+(\gamma+2)} - p(z) \prec h(z).$$

By Lemma 2, $-p(z) \prec h(z)$ for $\mathrm{Re}(\gamma+2) > d$; that is, $F \in M_a^c(h)$ for $\mathrm{Re}(\gamma+2) > d$.

COROLLARY 6.1. *For each fixed* γ *with* $\mathrm{Re}(\gamma+2) > d$, *the sequence* $\{(\gamma+1)/(\gamma+n)\}_{n=1}^{\infty}$ *preserves the property* $f \in M_a^c(h)$.

Proof. Corollary follows from Definition 3 with $d_n = (\gamma+1)/(\gamma+n)$.

DEFINITION 5. Let

$$f(z) = \frac{1}{z}\left(1 + \sum_{n=1}^{\infty} a_n z^n\right)$$

be in M. Define

$$F_p(z) = \sum_{n=1}^{\infty} \left(\frac{1+\gamma_1}{n+\gamma_1} \cdot \frac{1+\gamma_2}{n+\gamma_2} \cdots \frac{1+\gamma_p}{n+\gamma_p}\right) a_{n-1} z^{n-2},$$

$$F_{p+1}(z) = \sum_{n=1}^{\infty} \left(\frac{1+\gamma_1}{n+\gamma_1} \cdot \frac{1+\gamma_2}{n+\gamma_2} \cdot \ldots \cdot \frac{1+\gamma_p}{n+\gamma_p}\right)\left(\frac{1+\gamma_{p+1}}{n+\gamma_{p+1}}\right) a_{n-1} z^{n-2},$$

where $p = 1, 2, \ldots$, $\mathrm{Re}\,\gamma_p > -1$ and $F_0(z) \equiv f(z)$. Let

$$g(z) = \frac{1}{z}\left(1 + \sum_{n=1}^{\infty} b_n z^n\right),$$

$G_p(z)$, $G_{p+1}(z)$ be similarly defined, with identical γ as in $F_p(z)$ and $F_{p+1}(z)$ but with b_n replacing a_n. (The γ_i may or may not be distinct.)

THEOREM 7. *Let* $f(z)$, $g(z)$, $F_p(z)$, $F_{p+1}(z)$, $G_p(z)$, $G_{p+1}(z)$ *be defined as in Definition 5. Then, for* $p = 1, 2, \ldots$, *we have* $F_p \in M_a^s(h)$ *for* $\mathrm{Re}(\gamma_p+2) > d$ *provided* $(L_a * F_p)(z) \neq 0$, $p = 1, 2, \ldots$, *whenever* $f \in M_a^s(h)$. *Also if* $f \in M_a^c(h)$ *with respect to* $g \in M_a^s(h)$, *then* $F_p(z) \in M_a^c(h)$ *with respect to* $G_p(z) \in M_a^s(h)$ *for* $\mathrm{Re}(\gamma_p+2) > d$ *provided* $(L_a * G_p)(z) \neq 0$, $p = 1, 2, \ldots$

Proof. From the definition of $F_p(z)$ we have the following recursive relations

$$F_{p+1}(z) = (1+\gamma_{p+1}) z^{-(2+\gamma_{p+1})} \int_0^z t^{1+\gamma_{p+1}} F_p(t)\, dt.$$

We also have similar relation for $G_p(z)$. The results follow respectively from Theorem 2 and Theorem 6 with the above recursive relations.

DEFINITION 6. Let $M_{a,\alpha}^c(h)$, α any real number, denote the class of functions $f \in M$ such that

$$J_a(\alpha; f; \Phi) = \alpha\left(\frac{-z(L_{a+1}*f)'(z)}{(L_{a+1}*\Phi)(z)}\right)+(1-\alpha)\left(\frac{-z(L_a*f)'(z)}{(L_a*\Phi)(z)}\right) \prec h(z)$$

for some $\Phi \in M_a^s(h)$ satisfying $(L_{a+1}*\Phi)(z) \neq 0$, $z \in E$.

THEOREM 8. *Let* $f \in M_{a,\alpha}^c(h)$. (i) *If* $\alpha > 0$, *then* $f \in M_{a,0}^c(h) = M_a^c(h)$ *for* $a+1 > d$. (ii) *If* $\alpha < 0$, *then* $f \in M_{a,0}^c(h) = M_a^c(h)$ *for* $a+1 < m = \operatorname{Min}_{z\in E} \operatorname{Re} h(z)$.

Proof. Let

$$p(z) = \frac{z(L_a*f)'(z)}{(L_a*\Phi)(z)}, \quad \text{where } \Phi \in M_a^s(h).$$

Then, using (13), we find that

$$J_a(\alpha; f; \Phi) = \frac{-\alpha z p'(z)}{q(z)+(a+1)} - p(z).$$

If $f \in M_{a,\alpha}^c(h)$ and $\alpha > 0$, then, by Lemma 2, we have $-p(z) \prec h(z)$ for $a+1 > d$. That is, $f \in M_{a,0}^c(h) = M_a^c(h)$ for $a+1 > d$. (ii) If $f \in M_{a,\alpha}^c(h)$ and $\alpha < 0$, then, by Lemma 2, we have $-p(z) \prec h(z)$ for $\operatorname{Re}(-h(z)+a+1) < 0$, $z \in E$. That is, $f \in M_{a,0}^c(h) = M_a^c(h)$ for $a+1 < m$.

THEOREM 9. (i) *For* $\alpha > \beta \geqslant 0$, $M_{a,\alpha}^c(h) \subset M_{a,\beta}^c(h)$ *if* $a+1 > d$.

(ii) *For* $\alpha < \beta \leqslant 0$, $M_{a,\alpha}^c(h) \subset M_{a,\beta}^c(h)$ *if* $a+1 < m$.

Proof. (i) If $\beta = 0$, then this result reduces to part (i) of Theorem 8. Hence we assume that $\beta \neq 0$. Suppose $f \in M_{a,\alpha}^c(h)$. Then $J_a(\alpha; f; \Phi) \prec h(z)$. Let z_1 be any arbitrary point in E. Then

(16) $$J_a(\alpha; f; \Phi)(z_1) \in h(E).$$

By part (i) of Theorem 8,

$$\frac{-z(L_a*f)'(z)}{(L_a*f)(z)} \prec h(z) \quad \text{for } (a+1) > d;$$

so

(17) $$\frac{-z_1(L_a*f)'(z_1)}{(L_a*\Phi)(z_1)} \in h(E) \quad \text{for } a+1 > d.$$

Now

$$J_a(\alpha; f)(z) = \alpha\left(\frac{-z(L_{a+1}*f)'(z)}{(L_{a+1}*f)(z)}\right)+(1-\alpha)\left(\frac{-z(L_a*f)'(z)}{(L_a*f)(z)}\right) \prec h(z)$$

Since $\beta/\alpha < 1$ and $h(E)$ convex, $J_a(\beta; f; \Phi)(z_1) \in h(E)$ for $a+1 > d$ by virtue of (16) and (17). It implies that $J_a(\beta; f; \Phi)(z) \prec h(z)$ for $a+1 > d$. That is, $f \in M_{a,\beta}^c(h)$ for $a+1 > d$.

(ii) This part follows from part (ii) of Theorem 8 as part (i) follows from part (i) of Theorem 8.

DEFINITION 7. Let $M^s_{a,\alpha}(h)$, α being any real number, denote the class of functions $f \in M$ such that

$$J_a(\alpha; f)(z) = \alpha\left(\frac{-z(L_{a+1} * f)'(z)}{(L_{a+1} * f)(z)}\right) + (1-\alpha)\left(\frac{-z(L_a * f)'(z)}{(L_a * f)(z)}\right) \prec h(z)$$

with $(L_{a+1} * f)(z) \neq 0$ and $(L_a * f)(z) \neq 0$ for $z \in E$.

THEOREM 10. *Let* $f \in M^s_{a,\alpha}(h)$. (i) *If* $\alpha > 0$, *then* $f \in M^s_{a,0}(h) = M^s_a(h)$ *for* $a+1 > d$.

(ii) *If* $\alpha < 0$, *then* $f \in M^s_{a,0}(h) = M^s_a(h)$ *for* $a+1 < m$.

Proof. Let

$$p(z) = \frac{z(L_a * f)'(z)}{(L_a * f)(z)}.$$

Then, by (3), we have

$$J_a(\alpha; f)(z) = -\frac{\alpha z p'(z)}{p(z) + (a+1)} - p(z).$$

Since $f \in M^s_{a,\alpha}(h)$, Lemma 1 implies that $-p(z) \prec h(z)$ for $a+1 > d$ if $\alpha > 0$ and that $-p(z) \prec h(z)$ for $a+1 < m$ if $\alpha < 0$. That is, if $\alpha > 0$, then $f \in M^s_a(h)$ for $a+1 > d$ and if $\alpha < 0$, then $f \in M^s_a(h)$ for $a+1 < m$.

THEOREM 11. (i) *If* $\alpha > \beta \geqslant 0$, *then* $M^s_{a,\alpha}(h) \subset M^s_{a,\beta}(h)$ *for* $a+1 > d$.

(ii) *If* $\alpha < \beta \leqslant 0$, *then* $M^s_{a,\alpha}(h) \subset M^s_{a,\beta}(h)$ *for* $a+1 < m$.

Proof. Proof of this theorem is similar to that of Theorem 9.

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Reçu par la Rédaction le 15.04.1986