ANNALES
POLONICI MATHEMATICI
LIII.3 (1991)

Sinks, sources and saddles for expansive flows with the pseudo orbits tracing property

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Abstract. Classification of points and basic sets for an expansive flow having the pseudo orbits tracing property on a compact metric space is established. There are three different types of behavior like a sink, a source or a saddle.

1. Introduction. In this paper we examine expansive flows having the pseudo orbits tracing property on a compact metric space. Such flows have been distinguished by R. Thomas in [11] and examined in [11], [12] and [13]. They generalize Anosov flows, the restrictions of Axiom A flows to a basic set, the suspensions of subshifts of finite type (see Thomas' papers for more information). Earlier P. Walters ([14]) introduced expansive homeomorphisms having the pseudo orbits tracing property as a generalization of Anosov diffeomorphisms, the restrictions of Axiom A diffeomorphisms to a basic set and subshifts of finite type. Such homeomorphisms also appear in the theory of expanding maps ([10]). They have also been studied in [4]–[9]; we should note here that by a result in [5] the above class of homeomorphisms is the same as the class of Smale spaces ([14]) and the class of expansive homeomorphisms with canonical coordinates ([8], [9]).

We examine the behavior of orbits near a given point, near a periodic orbit and near a basic set (see Proposition 2.13 for definition) in terms of stable and unstable "manifolds". We show that an expansive flow having the pseudo orbits tracing property admits only three different types of behavior, similar to those in the differential and hyperbolic cases. We even use the terms sink, source and saddle for those types of behavior. Our results correspond partially to the similar results obtained for homeomorphisms in [6], [7] and [9]. We take advantage of the techniques introduced by Thomas [11]–[13] and by Bowen and Walters in [2]. We introduce (canonical) coordinates (Proposition 2.9), which are very useful in the sequel. The main results are stated in Theorems 3.7 and 3.12 and remarks after them.

1985 *Mathematics Subject Classification*: 58F15.

Supported by Polish Scientific Grant RP.1.10.

2. Basic definitions and lemmas. We will denote by X a compact metric space with a distance d and by F a flow (a continuous dynamical system) on X, i.e. $F\colon X\times\mathbf{R}\to X$ is continuous and for all $x\in\mathbf{R}$ and $t, s\in\mathbf{R}$, $F(x, 0)=x$, $F(F(x, t), s)=F(x, t+s)$ (see [1] for the theory of dynamical systems). For $x\in X$ and $t\in\mathbf{R}$ we shall often write xt instead of $F(x, t)$ and more generally for a set $I\subset\mathbf{R}$ we put $xI=\{xt\in X\colon t\in I\}$. The set $x\mathbf{R}$ is said to be the *orbit* of x. As we will consider expansive flows we assume without loss of generality that the flow does not have fixed points, i.e. for every x, $xt\neq x$ for some $t\in\mathbf{R}$ (see [2]).

The following two lemmas are easy to prove:

2.1. Lemma. *For every $T>0$ and $\varepsilon>0$ there exists $\delta>0$ such that if $d(x, y)\leqslant\delta$ and $|t|\leqslant T$ then $d(xt, yt)\leqslant\varepsilon$.*

2.2. Lemma. *For every $\varepsilon>0$ there is $\delta>0$ such that if $|t|\leqslant\delta$ then $d(xt, x)\leqslant\varepsilon$ for any $x\in X$.*

2.3. Lemma (cf. Lemma 2 in [2]). *There is $T_0>0$ such that for every $\tau>0$ there exists $\varepsilon>0$ such that for any $x\in X$, if $d(x, xt)\leqslant\varepsilon$ and $|t|\leqslant T_0$, then $|t|\leqslant\tau$.*

2.4. Definition ([2]). The flow is *expansive* if for every $r>0$ there is $\varepsilon>0$ such that for any $x, y\in X$ and every continuous map $h\colon\mathbf{R}\to\mathbf{R}$ with $h(0)=0$, if

$$d(yh(t), xt)\leqslant\varepsilon\quad\text{for all } t\in\mathbf{R}$$

then $y=xt$, where $|t|\leqslant r$.

The following lemma gives the basic property of expansive flows. It is, in fact, equivalent to expansiveness.

2.5. Lemma ([12, Lemma 8]). *Let F be an expansive flow. Then for every $r>0$ there exists $\varepsilon>0$ with the property that for every $\delta>0$ there exists $T>0$ such that for any $x, y\in X$ and every increasing continuous map $h\colon[-T, T]\to\mathbf{R}$ with $h(0)=0$, if*

$$d(yh(t), xt)\leqslant\varepsilon\quad\textit{for } |t|\leqslant T$$

then $d(y, xt)\leqslant\delta$ for some $|t|\leqslant r$.

Fix $r>0$ and $\varepsilon>0$. For given $T>0$ and $\delta>0$ define

$$\begin{aligned} V_T &= \{(x, y)\in X\times X\colon d(xh(t), yt)\leqslant\varepsilon,\ d(xt, yg(t))\leqslant\varepsilon,\\ &\qquad\text{for all } |t|\leqslant T \text{ with some continuous increasing maps}\\ &\qquad h, g\colon[-T, T]\to\mathbf{R},\ h(0)=g(0)=0\},\\ D_{\delta,r} &= \{(x, y)\in X\times X\colon d(x, yt_1)\leqslant\delta,\ d(xt_2, y)\leqslant\delta \text{ for some}\\ &\qquad |t_1|\leqslant r, |t_2|\leqslant r\}. \end{aligned}$$

Now, Lemma 2.5 can be stated as follows.

2.6. LEMMA. *Let F be an expansive flow. Then for every $r > 0$ there exists $\varepsilon > 0$ with the property that for every $\delta > 0$ there exists $T > 0$ such that $V_T \subset D_{\delta,r}$.*

Note that the above lemma corresponds to a basic lemma in the theory of expansive homeomorphisms (see [14, Lemma 2]).

Let $\tau > 0$ and $\delta > 0$. A pair of doubly infinite sequences $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots; \ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots)$ where $x_i \in X$ and $t_i \in \mathbf{R}$ is said to be a (δ, τ)-*pseudo orbit* ((δ, τ)-p.o.) if

$$(1) \qquad d(x_i t_i, x_{i+1}) \leqslant \delta \quad \text{and} \quad t_i \geqslant \tau, \text{ for all } i.$$

A pair of finite sequences $(x_0, \ldots, x_n; t_0, \ldots, t_{n-1})$ is a (δ, τ)-*chain* if (1) holds with $i = 0, 1, 2, \ldots, n-1$.

For a given (δ, τ)-p.o. or (δ, τ)-chain we shall denote by $x_0 * t$ the point on this p.o./chain t units from x_0. More precisely,

$$(2) \quad x_0 * t = \begin{cases} x_n\left(t - \sum_{i=0}^{n-1} t_i\right), & \text{when } \sum_{i=0}^{n-1} t_i \leqslant t < \sum_{i=0}^{n} t_i \quad \text{for } t \geqslant 0, \\ x_n\left(t + \sum_{i=n}^{-1} t_i\right), & \text{when } -\sum_{i=n}^{-1} t_i \leqslant t < -\sum_{i=n+1}^{-1} t_i \quad \text{for } t < 0. \end{cases}$$

Here $\sum_m^n (\ ) = 0$ if $n < m$.

A (δ, τ)-p.o. is ε-*traced* by $x \in X$ if there exists $h \in \operatorname{Rep}(\mathbf{R})$ (i.e. h is an increasing homeomorphism $\mathbf{R} \to \mathbf{R}$ with $h(0) = 0$) such that for all $t \in \mathbf{R}$

$$d\big(xh(t), x_0 * t\big) \leqslant \varepsilon.$$

Fix $\tau > 0$.

2.7. DEFINITION (cf. [11, Definition 1.2], see also [3]). The flow has the *pseudo orbits tracing property* (POTP) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any (δ, τ)-p.o. is ε-traced by some $x \in X$.

Thomas proved ([11, Proposition 1.4]) that this definition does not depend on $\tau > 0$. He also proved the following.

2.8. PROPOSITION ([11, Proposition 4.3]). *Assume the flow F has the following finite pseudo orbits tracing property. For any $\varepsilon > 0$ there exists $\delta > 0$ such that any $(\delta, 1)$-chain is ε-traced. Then F has POTP.*

From now on, we assume that the flow F is expansive and has POTP.

Now, we show that the flow has a property which corresponds to the one known for homeomorphisms as (canonical) coordinates. More details about coordinates will appear in another paper.

For an interval $I \subset \mathbf{R}$ containing the origin we denote by $\operatorname{Rep}(I)$ the set of increasing homeomorphisms $I \to I$ fixing the origin. For $\varepsilon > 0$ and $x \in X$ we define the ε-stable and ε-unstable "manifolds" at x:

$$W_\varepsilon^s(x) = \{y\in X:\ d(yh(t), xt) \leqslant \varepsilon, \text{ for all } t \geqslant 0, \text{ with some } h\in \mathrm{Rep}[0, \infty)\},$$
$$W_\varepsilon^u(x) = \{y\in X:\ d(yh(t), xt) \leqslant \varepsilon, \text{ for all } t \leqslant 0, \text{ with some } h\in \mathrm{Rep}(-\infty, 0]\}.$$

We also put

$$B_\delta = \{(x, y)\in X\times X:\ d(x, y) \leqslant \delta\}, \qquad O(F) = \{A \subset X:\ A \text{ lies on an orbit of } F\}.$$

2.9. PROPOSITION. *For any $r > 0$ there exist $\varepsilon_0 > 0$ with the property that for every $0 < \varepsilon \leqslant \varepsilon_0$ there are $\delta > 0$ and a map $[\ ,\ ]_{r,\varepsilon}$: $B_\delta \to O(F)$ such that:*

(a) *for any $(x, y)\in B_\delta$, $[x, y]_{r,\varepsilon} = W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \neq \emptyset$,*

(b) *for any $(x, y)\in B_\delta$ and $z, w\in [x, y]_{r,\varepsilon}$ there is $|t| \leqslant r$ such that $z = wt$,*

(c) *the following continuity condition holds: for every $\beta > 0$ there exists $\alpha > 0$ such that for every $(x, y), (x', y')\in B_\delta$ satisfying $d(x, x') \leqslant \alpha$, $d(y, y') \leqslant \alpha$ we have $[x, y]_{r,\varepsilon}\times [x', y']_{r,\varepsilon} \subset D_{\beta,r}$.*

Proof. We fix $r > 0$ and by expansiveness pick $\varepsilon_0 > 0$ such that $3\varepsilon_0$ corresponds to r according to Definition 2.4. For $0 < \varepsilon \leqslant \varepsilon_0$ we choose $\delta > 0$ corresponding to $\varepsilon > 0$ by POTP (Definition 2.7 with $\tau = 1$). Let $(x, y)\in B_\delta$. Define a $(\delta, 1)$-p.o. as follows:

$$x_n = \begin{cases} xn, & \text{for } n \geqslant 0, \\ yn, & \text{for } n < 0, \end{cases} \qquad t_n = 1 \quad \text{for all } n.$$

It is ε-traced by a point $z\in X$. That means $d(zh(t), x_0 * t) \leqslant \varepsilon$ for all $t\in \mathbf{R}$ and some $h\in \mathrm{Rep}(\mathbf{R})$. Hence, using (2), we have $z\in W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$, hence $W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \neq \emptyset$. Let $z, w\in W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$. From the triangle inequality and the definition of $W_\varepsilon^s(x)$ and $W_\varepsilon^u(y)$ we have, for $t \geqslant 0$, $d(zh_1(t), wh_2(t)) \leqslant 2\varepsilon$ with some $h_1, h_2\in \mathrm{Rep}[0, \infty)$, so $d(z(h_1\circ h_2^{-1})(t), wt) \leqslant 2\varepsilon_0$ and $h_1\circ h_2^{-1} \in \mathrm{Rep}[0, \infty)$. The same way $d(z(g_1\circ g_2^{-1})(t), wt) \leqslant 2\varepsilon_0$ for all $t \leqslant 0$ with some $g_1\circ g_2^{-1}\in \mathrm{Rep}(-\infty, 0]$. By expansiveness $z = wt$ with some $|t| \leqslant r$. So putting $[x, y]_{r,\varepsilon} = W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ we have proved (a) and (b).

In the proof of (c) we use the following result of Thomas. It is true for any flow without fixed points.

2.10. LEMMA ([13, Lemma 1.2]). *For any $\lambda > 0$ there is $\mu > 0$ with the property that for any $x, y\in X$, $I \subset \mathbf{R}$ an interval containing the origin and an increasing continuous map h: $I \to \mathbf{R}$ fixing the origin, if*

$$d(xh(t), yt) \leqslant \mu, \quad \text{for all } t\in I,$$

then

$$|h(t) - t| \leqslant \lambda |t|, \quad \text{for } t\in I,\ |t| > 1.$$

We prove condition (c). We change $\varepsilon_0 > 0$ in such a way that ε_0 corresponds to r in the sense of Lemma 2.6 and corresponds to $\lambda = 1/2$ in the above Lemma 2.10. Fix $\beta > 0$. By Lemma 2.6 there exists $T > 0$ with $V_T \subset D_{\beta,r}$.

Pick $\alpha > 0$ such that if $d(u, v) \leqslant \alpha$ and $|t| \leqslant 2T$ then $d(ut, vt) \leqslant \varepsilon$ (Lemma 2.1). Let (x, y), $(x', y') \in B_\delta$ and let $d(x, x') \leqslant \alpha$ and $d(y, y') \leqslant \alpha$. Let $z \in [x, y]$, $z' \in [x', y']$ (we omit the subscripts r, ε here).

There are $h, k \in \operatorname{Rep}(\mathbf{R})$ such that

$$(3) \qquad \begin{aligned} d\big(zh(t), xt\big) &\leqslant \varepsilon, \quad \text{for } t \geqslant 0,\\ d\big(zh(t), yt\big) &\leqslant \varepsilon, \quad \text{for } t \leqslant 0,\\ d\big(z'k(t), x't\big) &\leqslant \varepsilon, \quad \text{for } t \geqslant 0,\\ d\big(z'k(t), y't\big) &\leqslant \varepsilon, \quad \text{for } t \leqslant 0. \end{aligned}$$

From this and by the choice of α we have for $0 \leqslant t \leqslant 2T$

$$d\big(zh(t), z'k(t)\big) \leqslant d\big(zh(t), xt\big) + d(xt, x't) + d\big(x't, z'k(t)\big) \leqslant 3\varepsilon$$

and similarly for $-2T \leqslant t \leqslant 0$. So for $|t| \leqslant 2T$ we have

$$(4) \qquad d\big(zh(t), z'k(t)\big) \leqslant 3\varepsilon_0.$$

By Lemma 2.10 inequalities (3) imply

$$h(2T) \geqslant T, \quad h(-2T) \leqslant -T, \quad k(2T) \geqslant T, \quad k(-2T) \leqslant -T.$$

Now, (4) implies that $(z, z') \in V_T \subset D_{\beta,r}$. That means $[x, y] \times [x', y'] \subset D_{\beta,r}$.

The following proposition says that the positive orbit of a point from $W_\varepsilon^s(x)$ tends to the positive orbit of x. The same holds for $W_\varepsilon^u(x)$ so we omit the statement for that case.

2.11. Proposition. *Let $r > 0$ and let $\varepsilon > 0$ correspond to r in the sense of Lemma 2.5. Let $x, y \in X$, $h \in \operatorname{Rep}[0, \infty)$ and let*

$$d\big(yh(t), xt\big) \leqslant \varepsilon \quad \textit{for all } t \geqslant 0.$$

Then

$$\lim_{t \to \infty} \operatorname{dist} d\big(yh(t), x[t-r, t+r]\big) = 0.$$

Proof. Fix $\alpha > 0$ and choose $T > 0$ for this α according to Lemma 2.5. Fix $t \geqslant T$. Define $g\colon [-T, T] \to \mathbf{R}$ by $g(s) = h(t+s) - h(t)$. We have for $|s| \leqslant T$

$$d\big((yh(t))g(s), (xt)s\big) = d\big(yh(t+s), x(t+s)\big) \leqslant \varepsilon$$

as $t+s \geqslant 0$. By Lemma 2.5 there is $|t'| \leqslant r$ with $d\big(yh(t), x(t+t')\big) \leqslant \alpha$.

We define the following sets: the set of *periodic points*

$$\operatorname{Per} = \{x \in X\colon\ xt = x, \text{ for some } t \neq 0\};$$

the sets of *α-limit points* and *ω-limit points*

$$\alpha = \bigcup \{\alpha(x)\colon\ x \in X\}, \qquad \omega = \bigcup \{\omega(x)\colon\ x \in X\},$$

where $\alpha(x) = \{y \in X\colon xt_n \to y$, for some $t_n \to -\infty\}$ and $\omega(x) = \{y \in X\colon xt_n \to y$, for some $t_n \to \infty\}$; the *non-wandering set*

$$\Omega = \{x \in X\colon \text{for every neighborhood } U \text{ of } x \text{ and every } T > 0 \text{ there exists } t \geqslant T \text{ with } U \cap F(U, t) \neq \emptyset\};$$

the *chain recurrent set*

$$\mathrm{CR} = \{x \in X\colon \text{for every } \varepsilon > 0 \text{ and } \tau > 0 \text{ there is a non-trivial } (\varepsilon, \tau)\text{-chain from } x \text{ to } x\}.$$

It is easy to see that for any flow the sets defined above are invariant, also Ω and CR are closed and

$$\mathrm{Per} \subset \alpha \cap \omega \subset \alpha \cup \omega \subset \Omega \subset \mathrm{CR}.$$

It is also easy to prove for our flow F that $\mathrm{CR} \subset \mathrm{cl}(\mathrm{Per})$. Here $\mathrm{cl}(A)$ denotes the closure of A. Then we have

2.12. PROPOSITION. $\mathrm{cl}(\mathrm{Per}) = \mathrm{cl}\,\alpha = \mathrm{cl}\,\omega = \Omega = \mathrm{CR}$.

The following proposition contains some results from [13] (Lemma 4.1 and its proof).

2.13. PROPOSITION (Spectral Decomposition Theorem).

(a) $\Omega = \Omega_1 \cup \ldots \cup \Omega_n$, *where* Ω_i *are closed, invariant and pairwise disjoint,*

(b) *for every* $i = 1, \ldots, k$ *and* $x, y \in \Omega_i$, x *and* y *are chain equivalent* (*i.e. for every* $\varepsilon > 0$, $\tau > 0$ *there are* (ε, τ)*-chains from* x *to* y *and from* y *to* x),

(c) *every* Ω_i *contains a dense orbit* (*is topologically transitive*).

Any set Ω_i in the Spectral Decomposition Theorem is said to be a *basic set*.

3. Sinks, sources and saddles. We shall show in this section that under some natural assumptions we have for any point x and small $\varepsilon > 0$: either $x \in \mathrm{int}\, W_\varepsilon^s(x)$ or $x \in \mathrm{int}\, W_\varepsilon^u(x)$ or any neighborhood U of x contains points from $W_\varepsilon^s(x) \cap W_\varepsilon^u(x) \setminus x[-T_0, T_0]$. Each of such situations will be described in more detail. Particularly, in the first case we shall prove that the ω-limit set of x reduces to a periodic orbit which is a basic set. Next, we shall prove that any basic set is either a sink or a source or a saddle and we express this fact in terms of stable and unstable "manifolds" just as above for the point x. In the case of basic sets the description is even better than in the case of points. Moreover, we shall show that if $x \in \mathrm{int}\, W_\varepsilon^s(x)$ then the periodic orbit which is the ω-limit set of x is a sink and similarly if $x \in \mathrm{int}\, W_\varepsilon^u(x)$ then the periodic orbit which is the α-limit set of x is a source.

For any positive $r \leqslant T_0$, T_0 given in Lemma 2.3, we denote by ε_r a number chosen for this r in accordance with Definition 2.4, Lemmas 2.5 and 2.6 and Proposition 2.9.

In the sequel we fix $r > 0$.

3.1. PROPOSITION. *Let $0 < \varepsilon \leqslant \varepsilon_r$, and $x \in X$. The following two conditions are equivalent:*

(a) $x \in \operatorname{int} W^s_\varepsilon(x)$,

(b) *there exists a neighborhood H of x such that $W^u_\varepsilon(x) \cap H \subset x[-r, r]$.*

Proof. Assume (a). Let H be a neighborhood of x such that $x \in H \subset W^s_\varepsilon(x)$. Let $y \in W^u_\varepsilon(x) \cap H$ and $d(x, y) \leqslant \delta$, where δ is chosen for ε in Proposition 2.9. Then $y \in [x, x]_{r,\varepsilon} \subset x[-r, r]$.

Assume (b). By Lemma 2.2 we pick $r' > 0$ and $\beta > 0$ such that $r' < r$, $B(x[-r', r'], \beta) \subset H$ and

$$(5) \qquad d(ut, u) \leqslant \varepsilon/2, \quad \text{for all } u \in X,\ |t| \leqslant r'.$$

By Lemma 2.3 we have a positive $\varepsilon' \leqslant \min(\varepsilon/2, \varepsilon_{r'})$ such that

$$(6) \qquad d(xt, x) \leqslant \varepsilon' \text{ and } |t| \leqslant r \quad \text{imply} \quad |t| \leqslant r'.$$

Proposition 2.9 provides $\delta > 0$ and $\alpha > 0$, $\alpha \leqslant \delta$, such that the map $[\ ,\ ]_{r',\varepsilon'}$ is defined on B_δ and for any y with $d(y, x) \leqslant \alpha$ we have

$$(7) \qquad [y, x]_{r',\varepsilon'} \times [x, x]_{r',\varepsilon'} \subset D_{\beta,r'}.$$

We claim that $B(x, \alpha) \subset W^s_\varepsilon(x)$. Let $y \in B(x, \alpha)$ and let $z \in [y, x]_{r',\varepsilon'}$. Then (7) and the definition of $D_{\beta,r'}$ (in Section 2) provides t_1, $|t_1| \leqslant r'$, such that $d(z, xt_1) \leqslant \beta$. This means that $z \in H$. As $z \in [y, x]_{r',\varepsilon'} \subset W^u_{\varepsilon'}(x) \subset W^u_\varepsilon(x)$ we have $z \in x[-r, r]$ by (6). So $z = xt_2$, where $|t_2| \leqslant r$. Once again, as $z \in W^u_{\varepsilon'}(x)$, we have $d(z, x) \leqslant \varepsilon'$, hence $|t_2| \leqslant r'$ by (6). On the other hand, $z \in W^s_{\varepsilon'}(y)$ so there exists $h \in \operatorname{Rep}[0, \infty)$ such that

$$d(zh(t), yt) \leqslant \varepsilon' \leqslant \varepsilon/2, \quad \text{for all } t \geqslant 0.$$

By (5) we have for $t \geqslant 0$

$$d(zt, xt) = d((xt)t_2, xt) \leqslant \varepsilon/2,$$

so

$$d(xh(t), yt) \leqslant d(xh(t), zh(t)) + d(zh(t), yt) \leqslant \varepsilon \quad \text{for } t \geqslant 0.$$

This proves the claim and Proposition 3.1.

As $W^s_{\varepsilon_1}(x) \subset W^s_{\varepsilon_2}(x)$ and $W^u_{\varepsilon_1}(x) \subset W^u_{\varepsilon_2}(x)$ for $\varepsilon_1 < \varepsilon_2$, we have:

3.2. COROLLARY. *If $x \in \operatorname{int} W^s_{\varepsilon_0}(x)$ with some $0 < \varepsilon_0 \leqslant \varepsilon_r$, then $x \in \operatorname{int} W^s_\varepsilon(x)$ for all $0 < \varepsilon \leqslant \varepsilon_r$.*

In the proof of Proposition 3.4 we need the following:

3.3. LEMMA. *Let $x \in \Omega$, $0 < \varepsilon \leqslant \varepsilon_r$. If $x \in \operatorname{int} W^s_\varepsilon(x)$, then $x \in \operatorname{Per}$.*

Proof. By Proposition 2.12 any neighborhood of x contains periodic points. As $x \in \operatorname{int} W^s_{\varepsilon/2}(x)$ (see Corollary 3.2) we have a periodic point $p \in W^s_{\varepsilon/2}(x)$.

If $p \neq x$, then we can find another periodic point $q \neq p$, $q \in W^s_{\varepsilon/2}(x)$. By the triangle inequality, $p \in W^s_\varepsilon(q)$. Proposition 2.11 implies that the orbits of p and q are arbitrarily close to each other, which is a contradiction since they are compact. So $x = p \in \mathrm{Per}$.

3.4. PROPOSITION. *Let $0 < \varepsilon \leqslant \varepsilon_r$, $x \in X$, $y \in \omega(x)$. Assume that $x \in \operatorname{int} W^s_\varepsilon(x)$. Then*:

(a) *y is periodic*,
(b) $\omega(x) = y\mathbf{R}$,
(c) $y \in \operatorname{int} W^s_\varepsilon(y)$.

Proof. First we claim that $\omega(x) = \omega(y)$. As $\omega(x)$ is invariant and compact, $\omega(y) \subset \omega(x)$. To prove that $\omega(x) \subset \omega(y)$, let $z \in \omega(x)$ and fix small $\mu > 0$. By Corollary 3.2, $x \in \operatorname{int} W^s_{\mu/3}(x)$. Let $B(x, \lambda) \subset W^s_{\mu/3}(x)$ with some $\lambda > 0$. Pick $\delta > 0$ such that any δ-p.o. is $\min(\lambda, \mu/3)$-traced. As $y \in \omega(x)$, there exists $T > 0$ with $d(xT, y) \leqslant \delta$. So we can construct a $(\delta, 1)$-p.o. with $x_0 = x$ such that

$$x_0 * t = \begin{cases} xt, & \text{for } t < T, \\ y(t-T), & \text{for } t \geqslant T. \end{cases}$$

It is $\min(\lambda, \mu/3)$-traced by a point x' so we have $c \in \mathrm{Rep}(\mathbf{R})$ such that

$$d\big(x'c(t), xt\big) \leqslant \lambda, \quad \text{for } t \leqslant T, \tag{8}$$

$$d\big(x'c(T+t), yt\big) \leqslant \mu/3, \quad \text{for } t \geqslant 0. \tag{9}$$

From (8) we have $d(x', x) \leqslant \lambda$, so $x' \in W^s_{\mu/3}(x)$ and thus

$$d\big(xh(t), x't\big) \leqslant \mu/3 \quad \text{for } t \geqslant 0$$

where $h \in \mathrm{Rep}[0, \infty)$. As $h \circ c \in \mathrm{Rep}[0, \infty)$ and $z \in \omega(x)$, there is $t_0 > T$ with

$$d\big(xh(c(t_0)), z\big) \leqslant \mu/3. \tag{10}$$

Combining (8), (9) and (10) we have

$$d\big(y(t_0 - T), z\big) \leqslant d\big(y(t_0 - T), x'c(t_0)\big) + d\big(x'c(t_0), xh(c(t_0))\big) + d\big(xh(c(t_0)), z\big) \leqslant \mu.$$

This proves the claim.

Now, we prove (c). By Corollary 3.2, $x \in \operatorname{int} W^s_{\varepsilon/4}(x)$. Let $B(x, \lambda) \subset W^s_{\varepsilon/4}(x)$ where $\lambda > 0$. Let $\delta > 0$ correspond to $\min(\lambda, \varepsilon/4)$ according to POTP. We now show that $B(y, \delta/2) \subset W^s_\varepsilon(y)$, which will prove (c). Let $z \in B(y, \delta/2)$. There is $T > 0$ such that $d(xT, y) \leqslant \delta/2$ and thus $d(xT, z) \leqslant \delta$. We construct two $(\delta, 1)$-p.o. with initial points $y_0 = z_0 = x$ such that

$$y_0 * t = \begin{cases} xt, & \text{for } t < T, \\ y(t-T), & \text{for } t \geqslant T, \end{cases}$$

$$z_0 * t = \begin{cases} xt, & \text{for } t < T, \\ z(t-T), & \text{for } t \geqslant T. \end{cases}$$

They are $\min(\lambda, \varepsilon/4)$ traced by some points x_1 and x_2, respectively. So there are $c_1, c_2 \in \mathrm{Rep}(\mathbf{R})$ such that

$$d(x_1 c_1(t), xt) \leqslant \lambda, \tag{11}$$

$$d(x_2 c_2(t), xt) \leqslant \lambda, \tag{12}$$

for $t \leqslant T$, and

$$d(x_1 c_1(T+t), yt) \leqslant \varepsilon/4, \tag{13}$$

$$d(x_2 c_2(T+t), zt) \leqslant \varepsilon/4, \tag{14}$$

for $t \geqslant 0$.

(11) and (12) imply $d(x_1, x) \leqslant \lambda$, $d(x_2, x) \leqslant \lambda$, hence $x_1, x_2 \in W^s_{\varepsilon/4}(x)$. So, there are $h_1, h_2 \in \mathrm{Rep}[0, \infty)$ such that for $t \geqslant 0$ we have

$$d(xh_1(t), x_1 t) \leqslant \varepsilon/4, \tag{15}$$

$$d(xh_2(t), x_2 t) \leqslant \varepsilon/4. \tag{16}$$

Define increasing homeomorphisms $\alpha_i(t) = c_i^{-1}(t+c_i(T))-T$, $\beta_i(t) = h_i^{-1}(t) - c_i(T)$ and put $\varphi_i = \alpha_i \circ \beta_i$, $i = 1, 2$. We have for $t \geqslant 0$

$$\begin{aligned} d(y\varphi_1(t), z\varphi_2(t)) \leqslant\ & d\big(y\alpha_1(\beta_1(t)), x_1 c_1(T+\alpha_1(\beta_1(t)))\big) \\ & + d\big(x_1(\beta_1(t)+c_1(T)), xh_1(\beta_1(t)+c_1(T))\big) \\ & + d(xt, xt) + d\big(xh_2(\beta_2(t)+c_2(T)), x_2(\beta_2(t)+c_2(T))\big) \\ & + d\big(x_2 c_2(T+\alpha_2(\beta_2(t))), z\alpha_2(\beta_2(t))\big). \end{aligned}$$

By (13)–(16) we have $d(y\varphi_1(t), z\varphi_2(t)) \leqslant \varepsilon$ for $t \geqslant 0$, hence

$$d(y, z(\varphi_2 \circ \varphi_1^{-1})(t)) \leqslant \varepsilon.$$

As $\varphi_2 \circ \varphi_1^{-1} \in \mathrm{Rep}[0, \infty)$ we see that $z \in W^s_\varepsilon(y)$ and (c) is proved.

Now, (a) follows from (c) by Lemma 3.3.

Condition (a) implies that $\omega(y) = y\mathbf{R}$ so the claim at the beginning of the proof gives (b).

3.5. Proposition. *Let* $0 < \varepsilon \leqslant \varepsilon_r$, $x \in X$, $y \in \omega(x)$. *Assume that* $y \in \operatorname{int} W^s_\varepsilon(y)$. *Then* $x \in \operatorname{int} W^s_\varepsilon(x)$.

Proof. By Corollary 3.2 there is $\lambda > 0$ with $B(y, \lambda) \subset W^s_{\varepsilon/2}(y)$. As $y \in \omega(x)$ we can find T with $d(xT, y) \leqslant \lambda/2$. By Lemma 2.1 there is $\alpha > 0$ such that for any $z \in B(x, \alpha)$ we have

$$d(zt, xt) \leqslant \varepsilon \quad \text{for } 0 \leqslant t \leqslant T \tag{17}$$

and $d(zT, xT) \leqslant \lambda/2$. This implies $d(zT, y) \leqslant \lambda$, so $zT \in W^s_{\varepsilon/2}(y)$. Clearly also $xT \in W^s_{\varepsilon/2}(y)$. Thus by the triangle inequality

$$d\big(x(T+h_1(t)), z(T+h_2(t))\big) \leqslant \varepsilon \quad \text{for } t \geqslant 0$$

where $h_1, h_2 \in \mathrm{Rep}[0, \infty)$. This combined with (17) clearly means that $z \in W_\varepsilon^s(x)$. We have proved that $B(x, \alpha) \subset W_\varepsilon^s(x)$.

3.6. Proposition. *Let* $0 < \varepsilon \leqslant \varepsilon_r/2$, $x \in X$. *Assume that* $\operatorname{int} W_\varepsilon^s(x) \neq \emptyset$. *Then* $x \in \operatorname{int} W_\varepsilon^s(x)$.

Proof. Let $x' \in \operatorname{int} W_\varepsilon^s(x)$. By the triangle inequality, $x' \in \operatorname{int} W_{2\varepsilon}^s(x')$. Let $y \in \omega(x')$. From Proposition 3.4 it follows that $y \in \operatorname{int} W_{2\varepsilon}^s(y)$. On the other hand, Proposition 2.11 implies $\omega(x) = \omega(x')$. So by Proposition 3.5, $x \in \operatorname{int} W_{2\varepsilon}^s(x)$, and by Corollary 3.2, $x \in \operatorname{int} W_\varepsilon^s(x)$.

All the above results can be summarized in the following:

3.7. Theorem. *Let* $0 < \varepsilon \leqslant \varepsilon_r/2$, $x \in X$, $y \in \omega(x)$. *The following conditions are equivalent*:

(a) $\operatorname{int} W_\varepsilon^s(x) \neq \emptyset$,
(b) $x \in \operatorname{int} W_\varepsilon^s(x)$,
(c) *there exists a neighborhood* H *of* x *such that* $W_\varepsilon^u(x) \cap H \subset x[-r, r]$,
(d) y *is periodic and* $y \in \operatorname{int} W_\varepsilon^s(y)$,
(e) $\omega(x)$ *is a periodic orbit and* $z \in \operatorname{int} W_\varepsilon^s(z)$ *for any* $z \in \omega(x)$.

Moreover, if any of the above conditions holds for some positive $\varepsilon \leqslant \varepsilon_r/2$, *then all of them hold for all positive* $\varepsilon \leqslant \varepsilon_r/2$.

Now, it is reasonable to call a point $x \in X$ a *sink* in the situation described above, a *source* if x is a sink for the reverse flow $F'(x, t) = F(x, -t)$ and a *saddle* in all other cases. Assume that the space X and the flow satisfy the following quite natural assumption. There is $r > 0$ such that for any point x and any neighborhood U of x the set $U \setminus x[-r, r]$ is non-empty. Now, it is impossible for a point to be a sink and a source at the same time. Indeed, (c) of Theorem 3.7 and the condition $x \in \operatorname{int} W_\varepsilon^u(x)$ exclude each other. Moreover, we can state several equivalent conditions for a point to be a saddle. It is enough to consider the negation of a condition from (a) to (e) in Theorem 3.7 and the negation of a condition that we get in the dual case of the source. For example we have

3.8. Corollary. *A point* $x \in X$ *is a saddle if and only if there are small numbers* $r > 0$ *and* $\varepsilon > 0$ *such that any neighborhood* U *of* x *contains points from* $W_\varepsilon^s(x) \setminus x[-r, r]$ *and* $W_\varepsilon^u(x) \setminus x[-r, r]$ *and this holds if and only if* $\operatorname{int} W_\varepsilon^s(x) = \operatorname{int} W_\varepsilon^u(x) = \emptyset$ *for small* $\varepsilon > 0$.

As we have seen the ω-limit set of a sink is a periodic orbit. In the sequel we shall examine the behavior of the flow near periodic orbits. In fact, we shall examine the flow near a locally maximal set with the property that any two of its points are chain equivalent. As we shall see, any periodic orbit, as well as any basic set (see Proposition 2.13), is such a set.

We start with the following

3.9. PROPOSITION. *Let* $\emptyset \neq K \subset X$ *be a closed invariant set. The following conditions are equivalent*:

(a) *the restriction of the flow to* K *has POTP*,

(b) *there exists a constant* $e_K > 0$ *such that if an invariant set* L *satisfies* $K \subset L \subset B(K, e_K)$, *then* $K = L$,

(c) *there is a constant* $e_K > 0$ *such that if* $d(xt, K) \leqslant e_K$ *for all* $t \in \mathbf{R}$, *then* $x \in K$.

K is said to be *locally maximal* or *isolated* if it satisfies condition (b) and so, in our situation, conditions (a) and (c).

Proof. It is easy to see that (b) and (c) are equivalent.

(a) $\Rightarrow$ (c). Condition (a) provides $\delta > 0$ such that any $(\delta, 1)$-p.o. in K is $\varepsilon_{T_0}/2$-traced by a point from K. There is $\delta_1 > 0$ such that $d(x, y) \leqslant \delta_1$ and $0 \leqslant t \leqslant 1$ imply $d(xt, yt) \leqslant \min(\delta/2, \varepsilon_{T_0}/2)$. We put $e_K = \min(\delta/2, \delta_1)$. Assume that for $x \in X$ we have $d(xt, K) \leqslant e_K$, for all $t \in \mathbf{R}$. For any integer n there is $y_n \in K$ such that $d(x_n, y_n) \leqslant e_K$. By the choice of δ_1 we have

$$d(x(n+t), y_n t) \leqslant \varepsilon_{T_0}/2 \quad \text{for } 0 \leqslant t \leqslant 1 \tag{18}$$

and hence

$$d(y_n 1, y_{n+1}) \leqslant d(y_n 1, (xn)1) + d(x(n+1), y_{n+1}) \leqslant \delta.$$

The last condition means that the pair of sequences $(\ldots, y_{-1}, y_0, y_1, \ldots; \ldots, 1, 1, 1, \ldots)$ is a $(\delta, 1)$-p.o. It is $\varepsilon_{T_0}/2$-traced by a point $y \in K$. So we have

$$d(y_0 * t, yh(t)) \leqslant \varepsilon_{T_0}/2 \quad \text{for all } t \in \mathbf{R},$$

where $h \in \mathrm{Rep}(\mathbf{R})$. (18) also means that $d(y_0 * t, xt) \leqslant \varepsilon_{T_0}/2$ for all $t \in \mathbf{R}$, so $d(xt, yh(t)) \leqslant \varepsilon_{T_0}$ for all $t \in \mathbf{R}$. By expansiveness (Definition 2.4) $x = yt \in K$, so we have proved condition (c).

Now we prove that (c) implies (a). Let $0 < \varepsilon \leqslant e_K$ and pick δ for this $\varepsilon > 0$ according to POTP (Definition 2.7). Let $(\ldots, x_{-1}, x_0, x_1, \ldots; \ldots, t_{-1}, t_0, t_1, \ldots)$ be a $(\delta, 1)$-p.o. and $x_n \in K$ for all n. There are $x \in X$ and $h \in \mathrm{Rep}(\mathbf{R})$ such that $d(x_0 * t, xh(t)) \leqslant e_K$ for all $t \in \mathbf{R}$. As $x_0 * t \in K$ for all t and h is a homeomorphism, $d(K, xt) \leqslant e_K$ for all $t \in \mathbf{R}$ and $x \in K$ by (c). This proves (a).

Recall that x and y are chain equivalent if for any $\varepsilon > 0$ and $\tau > 0$ there are (ε, τ)-chains from x to y and from y to x (Proposition 2.13(b)).

3.10. PROPOSITION. *Let* K *be a basic set or a periodic orbit. Then* K *is locally maximal and any two points in* K *are chain equivalent.*

Proof. The last property is clearly satisfied by a periodic orbit and, by Proposition 2.13(b), by a basic set. We prove the first one.

Let $K = x\mathbf{R}$ be a periodic orbit. The restriction of the flow to K, $F|_{K \times \mathbf{R}}$, is the suspension flow of the homeomorphism defined on the one-point space $Y = \{y\}$. This homeomorphism has POTP and hence, by [11, Theorem 2], so does $F|_{K \times \mathbf{R}}$.

Let K be a basic set. To prove that $F|_{K\times\mathbf{R}}$ has POTP we apply Proposition 2.8. Let $0 < \varepsilon \leqslant \varepsilon_{T_0}/2$. Let $\delta > 0$ be chosen for ε by POTP (Definition 2.7, $\tau = 1$). Assume also that

$$\delta < \min\{\operatorname{dist}(K, \Omega_i)\colon \Omega_i \text{ is a basic set different from } K\}.$$

Let $(x_0, \dots, x_n; t_0, \dots, t_{n-1})$ be a $(\delta, 1)$-chain in K. Proposition 2.13(b) provides a $(\delta, 1)$-chain from x_n to x_0, say

$$(x_n, x_{n+1}, \dots, x_m = x_0, t_n, \dots, t_{m-1}), \quad n < m.$$

We construct an (infinite) $(\delta, 1)$-p.o. Extend the chain $(x_0, x_1, \dots, x_m = x_0; t_0, t_1, \dots, t_{m-1})$ backward and forward by putting

$$x_{m+1} = x_1, \qquad t_m = t_0,$$
$$x_{m+2} = x_2, \qquad t_{m+1} = t_1,$$

$$x_{-1} = x_{m-1}, \qquad t_{-1} = t_{m-1},$$
$$x_{-2} = x_{m-2}, \qquad t_{-2} = t_{m-2},$$

Now, it is easy to see that for each $t \in \mathbf{R}$, $x_0 * (t+T) = x_0 * t$, where $T = \sum_{i=0}^{m-1} t_i$. The above p.o. is ε-traced by a point $x \in X$. It is enough to show that $x \in K$. We have

$$d(x_0 * t, xh(t)) \leqslant \varepsilon, \qquad \text{for } t \in \mathbf{R},$$

where $h \in \operatorname{Rep}(\mathbf{R})$. As h is an increasing homeomorphism there is $n > 0$ such that $h(nT) > T_0$. We put $g(t) = h(nT+t) - h(nT)$. Clearly $g \in \operatorname{Rep}(\mathbf{R})$ and for any $t \in \mathbf{R}$ we have

$$d\big((xh(nT))g(t), xh(t)\big) \leqslant d\big(xh(nT+t), x_0 * (nT+t)\big) + d\big(x_0 * t, xh(t)\big) \leqslant 2\varepsilon \leqslant \varepsilon_{T_0}.$$

By expansiveness $xh(nT) = xt_1$, where $|t_1| \leqslant T_0$, so x is periodic as $h(nT) - t_1 > 0$. Hence $x \in \Omega$ and by Proposition 2.13(a) and the choice of δ, $x \in K$.

In the sequel we assume that K is locally maximal and any two points in K are chain equivalent.

We define ε-stable and ε-unstable "manifolds":

$$W^s(K) = \{x \in X\colon d(xt, K) \to 0,\ t \to \infty\} = \{x \in X\colon \omega(x) \subset K\},$$
$$W^u(K) = \{x \in X\colon d(xt, K) \to 0,\ t \to -\infty\} = \{x \in X\colon \alpha(x) \subset K\},$$
$$W^s_\varepsilon(K) = \{x \in K\colon d(xt, K) \leqslant \varepsilon, \text{ for } t \geqslant 0\},$$
$$W^u_\varepsilon(K) = \{x \in K\colon d(xt, K) \leqslant \varepsilon, \text{ for } t \leqslant 0\}.$$

It is easy to see that $W^s_\varepsilon(x) \subset W^s_\varepsilon(K)$ and $W^u_\varepsilon(x) \subset W^u_\varepsilon(K)$ for any $x \in K$ and $\varepsilon > 0$. Moreover, for any $\varepsilon > 0$ and a sequence $t_n \to \infty$,

$$W^s(K) \subset \bigcup_{n=0}^{\infty} F(W^s_\varepsilon(K), -t_n) \quad \text{and} \quad W^u(K) \subset \bigcup_{n=0}^{\infty} F(W^u_\varepsilon(K), t_n).$$

Also by Proposition 2.11, for $0 < \varepsilon \leqslant \varepsilon_{T_0}$ we have $W^s_\varepsilon(x) \subset W^s(K)$ and $W^u_\varepsilon(x) \subset W^u(K)$, for any $x \in K$. We prove more:

3.11. PROPOSITION. (a) *For every* $\varepsilon' > 0$ *there exists* $\varepsilon > 0$ *such that* $W^s_\varepsilon(K) \subset \bigcup_{x \in K} W^s_{\varepsilon'}(x)$.

(b) *There exists* $e_1 > 0$ *such that* $W^s_\varepsilon(K) \subset W^s(K)$ *for any* $0 < \varepsilon \leqslant e_1$.

(c) *For any* $0 < \varepsilon \leqslant e_1$ *and any sequence* $t_n \to \infty$

$$W^s(K) = \bigcup_{n=1}^{\infty} F(W^s_\varepsilon(K), -t_n).$$

Similar statements hold for the dual case.

Proof. (a) Given $\varepsilon' > 0$ we pick δ such that any $(\delta, 1)$-p.o. in K is $\varepsilon'/2$-traced by a point from K. Let $0 < \varepsilon \leqslant \delta/2$ be chosen in such a way that $d(x, y) \leqslant \varepsilon$ and $0 \leqslant t \leqslant 1$ imply $d(xt, yt) \leqslant \min(\delta/2, \varepsilon'/2)$. Let $x \in W^s_\varepsilon(K)$. There are points $y_0, y_1, y_2, \ldots$ in K with $d(x_n, y_n) \leqslant \varepsilon$, for $n = 0, 1, 2, \ldots$ As in the proof of Proposition 3.9 we can see that the pair of sequences

$$(\ldots, y_0(-2), y_0(-1), y_0, y_1, y_2, \ldots; \ldots, 1, 1, 1, \ldots)$$

is a $(\delta, 1)$-p.o. It is $\varepsilon'/2$-traced by a point $y \in K$. So, by the same arguments as in the mentioned proof we have $d(xt, yh(t)) \leqslant \varepsilon'$, for $t \geqslant 0$, where $h \in \mathrm{Rep}[0, \infty)$, hence $x \in W^s_{\varepsilon'}(K)$.

(b) Pick $\varepsilon' > 0$ such that $W^s_{\varepsilon'}(x) \subset W^s(K)$ for any $x \in K$; this is possible by the remarks preceding Proposition 3.11. Let $e_1 > 0$ correspond to this ε' according to (a). Now (b) is clear. Condition (c) follows from (b).

Now, we are ready to prove our main result about the set K.

3.12. THEOREM. *Let* $0 < \varepsilon \leqslant \min(e_K, e_1)$ *(see Propositions* 3.9 *and* 3.11). *The following conditions are equivalent*:

(a) $\operatorname{int} W^s_\varepsilon(K) \neq \emptyset$,
(b) $\operatorname{int} W^s(K) \neq \emptyset$,
(c) $K \subset \operatorname{int} W^s(K)$,
(d) $W^s(K)$ *is open*,
(e) $K \subset \operatorname{int} W^s_\varepsilon(K)$,
(f) *there is an open positively invariant set* G *such that* $K \subset G$ *and* $\bigcap_{t \geqslant 0} F(G, t) = K$,
(g) $W^u(K) = K$,
(h) $W^u_\varepsilon(K) = K$.

In particular, if any of conditions (a), (e), (h) *holds with some positive* $\varepsilon \leqslant \min(e_K, e_1)$, *then all three conditions hold for every positive* $\varepsilon \leqslant \min(e_K, e_1)$.

Proof. (a) $\Rightarrow$ (b) by Proposition 3.11(b).

(b) $\Rightarrow$ (c). Let $x_0 \in \operatorname{int} W^s(K)$, so $x_0 \in B(x_0, \delta) \subset W^s(K)$, where $\delta > 0$. Let $\delta_1 > 0$ correspond to $\min(\varepsilon/2, \delta)$ according to POTP. It is enough to show that $B(K, \delta_1/2) \subset W^s(K)$. Let $x \in B(K, \delta_1/2)$. We can find $p, q \in K$ and $T > 0$ such that $d(x, q) \leqslant \delta_1/2$ and $d(x_0 T, p) \leqslant \delta_1$. As p and q are chain equivalent we have a $(\delta_1/2, 1)$-chain from p to q. We stick together: the orbit segment $x_0(-\infty, T)$, the above $(\delta_1/2, 1)$-chain and the orbit segment $x[0, \infty)$. We thus get a $(\delta_1, 1)$-p.o. It is $\min(\varepsilon/2, \delta)$-traced by a point x'. Hence we have $h \in \operatorname{Rep}(\mathbf{R})$ such that $d(x'h(t), x_0 * t) \leqslant \min(\varepsilon/2, \delta)$ for all $t \in \mathbf{R}$. In particular, $d(x_0, x') \leqslant \delta$, which means that $x' \in W^s(K)$. Moreover, there is $T_1 > 0$ such that $d(x'h(T_1 + t), xt) \leqslant \varepsilon/2$ for all $t \geqslant 0$. Since $x't \to K$ as $t \to \infty$, and $h \in \operatorname{Rep}(\mathbf{R})$, we have $T_2 \geqslant 0$ such that $d(x'h(T_1 + t), K) \leqslant \varepsilon/2$ for $t \geqslant T_2$. That means $xT_2 \in W^s_\varepsilon(K) \subset W^s(K)$ (Proposition 3.11(b)) and then $x \in W^s(K)$.

(c) $\Rightarrow$ (e). Let $p \in K$. From (c) it follows that $p \in B(p, \delta_1) \subset W^s(K)$, where $\delta_1 > 0$. Let $\delta > 0$ be chosen for $\min(\varepsilon/2, \delta_1/2)$ by POTP and suppose $\delta < \delta_1/2$. It is enough to show that $B(p, \delta/2) \subset W^s_\varepsilon(K)$. Let $x_0 \in B(p, \delta/2)$. Since $p \in W^s(K)$ there exists $T \geqslant 1$ such that $d(x_0 t, K) \leqslant \min(\varepsilon/2, \delta/2) \leqslant \varepsilon$ for $t \geqslant T$. So, it is enough to show that

$$d(x_0 t, K) \leqslant \varepsilon \quad \text{for } 0 \leqslant t \leqslant T. \tag{19}$$

Pick $q \in K$ with $d(x_0 T, p) \leqslant \delta/2$ and a $(\delta/2, 1)$-chain from q to p, say $(q = y_0, \ldots, y_n = p; t_0, \ldots, t_{n-1})$. We construct the following $(\delta, 1)$-p.o.:

$$(\ldots, x_0(-2), x_0(-1), x_0, x_0 T, y_0, \ldots, y_n, x_0 T, y_0, \ldots;$$
$$\ldots, 1, 1, T, t_0, \ldots, t_{n-1}, T, t_0, \ldots).$$

By POTP we have a point x and $h \in \operatorname{Rep}(\mathbf{R})$ such that for all $t \in \mathbf{R}$

$$d(x_0 * t, xh(t)) \leqslant \min(\varepsilon/2, \delta_1/2). \tag{20}$$

This, in particular, means that $d(x_0, x) \leqslant \delta_1/2$ and as $d(x_0, p) \leqslant \delta/2 \leqslant \delta_1/2$ we have $d(x, p) \leqslant \delta_1$ and hence $x \in W^s(K)$. On the other hand, (20) implies that for any $t \in [0, T]$ and $n = 0, 1, 2, \ldots$ we have

$$d\big(x_0 t, xh(n(T+S)+t)\big) \leqslant \varepsilon/2,$$

where $S = \sum_{i=0}^{n-1} t_i$. Since $x \in W^s(K)$, for n large enough we have $d\big(xh(n(T+S)+t), K\big) \leqslant \varepsilon/2$, for $0 \leqslant t \leqslant T$. But this implies the required condition (19) so we have proved condition (e).

(e) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) are trivial.

(c) $\Rightarrow$ (d). Let $x \in W^s(K)$. There exists $T > 0$ with $xT \in \operatorname{int} W^s(K)$. By continuity, $yT \in W^s(K)$ for y from some neighborhood of x. So $y \in W^s(K)$ and we have (d).

(e) $\Rightarrow$ (f). Put $G = \bigcup_{t=0}^{\infty} F(\operatorname{int} W^s_\varepsilon(K), t)$. Clearly $K \subset G$, G is open and positively invariant. Moreover, for $x \in G$ we have $x(-t) \in W^s_\varepsilon(K)$, where $t > 0$, hence $x = x(-t)t \in B(K, \varepsilon)$. Thus $G \subset B(K, \varepsilon)$. Let $x \in \bigcap_{t=0}^{\infty} F(G, t)$. For all $t \geqslant 0$, $x(-t) \in G \subset B(K, \varepsilon)$. In particular, $x \in G$ and as G is positively invariant we have $xt \in G \subset B(K, \varepsilon)$ for $t \geqslant 0$. We have shown that $d(K, xt) \leqslant \varepsilon$ for all $t \in \mathbf{R}$. As $\varepsilon \leqslant e_K$ we have $x \in K$ by Proposition 3.9(c), which proves (f).

(f) $\Rightarrow$ (g). Let $x \notin K$. By (f), $x(-t) \notin G$ for all t large enough so $x \notin W^u(K)$.

(g) $\Rightarrow$ (h) follows from the dual statement to Proposition 3.11(b).

(h) $\Rightarrow$ (e). Let $\delta > 0$ correspond to $\varepsilon > 0$ by POTP. We show that $B(K, \delta) \subset W_\varepsilon^s(K)$. For $x_0 \in B(K, \delta)$ we have a $(\delta, 1)$-p.o. defined as follows:

$$x_0 * t = \begin{cases} yt, & \text{for } t < 0, \\ x_0 t, & \text{for } t \geqslant 0, \end{cases}$$

where $y \in K$ is a point such that $d(x_0, y) \leqslant \delta$. There is a point x and $h \in \operatorname{Rep}(\mathbf{R})$ such that

$$d\big(xh(t), yt\big) \leqslant \varepsilon \quad \text{for } t \leqslant 0, \tag{21}$$

$$d\big(xh(t), x_0 t\big) \leqslant \varepsilon \quad \text{for } t \geqslant 0. \tag{22}$$

(21) implies that $x \in W_\varepsilon^u(K) = K$. By (22), $x_0 \in W_\varepsilon^s(x) \subset W_\varepsilon^s(K)$. This proves (e).

We shall call K a *sink* in the situation described in the above Theorem 3.12, a *source* if K is a sink for the reverse flow $F'(x, t) = F(x, -t)$ and a *saddle* in all other cases. If we assume that K is non-isolated in X (i.e. given a neighborhood U of K there is $x \in U \backslash K$) we see that K cannot be a sink and a source at the same time (consider the conditions $K \subset \operatorname{int} W^s(K)$ for a sink and $W^s(K) = K$ for a source). Moreover, this assumption allows us to characterize the third situation, i.e. when K is a saddle. Namely we consider the negation of one of conditions (a)–(h) in Theorem 3.12 combined with the negation of a condition that we get in the dual case of the source.

We have

3.13. PROPOSITION. *If K is a sink/source then K is a basic set.*

Proof. Clearly $K \subset \mathrm{CR} = \Omega$ and as any two points in K are chain equivalent K must be contained in some basic set Ω_i. Since K is a sink/source, by Theorem 3.12(c) there exists a neighborhood U of K such that $(U \backslash K) \cap \Omega = \emptyset$. This implies $K = \Omega_i$.

Let us note that some conditions stated in Theorem 3.12 are known in classical stability theory (see [1] and references therein). For example, (c) means that K is an attractor, (e) means that K is (Lyapunov) stable, etc. For the flow F and the set K all those conditions are equivalent.

Combining Theorems 3.7 and 3.12 we have

3.14. COROLLARY. *Let $x \in X$, $y \in \omega(x)/y \in \alpha(x)$. Then x is a sink/source if and only if y lies on a periodic orbit which is a sink/source.*

Proof. If x is a sink then condition (d) in Theorem 3.7 holds. So, the orbit K of y is periodic and $y \in \operatorname{int} W_\varepsilon^s(y) \subset \operatorname{int} W_\varepsilon^s(K)$, hence condition (e) in Theorem 3.12 holds and K is a sink. Assume now that y lies on a periodic orbit K which is a sink. Let $r > 0$ and $T_0 = T/2$, where T is the smallest positive period of y. Pick $\varepsilon > 0$ so that $d(u, vt) \leqslant \varepsilon$ with $|t| \leqslant T_0$ implies $|t| \leqslant r$ (Lemma 2.3) and $W_\varepsilon^u(K) = K$ (Theorem 3.12(h)). Let $z \in W_\varepsilon^u(y)$. Then $z \in W_\varepsilon^u(K) = K$. So $z = yt$, where $|t| \leqslant T_0$, hence $|t| \leqslant r$. We have proved that $W_\varepsilon^u(y) \subset y[-r, r]$, which implies condition (c) in Theorem 3.7 for y. It is equivalent to condition (b) in this theorem for y. As y is periodic we have condition (d) in the same theorem. This means that x is a sink.

The proof for a source is straightforward.

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Reçu par la Rédaction le 03.01.1989
Révisé le 10.10.1989