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GENERIC PROPERTIES OF PERIODIC REFLECTING RAYS AND THE POISSON RELATION

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Let $\Omega \subset \boldsymbol{R}^n$, $n \geqslant 2$, be a compact domain with C^∞ smooth boundary $X = \partial\Omega$, and let $\{\lambda_j^2\}_{j=1}^\infty$ be the eigenvalues corresponding to the Dirichlet problem for the Laplace equation

$$-\Delta u = \lambda^2 u \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

Consider the distribution

$$\sigma(t) = \sum_{j=1}^{\infty} \cos\lambda_j t \in S'(\boldsymbol{R}). \tag{2}$$

It turns out that $\operatorname{sing\,supp}\sigma$ is relevant to the periods of periodic geodesics in Ω.

Let $\gamma = \bigcup_{k=1}^{s} l_k$ be a curve in Ω such that every l_k either is an arc $x_k x_{k+1}$ which is a geodesic on the smooth surface $X = \partial\Omega$ or is a straightline segment $[x_k, x_{k+1}]$, and the points $x_1, x_2, \ldots, x_s, x_{s+1} = x_1$ belong to X. Set for convenience $x_0 = x_s$, $l_0 = l_s$, $l_{s+1} = l_1$. The curve γ will be called a *periodic geodesic* in Ω if the following conditions hold:

(i) if $l_k = \widehat{x_k x_{k+1}}$ is a geodesic on $\partial\Omega$, then the curvature of $\partial\Omega$ vanishes at x_k and x_{k+1},

(ii) for every $k = 1, \ldots, s$ either l_k or l_{k+1} is a segment, and if both l_k and l_{k+1} are segments, then they are not tangent to $\partial\Omega$ at x_{k+1} and satisfy the usual law of reflection at x_{k+1} with respect to $\partial\Omega$,

(iii) if $l_{k-1} = \widehat{x_{k-1} x_k}$ is a geodesic on $\partial\Omega$ and $l_k = [x_k, x_{k+1}]$ is a segment, then l_k is tangent to l_{k-1} (and then to $\partial\Omega$) at x_k and $\overrightarrow{x_k x_{k+1}}$ is an asymptotic direction for $\partial\Omega$ at x_k,

(iv) if $l_{k-1} = [x_{k-1}, x_k]$ is a segment and $l_k = \widehat{x_k x_{k+1}}$ is a geodesic on $\partial\Omega$, then l_{k-1} is tangent to l_k (and then to $\partial\Omega$) at x_k and $\overrightarrow{x_{k-1} x_k}$ is an asyptotic direction for $\partial\Omega$ at x_k.

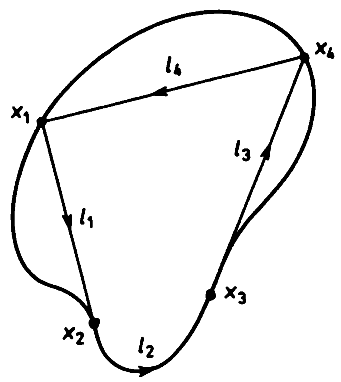

Fig. 1

The number $T_\gamma = \sum_{k=1}^{s} |l_k|$, where $|l_k|$ is the length of l_k, is called the *period* (*length*) of γ. By $\mathscr{L}_\Omega$ we denote the set of all periodic geodesics in Ω.

It follows from the results of Anderson–Melrose [1] and Melrose–Sjöstrand [13] that if for every $x \in \partial\Omega$ the curvature of $\partial\Omega$ at x does not vanish of infinite order, then

(3) $$\operatorname{sing\,supp} \sigma \subset \{-T_\gamma:\ \gamma \in \mathscr{L}_\Omega\} \cup \{0\} \cup \{T_\gamma:\ \gamma \in \mathscr{L}_\Omega\}.$$

This is a special case of the so called *Poisson relation* for manifolds with boundary. In this connection we should mention the following question raised by M. Kac [6]: is it true that if Ω_1 and Ω_2 are two domains in $\boldsymbol{R}^2$ with the same spectrum of the Laplacian, then Ω_1 and Ω_2 are congruent? This problem seems to be very difficult, although it is well known that the spectrum $\{\lambda_j^2\}$ carries some information about the geometry of Ω (cf. [6], [3], [5], [10], [12]).

In this paper some results of V. Petkov and the author are presented, which aim at showing that for generic domains Ω in $\boldsymbol{R}^2$ the Poisson relation (3) becomes an equality.

To explain what we mean by "generic" we will use the space $C^\infty_{\text{emb}}(X, \boldsymbol{R}^n)$ of all C^∞ embeddings $X \to \boldsymbol{R}^n$ endowed with the Whitney C^∞ topology (cf. [2]). This is a Baire space, therefore every *residual* subset (a set containing a countable intersection of open dense sets) of $C^\infty_{\text{emb}}(X, \boldsymbol{R}^n)$ is dense in it. If Y is a smooth connected compact $(n-1)$-dimensional submanifold of $\boldsymbol{R}^n$, by Ω_Y we denote the compact domain in $\boldsymbol{R}^n$ with $\partial\Omega_Y = Y$. A property $\mathscr{P}$ of the smooth connected compact $(n-1)$-dimensional submanifolds X of $\boldsymbol{R}^n$ is called *generic* if, for every such X, the set of all $f \in C^\infty_{\text{emb}}(X, \boldsymbol{R}^n)$ such that $Y = f(X)$ has property $\mathscr{P}$ is a residual subset of $C^\infty_{\text{emb}}(X, \boldsymbol{R}^n)$. By generic domains we mean domains of type Ω_X for generic X.

Let $\gamma = \bigcup_{k=1}^{s} l_k$ be a periodic geodesic in Ω. If all l_k are segments, then γ is called a *periodic reflecting ray* (or a closed billiard trajectory) in Ω. In this case the points $x_1, \ldots, x_s$ are called *reflection points* of γ. If, moreover, some $l_k = [x_k, x_{k+1}]$ is orthogonal to X at x_k or x_{k+1}, then γ is called a *symmetric reflecting ray* ([10]). Otherwise, γ is called a *non-symmetric reflecting ray*. In the first case either $s = 2$ or $s > 2$ and exactly two segments of γ are orthogonal to X at some of their endpoints. Denote by m the number of all different segments of γ (for symmetric γ we should take only one half of all segments in γ), and let t be the number of all different reflection points of γ. The number $d(\gamma)$, defined by $d(\gamma) = m - t$ for non-symmetric γ and by $d(\gamma) = m + 1 - t$ for symmetric γ, is called the *defect* of γ ([10]). To explain the geometrical meaning of $d(\gamma)$, observe that if γ is non-symmetric, then $d(\gamma) = 0$ iff γ passes only once through each of its reflection points. Clearly, in general $d(\gamma)$ could be non-zero.

If every segment l_k of a periodic reflecting ray γ in Ω is not tangent to $\partial\Omega$ at any of its (interior) points, then γ will be called an *ordinary reflecting ray*. There is a very important object related to such a ray γ namely, the *linear Poincaré map* P_γ. To define this map consider a point x lying on the open segment $\mathring{l}_1 = (x_1, x_2)$ and a hyperplane H passing through x and orthogonal to l_1. Set $u = (x_2 - x_1)/\|x_2 - x_1\|$. Taking $(x', u') \in H \times S^{n-1}$, consider the ray γ' starting at x' in direction u' and reflecting from $\partial\Omega$ following the usual law of reflection. It is easy to see that there is an open neighbourhood U of (x, u) in $H \times S^{n-1}$ such that for $(x', u') \in U$ after s reflections on $\partial\Omega$ the ray γ' intersects H transversally at some point y in some direction $v \in S^{n-1}$ (see Figure 2). The map $\mathscr{P}_\gamma$: $U \to H \times S^{n-1}$, defined by $\mathscr{P}_\gamma(x', u') = (y, v)$ is called the Poincaré map related to γ. The linear Poincaré map is given by

$$P_\gamma = d\mathscr{P}_\gamma(x, u)$$

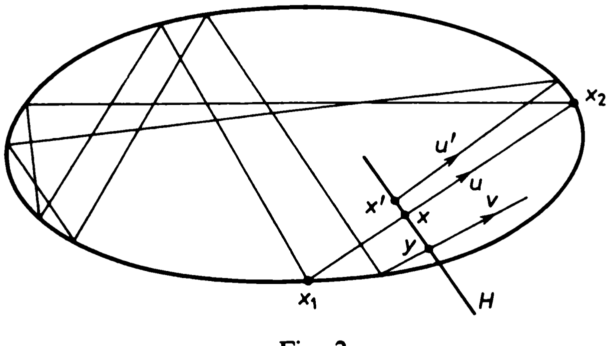

Fig. 2

(cf. [4] or [16]). It is known that, choosing another H, we get a new map P_γ which is conjugate to the previous one. Therefore the spectrum $\operatorname{spec} P_\gamma$ of P_γ does not depend on the choice of H. If $\operatorname{spec} P_\gamma$ does not contain roots of 1, then γ is called a *non-degenerate* periodic reflecting ray. It is well known that $\operatorname{spec} P_\gamma$ gives some information on the dynamical properties of γ. For example, non-degenericity of γ implies γ is isolated among all periodic reflecting rays with s reflection points. This means that there is a neighbourhood V of (x, u) in U such that if some $(x', u') \in V$ generates a periodic reflecting ray γ', then γ' has more than s reflection points.

The starting point of our work is an important theorem of Guillemin and Melrose [4] which says, in particular, that if $\gamma \in \mathscr{L}_\Omega$ is a non-degenerate (ordinary) periodic reflecting ray in Ω and $T_\gamma \neq T_\delta$ for every $\delta \in \mathscr{L}_\Omega \setminus \{\gamma\}$, then T_γ belongs to singsupp $\sigma(t)$.

It is quite natural to ask if the assumptions of this theorem are fulfilled for generic domains Ω.

THEOREM 1. *The following properties are generic for the compact domains Ω in $\boldsymbol{R}^n$:*

(a) ([15]) *Every two different periodic reflecting rays in Ω have rationally independent periods;*

(b) ([19]) $d(\gamma) = 0$ *for every periodic reflecting ray γ in Ω;*

(c) ([14], [16], [17]) *Every periodic reflecting ray in Ω is ordinary and non-degenerate;*

(d) ([17]) *For any integer $s \geqslant 2$ the number $N_\Omega(s)$ of the periodic reflecting rays in Ω with s reflection points is finite;*

(e) ([14], [15], *only for* $n = 2$) *There are no points on $\partial\Omega$ where the curvature of $\partial\Omega$ vanishes of second order, and every periodic geodesic in Ω which does not coincide with $\partial\Omega$ is a periodic reflecting ray.*

The proofs of the above results are based on the multijet transversality theorem (cf. ch. II, theorem 4.13 in [2]). Using different methods Lazutkin [10] proved that for generic strictly convex domains Ω in $\boldsymbol{R}^2$ every periodic reflecting ray γ in Ω is non-degenerate and $d(\gamma) \leqslant 2$. It should be mentioned that our proof of the second part of (c) (which is, in fact, the first part of a Kupka–Smale type theorem for billiards) uses essentially the representation of the Poincaré map P_γ found by Petkov and Vogel [18].

As regards (d), it will be interesting to find the asymptotic of $\{N_\Omega(s)\}_{s=1}^{\infty}$. As is well known, the growth of the number $P(T)$ of periodic geodesics of length less than T has been studied for Riemannian manifolds without boundary. In particular, for manifolds of negative curvature $\lim_{T \to \infty} \log P(T)/T$ exists and equals the topological entropy of the geodesic flow (cf. [11], [7]). For some domains $\Omega \subset \boldsymbol{R}^2$ the growth of $P(T)$ for the billiard flow can be obtained from the estimate from above of the metric entropy (cf. [8]).

Probably, the assertion in (e) remains true for $n = 3$. However, it will be quite surprising if it turns out to be true for some $n \geqslant 4$.

Using Theorem 1 and the result of Guillemin and Melrose [4] mentioned above, we get the following.

THEOREM 2 ([14], [15]). *For generic domains* Ω *in* $\boldsymbol{R}^2$ *we have*

$$\operatorname{sing\,supp}\sigma(t) = \{-T_\gamma\colon \gamma\in\mathscr{L}_\Omega\}\cup\{0\}\cup\{T_\gamma\colon \gamma\in\mathscr{L}_\Omega\},$$

where $\sigma(t)$ *is defined by* (2) *and* $\{\lambda_j^2\}_{j=1}^{\infty}$ *is the spectrum of the problem* (1). *Moreover, for every* $\gamma\in\mathscr{L}_\Omega$ *we can recover the spectrum of the Poincaré map* P_γ *from* $\{\lambda_j^2\}_{j=1}^{\infty}$.

The last statement means that if Ω_1 and Ω_2 are two generic domains in $\boldsymbol{R}^2$ with one and the same spectrum of the Laplacian, then there is a bijection $\varphi\colon \mathscr{L}_{\Omega_1}\to\mathscr{L}_{\Omega_2}$ such that $T_{\varphi(\gamma)} = T_\gamma$ and $\operatorname{spec}P_{\varphi(\gamma)} = \operatorname{spec}P_\gamma$ for every $\gamma\in\mathscr{L}_{\Omega_1}$.

Note that in the above theorem Ω is not assumed to be convex. Marvizi and Melrose [12] proved that for every strictly convex Ω there exists an integer $N(\Omega)$ such that if γ is a periodic reflecting ray in Ω with n reflection points, $n > N(\Omega)$, and rotation number 1, then T_γ belongs to $\operatorname{sing\,supp}\sigma$.

Probably, the statement of Theorem 2 is true for $n > 2$ too. To prove this one might use again the result of Guillemin and Melrose [4] and some parts of Theorem 1. However, in this case some additional difficulties appear. For example, $\mathscr{L}_\Omega$ includes the periodic geodesics lying on $\partial\Omega$, and one should prove that for generic Ω, T_γ/T_δ is not rational for any two different elements γ and δ of $\mathscr{L}_\Omega$. There are three possibilities for γ and δ: (1) γ and δ are periodic reflecting rays; (2) γ and δ are geodesics on $\partial\Omega$; (3) γ is a periodic reflecting ray in Ω and δ is a geodesic on $\partial\Omega$. Klingenberg and Takens [9] proved that given a compact smooth manifold X there is a residual set $\mathscr{R}$ of Riemannian metrics on X ($\mathscr{R}$ is residual in the space of all Riemannian metrics on X) such that for $g\in\mathscr{R}$, every geodesic γ on (X, g) has a non-degenerate Poincaré map. In connection with the problem discussed above, it would be important to prove a similar result for $X\subset\boldsymbol{R}^n$ considering only the Riemannian metrics on X induced by smooth embeddings $f\colon X\to\boldsymbol{R}^n$.

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