COLLOQUIUM MATHEMATICUM

XXVI DÉDIÉ À M. WŁADYSŁAW ŚLEBODZIŃSKI 1972

APPLICATIONS OF LIE DERIVATIVES TO SYMMETRIES, GEODESIC MAPPINGS, AND FIRST INTEGRALS IN RIEMANNIAN SPACES

BY

GERALD H. KATZIN AND JACK LEVINE (RALEIGH, NORTH CAROLINA)

1. Introduction. In this paper we give a survey of the recent work on the applications of Lie derivatives to symmetries, first integrals of geodesics, and geodesic mappings of a Riemannian space V_n. In addition several new results related to these topics are obtained.

It is generally recognized that the knowledge of conservation laws is of fundamental importance in the physical description of nature. It has also been observed that the existence of certain geometric symmetries definable in terms of Lie derivatives leads to conservation laws expressible in the form of first integrals (i.e., constants of the motion) of dynamical systems whose trajectories are geodesics in a Riemannian space V_n. This class of trajectories is of particular importance in the theory of general relativity in that it includes the description of the motion of mass-pole test particles and light rays.

Section 2 includes a list of the standard symmetries together with their geometric interpretations and interrelationships. In addition a new symmetry called a null geodesic collineation is introduced based on the idea of mapping null geodesics to geodesics. Various properties of this symmetry are described in several of the theorems of this section. A diagram is also included (Fig. 1) which summarizes the interrelationships of the symmetries discussed in this section.

Section 3 discusses the relations between symmetries and first integrals induced by them. New results include first integrals induced by conformal collineations and by null geodesic collineations.

Section 4 discusses the Related Integral Theorem for which we give a new simplified proof. In addition we present a new related integral theorem based upon the existence of a conformal motion and a quadratic first integral of null geodesics. This is stated as Theorem 4.1.

In Section 5 a summary of various properties of curvature collineations is given.

Throughout this paper when reference is made to "null geodesic" it is assumed the V_n is of indefinite form.

The expression "preservation of parameter", when referring to geodesic mappings, is understood to mean "preservation of geodesic parameter to within a linear transformation".

2. Symmetries expressed in terms of Lie derivatives. In this section we give a graphical presentation (Fig. 1) of the standard symmetries which show their interrelationships. Where available, geometric interpretations of the various symmetries are included along with their characterization in terms of Lie derivatives. In this presentation we introduce a new symmetry called semi-null geodesic collineation.

Projective Collineations (PC). Such a collineation is basically defined as a point transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta t, \tag{2.1}$$

where δt is an infinitesimal, which maps the geodesics of a V_n into geodesics of the V_n without necessarily preserving the geodesic parameter. A PC does not distinguish between null and non-null geodesics in the mapping and hence allows for mappings between geodesics of different types. Thus a PC may be considered as a general geodesic collineation.

To derive the conditions for PC [8] we use the parameter-independent form of the geodesic equation which may be expressed as

$$p^j(Dp^i/d\tau) - p^i(Dp^j/d\tau) = 0, \quad p^i \equiv dx^i/d\tau, \tag{2.2}$$

where τ is an arbitrary parameter and $D/d\tau$ denotes intrinsic differentiation along a geodesic. The PC requirement as stated above is expressible in the form

$$\pounds[p^j(Dp^i/d\tau) - p^i(Dp^j/d\tau)] = 0, \tag{2.3}$$

where $\pounds$ indicates Lie differentiation with respect to the vector ξ^i.

From the formal properties of Lie differentiation (cf. [24], p. 89) it follows that

$$\pounds p^i = -p^i(\pounds d\tau/d\tau) \tag{2.4}$$

and

$$\pounds(Dp^i/d\tau) = (\pounds\Gamma^i_{jk})p^j p^k - 2(Dp^i/d\tau)(\pounds d\tau/d\tau) - p^i D(\pounds d\tau/d\tau)/d\tau, \tag{2.4a}$$

where Γ^i_{jk} denotes the Christoffel symbol of the V_n. By use of this equation, it can be shown that (2.3) takes the form

$$(\delta^j_m \pounds\Gamma^i_{hk} - \delta^i_m \pounds\Gamma^j_{hk})p^k p^h p^m = 0. \tag{2.4b}$$

From (2.4b) there follows the well-known condition, for PC in V_n,

$$\pounds \Gamma^i_{jk} = \delta^i_j \varphi_{,k} + \delta^i_k \varphi_{,j}, \tag{2.5}$$

where gradient $\varphi_{,i}$ is defined by

$$\varphi_{,i} \equiv (n+1)^{-1} \pounds \Gamma^j_{ji} = (n+1)^{-1} \xi^j_{;ji} = (n+1)^{-1} (\xi^j_{;j})_{,i}, \tag{2.6}$$

and where comma (,) denotes partial differentiation and semicolon (;) denotes covariant differentiation. (An equivalent form to (2.5) is given by (3.5).)

It can be shown [8] that if the geodesic equation is written in parameter-dependent form

$$DP^i/du = 0, \quad P^i \equiv dx^i/du, \tag{2.7}$$

with u the geodesic-parameter, then the PC induces a change in u such that

$$\pounds du = (2\varphi + c)\, du, \quad c \equiv \text{constant}. \tag{2.8}$$

Affine Collineation (AC). An AC is a special case of a PC for which the geodesic parameter is preserved. As is well known the geodesic equations (2.7) remain unchanged in form under a linear change in parameter. Hence, from Yano [24], p. 90, we may write $\pounds du/du = \text{constant}$, which by (2.8) requires $\varphi = \text{constant}$, and hence by (2.5) gives the familiar condition for AC (cf. [24], p. 52, [11] and [4], p. 136),

$$\pounds \Gamma^k_{ij} \equiv \xi^k_{;ji} + \xi^m R^k_{jmi} \equiv \tfrac{1}{2} g^{ka} (h_{ai;j} + h_{aj;i} - h_{ij;a}) = 0, \tag{2.9}$$

where $h_{ij} \equiv \pounds g_{ij} = \xi_{i;j} + \xi_{j;i}$. An equivalent condition to (2.9) is

$$h_{ij;k} = 0. \tag{2.10}$$

Motions (M). A motion is an infinitesimal point transformation which preserves distance. This implies $\pounds ds = 0$, i.e.,

$$\pounds(ds^2) = \pounds(g_{ij}\, dx^i dx^j) = h_{ij}\, dx^i dx^j = 0. \tag{2.11}$$

This in turn implies the familiar Killing equation condition for a motion

$$h_{ij} \equiv \pounds g_{ij} = \xi_{i;j} + \xi_{j;i} = 0. \tag{2.12}$$

From (2.12) and (2.9) it follows immediately that every M is an AC.

Conformal Motion (Conf M). A Conf M is defined by ([24], p. 32) $\pounds G_{ij} = 0$, where $G_{ij} \equiv g^{-1/n} g_{ij}$, $g \equiv |g_{ij}|$. It is easily shown this is equivalent to the usual formulation

$$h_{ij} = 2\sigma g_{ij}, \quad \sigma = (1/n)\, \xi^i_{;i}. \tag{2.13}$$

It follows from (2.13) that for a Conf M

$$\pounds \Gamma^i_{jk} = \delta^i_j \sigma_{,k} + \delta^i_k \sigma_{,j} - g_{jk} g^{im} \sigma_{,m}. \tag{2.14}$$

It can be shown that the mapping defined by a Conf M prerveses angles between two directions at a point. We now derive an alternative characterization of Conf M in terms of mappings of null-geodesics. Consider, then, the infinitesimal transformation (2.1) which maps null-geodesics (NG) into NG. An NG may be defined as a geodesic satisfying (2.2) and

$$g_{ij}\, p^i p^j = 0. \tag{2.15}$$

To preserve the null character of the geodesics under (2.1) we must require that (2.3) hold and

$$\pounds (g_{ij} p^i p^j) = 0. \tag{2.16}$$

If (2.16) be expanded and use be made of (2.4) and (2.15), we obtain

$$(\pounds g_{ij}) p^i p^j = 0. \tag{2.17}$$

Since (2.17) must hold for all values of p^i which satisfy (2.15), it follows that $(\pounds g_{ij})p^i p^j = 2\sigma g_{ij} p^i p^j$ (for some scalar σ). This implies (2.13); thus a necessary condition that NG be mapped into NG by (2.1) is that the mapping vector ξ^i define a Conf M. Condition (2.13) for this type of mapping is also sufficient. This can be shown by expanding the left-hand side of (2.3) by making use of (2.4) and (2.4a). Then, by using (2.2), (2.14) and (2.15), it follows that (2.3) is satisfied identically. Next, if the left-hand side of (2.16) be expanded by use of (2.4), followed by use of (2.13) and (2.15), it is seen that (2.16) is satisfied identically. Hence we state

THEOREM 2.1. *A necessary and sufficient condition that the infinitesimal transformation* $\bar{x}^i = x^i + \xi^i \delta t$ *map null geodesics into null geodesics* (*with general change in parameter*) *is that the vector* ξ^i *define a* Conf M.

Thus we may consider a Conf M as a null-null geodesic collineation.

Homothetic Motion (HM). An HM is a special case of a Conf M in which $\sigma = \sigma_0 =$ constant. For an HM

$$\pounds(ds^2) = \pounds(g_{ij} dx^i dx^j) = (\pounds g_{ij}) dx^i dx^j = 2\sigma_0 g_{ij} dx^i dx^j = 2\sigma_0 ds^2.$$

This implies $\pounds ds = \sigma_0 ds$. Hence HM not only preserves angles but also scales all distances by the same constant factor.

An additional interpretation of HM relative to null geodesic mappings is stated in the theorem to follow:

THEOREM 2.2. *A necessary condition that the infinitesimal transformation* $\bar{x}^i = x^i + \xi^i \delta t$ *map null geodesics into null geodesics with assumed preservation of parameter is that the vector* ξ^i *define an* HM. *A sufficient* *condition that null geodesics map into null geodesics is that the vector ξ^i define an* HM. *In this case the parameter will be preserved.*

Proof. Necessity. Consider a mapping of NG of the type stated in Theorem 2.2. We show this must be an HM. We take ([20], p. 46) the equations of NG in the form

$$DP^i/du = 0, \tag{2.18a}$$

$$g_{ij}P^iP^j = 0. \tag{2.18b}$$

Hence, requiring

$$\pounds(DP^i/du) = 0, \tag{2.19a}$$

$$\pounds(g_{ij}P^iP^j) = 0, \tag{2.19b}$$

we find, by use of (2.4a), (2.19a), (2.19b) and the condition $(\pounds du)/du = \text{constant}$, that

$$(\pounds\Gamma^i_{jk})P^jP^k = 0, \tag{2.20a}$$

$$(\pounds g_{ij})P^iP^j = 0 \tag{2.20b}$$

must hold as a consequence of (2.18b). This implies that we must have

$$\pounds\Gamma^i_{jk} = \psi^i g_{jk}, \tag{2.21a}$$

$$\pounds g_{ij} = 2\sigma g_{ij}, \tag{2.21b}$$

where ψ^i is a contravariant vector, and σ is a scalar.

It is now shown that (2.21a) and (2.21b) imply $\sigma = \text{constant}$; thus showing the mapping is an HM. If in (2.21a) we contract on i and j and use the first part of (2.9), we obtain

$$\xi^i_{;ik} = (\xi^i_{;i})_{,k} = \psi_k \equiv \psi_{,k} \qquad (\psi_k \equiv g_{jk}\psi^j). \tag{2.22}$$

Since (2.21b) implies (2.14), we infer from (2.14), (2.21a) and (2.22) that

$$\delta^i_j\sigma_{,k} + \delta^i_k\sigma_{,j} - g_{jk}g^{im}\sigma_{,m} = g_{jk}g^{im}\psi_{,m}, \tag{2.23}$$

from which we easily find

$$n\sigma_{,k} = \psi_{,k}. \tag{2.24}$$

Next, in (2.23) we multiply by $g^{jk}g_{im}$ and sum on i, j and k to obtain

$$(2-n)\sigma_{,k} = n\psi_{,k}. \tag{2.25}$$

Equations (2.24) and (2.25) imply $\sigma_{,i} = 0$, i.e., $\sigma = \sigma_0 = \text{constant}$ and $\psi^i = 0$; hence the mapping is an HM as follows from (2.21b). In addition, (2.21a) reduces to $\pounds\Gamma^i_{jk} = 0$, which is satisfied identically as a consequence of the HM. This completes the necessity proof.

Sufficiency. Since an HM is a Conf M, by Theorem 2.1 a sufficient condition that NG map into NG (with possible general change in parameter) is that the mapping be an HM. It remains to show that the change in parameter is at most linear. Hence assume that (2.19a) holds and the mapping is HM. This implies, by (2.4a) and (2.18a), that $(\pounds du)/du =$ constant (since $\pounds \Gamma^i_{jk} = 0$, because of HM).

Conformal Collineation (Conf C). A space V_n is said to admit a Conf C [21] if it admits a vector ξ^i which satisfies (2.14) (or equivalently (3.9)). From the remarks following (2.13) it follows that all Conf M are Conf C but not conversely. A Conf C is said to be *proper* if it is not a Conf M.

We now show that a sufficient condition for transformation (2.1) to map NG into non-null geodesics (non-NG) is that the mapping be a proper Conf C. To show this we use (2.14) to substitute for $\pounds \Gamma^i_{jk}$ in (2.4b), and also make use of (2.15). This results in the left-hand side of (2.4b) vanishing identically. This implies that NG map into geodesics. Since the Conf C is proper (i.e., not a Conf M), the NG must be mapped into non-NG as a consequence of Theorem 2.1([1]). Hence we can state

THEOREM 2.3. *A sufficient condition that* NG *be mapped into non*-NG *by the point transformation* $\bar{x}^i = x^i + \xi^i \delta t$ *is that the vector* ξ^i *define a proper* Conf C([1]). *In general, the geodesic parameter will not be preserved by this mapping.*

The last sentence of this theorem follows from the fact that the geodesic equations were taken in parameter-independent form.

Null Geodesic Collineations (NC). We consider now the general problem of mapping null geodesics into geodesics with preservation of parameter. Consider then the null geodesic equations expressed in form (2.18). It follows from (2.19a), by use of (2.18a), (2.4a) and the requirement $(\pounds du)/du =$ constant, that (2.20a) must hold as a consequence of (2.18b). Hence we must obtain (2.21a) as a necessary condition that NG map into geodesics (null or non-null). Hence we may state

THEOREM 2.4. *A necessary condition that the transformation* $\bar{x}^i = x^i + \xi^i \delta t$ *map* NG *into geodesics with* (*assumed*) *preservation of parameter is that* $\pounds_\xi \Gamma^i_{jk} = g^{im} g_{jk} \psi_{,m}$. *Such a mapping will be called a null geodesic collineation* (NC).

If we assume (2.21a) holds and we take the equations of NG in parameter independent form (2.2) and (2.15), it then follows that (2.3) is identically satisfied if use be made of (2.4), (2.4a), (2.15) and (2.21a). This implies that (2.21a) is a sufficient condition to map NG into geodesics. Hence, if the equations of NG be taken in parameter dependent form (2.18a) and (2.18b), it follows that (2.19a) is satisfied if (2.21a) is assumed.

([1]) Possible exceptions may occur for NG along which (2.21b) is satisfied.

Thus the left-hand side of (2.4a) (in which we consider $p^i \to P^i$ and $\tau \to u$) is zero. The first two terms of the right-hand side of (2.4a) will also be zero, by reason of (2.21a), (2.18a) and (2.18b). This implies that $\pounds du/du$ = constant under this mapping of NG to geodesics determined by (2.21a). Thus we have

THEOREM 2.5. *A sufficient condition that* NG *map into geodesics is that* (2.21a) *hold. In this mapping the parameter will be preserved.*

Semi-null Geodesic Collineation (Semi-NC). We continue the study of the symmetry defined by (2.21a) and consider the two cases $\psi^i = 0$ and $\psi^i \neq 0$. In the case $\psi^i = 0$, the above-stated condition (2.21a) reduces to an AC, and hence will not be considered further here. We thus are led to the case of main interest, $\psi^i \neq 0$. For this case we shall show that (2.21a) (with $\psi^i \neq 0$) is a sufficient condition that NG be mapped into non-NG with preservation of parameter.

By Theorem 2.5 the mapping (2.21a) takes NG into geodesics with preservation of parameter. We now show that if $\psi^i \neq 0$, (2.21a) cannot define a Conf M and hence, by Theorem 2.1, the mapping (2.21a) ($\psi^i \neq 0$) takes NG into non-NG [1]. To show this we assume (2.21a) and (2.21b) hold with $\psi^i \neq 0$. It then follows from the discussion after (2.22) that we obtain the contradiction $\psi^i = 0$. Hence we may state

THEOREM 2.6. *A sufficient condition that* $\bar{x}^i = x^i + \xi^i \delta t$ *map* NG *into non*-NG *is that* [1]

$$\pounds \Gamma^i_{jk} = \psi^i g_{jk}, \quad \psi^i \neq 0. \tag{2.26}$$

In this case the parameter will be preserved. We refer to the symmetry defined by (2.26) *as a semi-null geodesic collineation.*

An equivalent form for (2.21a) is easily obtained. By use of (2.9) and (2.22), we may express (2.21a) in the covariant form

$$h_{ij;k} + h_{ik;j} - h_{jk;i} = 2g_{jk}\psi_{,i}. \tag{2.27}$$

If i and j be interchanged in (2.27) and the resulting equation be added to (2.27), we obtain

$$h_{ij;k} = g_{jk}\psi_{,i} + g_{ik}\psi_{,j}, \tag{2.28}$$

which can easily be shown to be equivalent to (2.21a).

The block diagram, shown in Fig. 1, summarizes relationships between symmetries (including some not discussed in this paper). The diagram should be read in the following sense. When it exists (i.e., is admitted by the V_n), the symmetry described in any given block is automatically a subcase of the symmetries described in those adjacent blocks indicated by the arrows leading from the given block. Thus, for example, if an HM is admitted by the V_n, then the transformation which defines the

HM also satisfies the requirements for being a Conf M and AC and so on through the diagram.

Note that the dashed arrows should only be considered when the Ricci tensor vanishes. The dash-dot arrow applies under conditions described in Section 5 under the heading Ricci Flat Spaces.

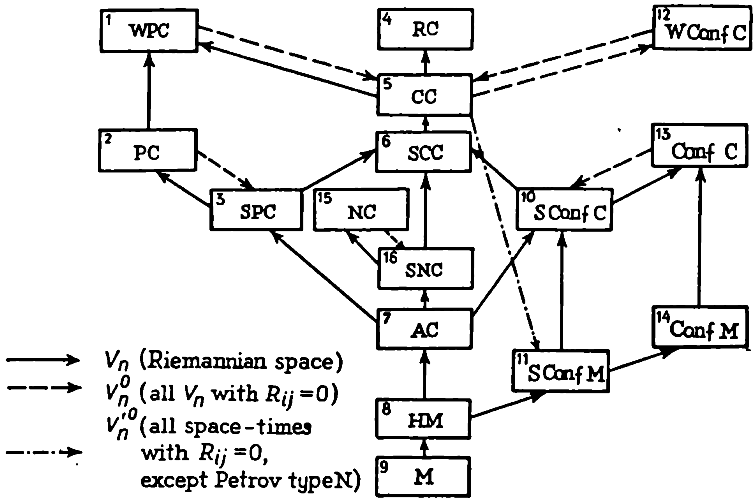

Fig. 1. Relations between symmetries

1. WPC — Weyl Projective Collineation: $\pounds W^i_{jkl} = 0$ $(n > 2)$.
2. PC — Projective Collineation: $\pounds \Gamma^i_{jk} = \delta^i_j \varphi_{;k} + \delta^i_k \varphi_{;j}$.
3. SPC — Special Projective Collineation: $\pounds \Gamma^i_{jk} = \delta^i_j \varphi_{;k} + \delta^i_k \varphi_{;j}$, $\varphi_{;jk} = 0$.
4. RC — Ricci Collineation: $\pounds R_{ij} = 0$.
5. CC — Curvature Collineation: $\pounds R^i_{jkl} = 0$.
6. SCC — Special Curvature Collineation: $(\pounds \Gamma^i_{jk})_{;l} = 0$.
7. AC — Affine Collineation: $\pounds \Gamma^i_{jk} = 0$.
8. HM — Homothetic Motion: $\pounds g_{ij} = 2\sigma g_{ij}$, $\sigma =$ constant.
9. M — Motions: $\pounds g_{ij} = 0$.
10. S Conf C — Special Conformal Collineation: $\pounds \Gamma^i_{jk} = \delta^i_j \sigma_{;k} + \delta^i_k \sigma_{;j} - g_{jk} g^{il} \sigma_{;l}$, $\sigma_{;jk} = 0$.
11. S Conf M — Special Conformal Motions: $\pounds g_{ij} = 2\sigma g_{ij}$, $\sigma_{;jk} = 0$.
12. W Conf C — Weyl Conformal Collineation: $\pounds C^i_{jkl} = 0$ $(n > 3)$.
13. Conf C — Conformal Collineation: $\pounds \Gamma^i_{jk} = \delta^i_j \sigma_{;k} + \delta^i_k \sigma_{;j} - g_{jk} g^{il} \sigma_{;l}$.
14. Conf M — Conformal Motion: $\pounds g_{ij} = 2\sigma g_{ij}$.
15. NC — Null Geodesic Collineation: $\pounds \Gamma^i_{jk} = g_{jk} g^{im} \psi_{;m}$.
16. SNC — Special Null Geodesic Collineation: $\pounds \Gamma^i_{jk} = g_{jk} g^{im} \psi_{;m}$, $\psi_{;jk} = 0$.

This diagram corrects an error made in the previous diagram (cf. [11], Fig. 1) by removing the solid line which connected block 13 to block 12. In addition, two new symmetry blocks (blocks 15 and 16) have been introduced as well as several new connecting lines.

3. Relations between symmetries and first integrals of geodesics. In this section we show how V_n which admit certain types of symmetries also admit first integrals of the geodesics. A necessary and sufficient condition that the geodesics (2.7) admit the mth order first integral (mFI)

$$A_{i_1 \cdots i_m} P^{i_1} \cdots P^{i_m} = \text{constant} \tag{3.1}$$

is (cf. [9])

$$P[A_{i_1 \cdots i_m; i_{m+1}}] = 0, \tag{3.2}$$

where the symbol $P[\ ]$ indicates the sum of the terms obtained by forming all cyclic permutations of the indices that are not summed completely within the brackets.

Motions. For the case of $m = 1$, equations (2.12) and (3.2) show that

$$\xi_i P^i = \text{constant} \tag{3.3}$$

is a *linear first integral* (LFI) of the geodesics.

Homothetic Motion (HM). From (2.13) we obtain the condition for HM by taking $\sigma = \sigma_0 = \text{constant} \ (\neq 0)$. This implies (2.10), a special case of (3.2). Hence $h_{ij} P^i P^j = \sigma_0 g_{ij} P^i P^j = \text{constant}$ is a *quadratic first integral* (QFI) of the geodesics. However, this is linearly dependent on the metric QFI and hence gives nothing new.

Affine Collineation (AC). If a V_n admits an AC, then (2.10) holds, which implies that

$$h_{ij} P^i P^j = (\pounds g_{ij}) P^i P^j = (\xi_{i;j} + \xi_{j;i}) P^i P^j = \text{constant} \tag{3.4}$$

is a QFI of the geodesics.

Projective Collineation (PC). If a V_n admits a PC, then (2.5) may be expressed equivalently as (cf. [4], p. 135)

$$h_{ij;k} = 2 g_{ij} \varphi_{,k} + g_{jk} \varphi_{,i} + g_{ik} \varphi_{,j}. \tag{3.5}$$

Cyclic permutations of the indices in (3.5) and addition of the resulting equations gives

$$P[(h_{ij} - 4\varphi g_{ij})_{;k}] = 0 \tag{3.6}$$

(cf. [3]), which implies the existence of a QFI of the form

$$(h_{ij} - 4\varphi g_{ij}) P^i P^j = \text{constant}. \tag{3.7}$$

(As will be shown in the next section, (3.7) may be derived by an alternate method based on a geometric approach.)

Conformal Collineation (Conf C). From condition (2.14) for the existence of a Conf C we write (cf. [24], p. 52) equivalently

$$\tfrac{1}{2} g^{im} [h_{mj;k} + h_{mk;j} - h_{jk;m}] = \delta^i_j \sigma_{,k} + \delta^i_k \sigma_{,j} - g_{jk} g^{im} \sigma_{,m}. \tag{3.8}$$

From (3.8) we easily obtain the equivalent form for a Conf C:

$$h_{ij;k} = 2\sigma_{,k} g_{ij}. \tag{3.9}$$

From (3.9) we find

$$(h_{ij} - 2\sigma g_{ij})_{;k} = 0. \tag{3.10}$$

Hence, in the case of a proper Conf C, i.e., $h_{ij} \neq 2\sigma g_{ij}$, we obtain a QFI of the geodesics of the form

$$(h_{ij} - 2\sigma g_{ij}) P^i P^j = \text{constant}. \tag{3.11}$$

Semi-null geodesic collineation. From equation (2.28) ($\psi_{,i} \neq 0$) defining this type of symmetry, we obtain, by cyclic permutations of the indices,

$$P[(h_{ij} - 2\psi g_{ij})_{;k}] = 0. \tag{3.12}$$

Hence, if a V_n admits a Semi-NC, it follows that the geodesics admit the QFI,

$$(h_{ij} - 2\psi g_{ij}) P^i P^j = \text{constant}. \tag{3.13}$$

(From the discussion preceding the statement of Theorem 2.6 it follows that in (3.13) $h_{ij} - 2\psi g_{ij} \neq 0$, since we are assuming $\psi^i \neq 0$.)

Conformal Motion (Conf M). The existence of such a symmetry in a V_n implies that the null geodesics admit an LFI,

$$\xi_i P^i = \text{constant}. \tag{3.14}$$

This can be easily shown as follows. Assume first that (3.14) defines an LFI of the null geodesics (2.18a) and (2.18b). Hence, from $D(\xi_i P^i)/du = 0$ we obtain, using (2.18a),

$$\xi_{i;j} P^i P^j = 0, \tag{3.15}$$

which is to hold as a consequence of (2.18b). This implies the necessary condition

$$\xi_{i;j} + \xi_{j;i} = 2\sigma g_{ij} \tag{3.16}$$

(for some scalar σ), which defines a Conf M. Conversely, from (3.16) it is easily shown that (3.14) holds along null geodesics. We may thus state

THEOREM 3.1. *A necessary and sufficient condition that null geodesics admit a linear first integral* $\xi_i P^i = \text{constant}$ *is that the vector* ξ^i *define a* Conf M, *i.e.,* $\xi_{i;j} + \xi_{j;i} = 2\sigma g_{ij}$. *In case* $\sigma = 0$, *the* Conf M *reduces to an* M *and all geodesics will admit this* LFI.

Special Curvature Collineations (SCC). (See Section 5 for a discussion of curvature collineations.) The condition for an SCC is

$$h_{ij;km} = 0. \tag{3.7}$$

Hence, if a V_n admits an SCC, the geodesics admit a cubic first integral of the form

$$h_{ij;k}P^iP^jP^k = \text{constant}. \tag{3.18}$$

For general discussion of the enumeration problem relating to the number of linearly independent first integrals of the geodesics in a V_n, the following references may be consulted: [5]-[7], [15], [16], [18] and [22].

4. Related integral theorems. As mentioned above in Section 3, Davis and Moss [3] found that the existence of a PC (including AC) in a V_n implied the existence of a QFI. In a later paper [9], we extended and unified this method for obtaining first integrals of the geodesics based upon PC. This extension we called the *Related Integral Theorem* (RIT). In the present paper, we further extend this method of obtaining first integrals by using Conf M symmetries. In this extension, the first integrals will however be restricted to null geodesics only.

Derivation of RIT *based on* PC. Assume a V_n admits an mFI (3.1) and also a PC. As a result of the PC mapping the mFI (3.1) is "dragged along" in the sense of Schouten [19], p. 102, as geodesics map into geodesics. The mFI is deformed by this process and takes the form

$$\bar{A}_{i_1\ldots i_m}\bar{P}^{i_1}\ldots\bar{P}^{i_m} = A_{i_1\ldots i_m}P^{i_1}\ldots P^{i_m}+\pounds(A_{i_1\ldots i_m}P^{i_1}\ldots P^{i_m})\,\delta t = \bar{k}, \tag{4.1}$$

where $\bar{P}^i \equiv dx^i/d\bar{u}$ and

$$\bar{k} \equiv k+(\pounds k)\,\delta t. \tag{4.2}$$

It follows by (3.1), (4.1) and (4.2) that the deformed mFI (4.1) can be expressed as

$$\pounds(A_{i_1\ldots i_m}P^{i_1}\ldots P^{i_m}) = \pounds k. \tag{4.3}$$

If (4.3) be expanded and use be made of (2.4), (2.8) and the definition

$$\pounds du \equiv \lim_{\delta t\to 0}(d\bar{u}-du)/\delta t,$$

we find that

$$[(\pounds A_{i_1\ldots i_m})-2m\varphi A_{i_1\ldots i_m}]P^{i_1}\ldots P^{i_m} = cmk+\pounds k \equiv k'. \tag{4.4}$$

Since $(\pounds k)\,\delta t$ represents the change in k induced by an infinitesimal point transformation of the form (2.1), we may consider $\pounds k$ expressible in the form $\pounds k = f(k)$, and hence $\pounds k$ may be considered as a constant along a geodesic. Thus, the right-hand side of (4.4) is constant along a geodesic, and hence the left-hand side of (4.4) defines an mFI of the geodesics. This new integral is generically related to the original mFI (3.1), and may be considered as being induced by the PC.

Simplified alternate proof. In [9] an alternate derivation of RIT was given. We here give a simplified version of this alternate proof which is essentially a direct verification that the left-hand side of (4.4) satisfies (3.2). Thus we show that

$$I \equiv P[(\pounds A_{i_1\ldots i_m} - 2m\varphi A_{i_1\ldots i_m})_{;i_{m+1}}] = 0. \tag{4.5}$$

By expanding the covariant derivative in the left-hand side of (4.5) and then making use of (3.2), we obtain

$$I = P[(\pounds A_{i_1\ldots i_m})_{;i_{m+1}} - 2m\varphi A_{i_1\ldots i_m}], \qquad \varphi_i \equiv \varphi_{;i}. \tag{4.6}$$

From Yano [24], p. 16, we have

$$(\pounds A_{i_1\ldots i_m})_{;i_{m+1}} = \pounds(A_{i_1\ldots i_m;i_{m+1}}) + (\pounds\Gamma^j_{i_1 i_{m+1}})A_{j i_2\ldots i_m} + \\ + \ldots + (\pounds\Gamma^j_{i_m i_{m+1}})A_{i_1\ldots i_{m-1}j}. \tag{4.7}$$

By replacing the $\pounds\Gamma$ terms in (4.7) by expressions of the form (2.5), we obtain

$$(\pounds A_{i_1\ldots i_m})_{;i_{m+1}} = \pounds(A_{i_1\ldots i_m;i_{m+1}}) + m\varphi_{i_{m+1}}A_{i_1\ldots i_m} + \varphi_{i_1}A_{i_{m+1}i_2\ldots i_m} + \\ + \ldots + \varphi_{i_m}A_{i_1\ldots i_{m-1}i_{m+1}}. \tag{4.8}$$

By substitution of (4.8) into (4.6) if follows that $I = 0$, thus completing the proof.

Further details concerning properties of derived integrals including dependency relations resulting from the group structure of the underlying symmetry group are given in [9].

In the special case $m = 2$, with $A_{ij} = g_{ij}$, the first derived integrals turn out to be those previously obtained by direct construction by Davis and Moss [3], and are given by (3.7).

It is of interest to compare RIT with the well-known Poisson theorem of constants of motion which generates a first integral of order $m+q-1$ from two given integrals of order m and q, respectively. This comparison can be made for the case $q = 1$. In this case, it is shown (cf. [9], p. 13) that the derived first integrals obtained by both methods are the same provided the RIT was based upon a motion as the underlying symmetry. Other than this choice of $q = 1$ and a motion, the two methods are in general distinct for determining derived integrals.

An RIT *for null geodesics based on* Conf M. Since Conf M map null geodesics into null geodesics, this suggests the possibility of deriving an RIT dependent on this property. We give below the the outline of the proof of such a new theorem based on a Conf M and the existence of a QFI ($m = 2$) of null geodesics.

Consider then equations (2.18a) and (2.18b) assumed to admit the QFI

$$A_{ij}P^iP^j = \text{constant}. \tag{4.9}$$

We first derive the conditions that (4.9) be a QFI of null geodesics. Equation (4.9) implies the condition

$$A_{ij;k}P^iP^jP^k = 0 \tag{4.10}$$

must hold as a consequence of (2.18b). This implies

$$A_{ij;k}P^iP^jP^k = \varrho g_{ij}P^iP^j, \tag{4.11}$$

where scalar $\varrho \equiv -\mu_i P^i$ (μ_i is some covariant vector). Hence we must have

$$P[A_{ij;k} + \mu_i g_{jk}] = 0 \tag{4.12}$$

as a required condition for (4.9) to be a QFI of null geodesics. This condition is easily shown to be sufficient (see also [23], p. 83). It follows from (4.12) that

$$\mu_k = -(n+2)^{-1}(A^m_{m;k} + 2A^m_{k;m}), \quad A^i_j \equiv g^{ik}A_{kj}. \tag{4.13}$$

In a manner similar to the derivation of (4.4), we obtain from the assumed QFI (4.9) of the null geodesics the derived first integral

$$(\pounds A_{ij} - 4\sigma A_{ij})P^iP^j = \text{constant}, \tag{4.14}$$

where σ is defined in (2.13) (the conditions of Conf M), and where $\pounds$ is based on the Conf M vector ξ^i.

We give now a sketch of a direct verification that (4.14) is indeed a first integral of the null geodesics.

In order for the left-hand side of (4.14) to define a QFI of null geodesics it is necessary (and sufficient) that it satisfy a condition of the form (4.12), i.e., we must show that

$$Q_{ijk} \equiv P[B_{ij;k} + \lambda_k g_{ij}] = 0, \tag{4.15}$$

where

$$B_{ij} \equiv \pounds A_{ij} - 4\sigma A_{ij}, \tag{4.16}$$

and

$$\lambda_k = -(n+2)^{-1}(B^m_{m;k} + 2B^m_{k;m}). \tag{4.17}$$

We first evaluate λ_k of (4.17), by use of (4.16), (2.14), (4.12) and (4.13), and obtain after a lengthy calculation

$$\lambda_k = \pounds\mu_k - 2\sigma\mu_k + 2A^m_k \sigma_{,m}. \tag{4.18}$$

It then follows by use of (4.18) that Q_{ijk} can be shown to be identically zero. Hence we may state

THEOREM 4.1. *If a V_n admits a* Conf M, $\pounds g_{ij} = 2\sigma g_{ij}$, *and if the null geodesics* (2.18a) *and* (2.18b) *of the V_n admit a* QFI $A_{ij}P^iP^j$ = constant, *then the null geodesics admit a derived* QFI $(\pounds A_{ij} - 4\sigma A_{ij})P^iP^j$ = constant, *where Lie differentiation is formed with respect to the vector ξ^i defined by the* Conf M.

A more detailed proof of this new related integral theorem based upon a given mFI of null geodesics will be published elsewhere.

5. Curvature Collineations (CC). The curvature collineation symmetry which was recently introduced [11] is defined by the condition that the V_n admit a vector ξ^i such that

$$\pounds R^i_{jkm} = 0. \tag{5.1}$$

The investigation of this symmetry was strongly motivated by the all-important role of the Riemannian curvature tensor in the theory of general relativity. As shown in Fig. 1, several of the previously known symmetries are special cases of CC and others are closely related to CC.

It was shown in Section 3 that the existence of a certain type of CC, namely *Special Curvature Collineations* (SCC), implies the V_n admits a cubic first integral of the geodesics. In addition, it was also found [11] that if a V_4 with non-vanishing Ricci tensor R_{ij} and with vanishing scalar curvature R admits a CC, then a field conservation law results in the theory of general relativity. Furthermore, it turns out that the identity of Komar [14], which serves as a covariant generator of field conservation laws in the theory of general relativity appears in a natural manner as an essentially trivial necessary condition for existence of CC in a V_n.

By expansion of (5.1) we obtain

$$\pounds R^k_{jhi} = (\pounds \Gamma^k_{ij})_{;h} - (\pounds \Gamma^k_{hj})_{;i} = 0. \tag{5.2}$$

From (2.9) $\pounds \Gamma^k_{ij}$ may be expressed in terms of $h_{ij} \equiv \pounds g_{ij}$ so that (5.1) may then be expressed in the equivalent form

$$(h_{im;j} + h_{mj;i} - h_{ij;m})_{;k} - (h_{km;j} + h_{mj;k} - h_{kj;m})_{;i} = 0. \tag{5.3}$$

A necessary condition for (5.3) to hold may be obtained from (5.3) and is given by

$$h_{jm;ik} - h_{jm;ki} = 0. \tag{5.4}$$

The necessary condition (5.4) for a CC leads directly to the Komar identity (mentioned above) by multiplication of (5.4), by $g^{1/2}g^{jk}g^{mi}$ ($g \equiv |g_{ij}|$), to obtain

$$[g^{1/2}(\xi^{i;j} - \xi^{j;i})]_{;ji} \equiv \{[g^{1/2}(\xi^{i;j} - \xi^{j;i})]_{;j}\}_{,i} \equiv 0.$$

For the special case of a CC for which

$$(\pounds \Gamma^k_{ij})_{;h} = 0, \tag{5.5}$$

the CC is the special CC referred to above. It easily follows that condition (5.5) for an SCC (which is also sufficient) may be expressed in the equivalent form

$$h_{im;jk} = 0. \tag{5.6}$$

It is of interest to note that (5.1) implies $\pounds R_{ij} = 0$, which is called a *Ricci Collineation* (RC).

We now consider CC in special spaces.

Ricci flat spaces (V^0_n). Such spaces are defined by $R_{ij} = 0$. It is shown in [11] that in a V^0_n every PC and Conf C is a CC. However, the proof of Theorem 4.2 given in [11] that every Conf C is a CC in a V^0_n is incorrect. We give a corrected proof. Thus we assume a V^0_n admits a Conf C. It then follows from (2.14) and the first part of (5.2) that

$$\begin{aligned}\pounds R^i_{kmj} &= (\pounds \Gamma^i_{jk})_{;m} - (\pounds \Gamma^i_{mk})_{;j} \\ &= \delta^i_j \sigma_{;km} + \delta^i_k \sigma_{;jm} - g^{ia} g_{jk} \sigma_{;am} - \delta^i_m \sigma_{;kj} - \delta^i_k \sigma_{;mj} + g^{ia} g_{mk} \sigma_{;aj}.\end{aligned} \tag{5.7}$$

In (5.7), if we contract on i and j and use the fact that $R_{ij} = 0$, we obtain

$$(n-2)\sigma_{;km} + g_{km} g^{ia} \sigma_{;ia} = 0. \tag{5.8}$$

This implies $\sigma_{;ij} = 0$. It now follows from (5.7) that $\pounds R^i_{kmj} = 0$, and hence in a V^0_n every Conf C is a CC.

It now follows, since, in a V^0_n ($n > 3$), $C^i_{jkm} = R^i_{jkm}$, where C^i_{jkm} is the conformal curvature tensor, that every vector ξ^i which defines a Conf C (in a V^0_n) will also satisfy $\pounds C^i_{jkm} = 0$ in such a space. This corrects an erroneous statement made in [11], (1.15), and the line immediately above this equation.

Collinson [2] outlines the proof of a theorem stating that the only CC admitted by a V^0_4, not of Petrov type N, are Conf M. Based on this result and [11], Corollary 4.3, it can be shown that such a Conf M must satisfy $\sigma_{;jk} = 0$ and hence is a special Conf M (see Fig. 1). This result allows us to connect symmetry block 5 to block 11 by a dash-dot arrow.

Einstein space (E_n). An E_n (defined by $R_{ij} = (R/n)g_{ij}$) has the property that every RC and every CC ($R \neq 0$, $n > 2$) is a motion. Further results of this nature may be found in [11], Section 4.

Flat spaces (S_n). The necessary and sufficient conditions (5.3) for a space to admit a CC are found to be satisfied by any arbitrary vector ξ^i if the space is an S_n. However, condition (5.6) for an SCC is not identically satisfied for an S_n. It follows [10] that for an S_n with fundamental form

$$\Phi = \sum_i e_i (dx^i)^2, \quad e_i = \pm 1,$$

the most general SCC vector is given by

$$\xi^i = a^i_{jm} x^j x^m + b^i_j x^j + c^i, \tag{5.9}$$

where a^i_{jm}, b^i_j and c^i $(a^i_{jm} = a^i_{mj})$ are $N = n(n+1)(n+2)/2$ arbitrary constants. This vector is expressible in the form

$$\xi^i = e_j (a^j_{km} S^{(k)} S^{(m)} + b^j_k S^{(k)} + c^j) \lambda^{(j)i},$$

where $\lambda^{(j)i} = e_i \delta^j_i$ are a set of n parallel vector fields admitted by the S_n, and where the n scalars $S^{(j)} \equiv x^j$.

The choices

$$a^i_{jm} \equiv (\delta^i_j a_m + \delta^i_m a_j)/2 \quad \text{and} \quad a^i_{jm} \equiv (A_j \delta^i_m + A_m \delta^i_m)/2 - e_i A_i \delta_{jm} (e_j + e_m)/4$$

define, respectively, the $n(n+2)$-parameter group of PC and $n(n+2)$-parameter group of Conf C admitted by the S_n, where a_i and A_i are arbitrary constants.

Particular choices of the constants in the general SCC vector (5.9) define a restricted class of SCC vectors ξ'^i which leave invariant the electromagnetic field tensor F_{ij} for a plane wave in that $\pounds_{\xi'} F_{ij} = 0$ (cf. [10]).

Conformally flat spaces (C_n). The problem of finding CC in C_n conveniently divides into two cases: (1) CC $\neq$ Conf M [12], (2) CC $=$ Conf M [17], where in each case E_n spaces are excluded in that CC in E_n (as discussed above) have been treated [11]. A C_n space $(\neq E_n)$ which admits a CC $\neq$ Conf M is called a C'_n. In a C'_n the vector ξ^i, defining a CC of this type, must satisfy $\pounds g_{ij} = \varrho R_{ij} + 2\varphi g_{ij}$, where ϱ and φ are scalars, with $\varrho \neq 0$, and, in addition, $R^i_a R^a_j - (n-1)^{-1} R R^i_j = 0$, $R^2 = (n-1) R^i_a R^a_i$, and $R_{ij;km} - R_{ij;mk} = 0$.

An example of such a C'_n is the reducible space $K_1 \times K_{n-1}$, where K_r denotes a space of constant curvature of r dimensions [12].

For the second case, in which CC $=$ Conf M, canonical forms of such spaces are given in [17] together with the complete groups of such symmetries.

V_n with parallel vector fields [13]. Assume a V_n admits a parallel vector field λ_i $(\lambda_{i;j} = 0)$, $\lambda_i \equiv S_{,i}$, where S is a non-constant scalar. Let $F(S)$ be an arbitrary function of S which is four times differentiable. If

$$\xi_i \equiv F_{,i} \equiv F' \lambda_i \quad (F' \equiv dF/dS), \tag{5.10}$$

then we find from definition (2.12) of h_{ij} that

(5.11)

(a) $h_{ij} = 2F''\lambda_i\lambda_j$; (b) $h_{ij;k} = 2F'''\lambda_i\lambda_j\lambda_k$; (c) $h_{ij;km} = 2F''''\lambda_i\lambda_j\lambda_k\lambda_m$.

It immediately follows from (5.10) and (5.11) and the definitions of CC, SCC, AC and M that a V_n which admits a parallel vector field $\lambda_i = S_{,i}$ admits (a) a Prop SCC (SCC $\neq$ AC): $\xi^{(2)i} \equiv S^2\lambda^i$; (b) Prop AC ($AC \neq M$): $\xi^{(1)i} \equiv S\lambda^i$; and (c) an M: $\xi^{(0)i} \equiv \lambda^i$. The generators $X_a \equiv \xi^{(a)i}\partial/\partial x^i$ define a 3-parameter group of Prop SCC, which is Abelian if λ^i is a null vector ($\Lambda \equiv \lambda_i\lambda^i = 0$). The V_n also admits a Prop CC (CC $\neq$ SCC) defined in terms of a general $F(S)$, where $F'''' \neq 0$ with $\xi^i \equiv \equiv F'(S)\lambda^i$. In this case if $\Lambda = 0$, the V_n will admit an r-parameter Abelian group of Prop CC defined by the r vectors $\xi^{(a)i} = F_{(a)}\lambda^i$, where $F_{(a)}(S)$, $(F''''_{(a)}(S) \neq 0)$, $a = 1, 2, \ldots, r$, is any given set of r linearly independent but otherwise arbitrary scalar functions for any $r = 1, 2, \ldots$

It has been shown [13] that every V_4 space-time which admits an expansion-free, shear-free, rotation-free geodesic congruence of curves defined with respect to the vector field u^i admits a group of CC's as described above, where the parallel vector field λ^i is taken to be u^i. Since plane-fronted pure gravitational wave space-times admit a parallel vector field defined by the wave propagation vector, it follows that such space-times admit groups of CC's as described above. For a complete analysis of such space-times refer to [1].

Added in proof. In Fig. 1 a solid line arrow should go from Block 14 to Block 12.

REFERENCES

[1] P. C. Aichelburg, *Curvature collineations for gravitational pp waves*, Journal of Mathematical Physics 11 (1970), p. 2458-2459.

[2] C. D. Collinson, *Curvature collineations in empty space-times*, ibidem 11 (1970), p. 818-819.

[3] W. R. Davis and M. K. Moss, *Conservation laws in general relativity, II. Space-times admitting certain symmetry properties more general than motions*, Nuovo Cimento 38 (1965), p. 1558-1569.

[4] L. P. Eisenhart, *Non-Riemannian geometry*, American Mathematical Society Colloquium Publications 8 (1927).

[5] G. H. Katzin and J. Levine, *Quadratic first integrals of the geodesics in spaces of constant curvature*, Tensor, New Series, 16 (1965), p. 97-104.

[6] — *Cubic first integrals of the geodesics in spaces of constant curvature*, ibidem 18 (1967), p. 32-39.

[7] — *Note on the number of linearly independent m^{th}-order first integrals in space of constant curvature*, ibidem 19 (1968), p. 42-44.

[8] — *Note on the equations of projective collineation*, ibidem 19 (1968), p. 162-164.

[9] — *Related first integral theorem: A method for obtaining conservation laws of dynamical systems with geodesic trajectories in Riemannian spaces admitting symmetries*, Journal of Mathematical Physics 9 (1968), p. 8-15.
[10] — *Special curvature collineations in a flat space*, Tensor, New Series, 22 (1971), p. 64-68.
[11] — and W. R. Davis, *Curvature collineations: A fundamental symmetry property of space-times of general relativity defined by the vanishing Lie derivative of the Riemann curvature tensor*, Journal of Mathematical Physics 10 (1969), p. 617-629.
[12] — *Curvature collineations in conformally flat spaces, I*, Tensor, New Series, 21 (1970), p. 51-61.
[13] — *Groups of curvature collineations in Riemannian space-times which admit fields of parallel vectors*, Journal of Mathematical Physics 11 (1970), p. 1578-1580.
[14] A. Komar, *Covariant conservation laws in general relativity*, Physical Review 113 (1959), p. 934-936.
[15] J. Levine and G. H. Katzin, *Quadratic first integrals in affinely connected spaces*, Tensor, New Series, 18 (1967), p. 106-112.
[16] — *On the number of special quadratic first integrals in affinely connected and Riemannian spaces*, ibidem 19 (1968), p. 113-118.
[17] — *Curvature collineations in conformally flat spaces, II*, ibidem 21 (1970), p. 319-329.
[18] A. Nijenhuis, *A note on first integrals of geodesics*, Koninklijke Nederlandse Akademie van Watenschappen, Proceedings, Serie A, 52 (1967), p. 141-145.
[19] J. A. Schouten, *Ricci-calculus*, 2nd ed., Berlin 1954.
[20] J. L. Synge and A. Schild, *Tensor calculus*, Toronto 1949.
[21] Y. Tashiro, *Conformal collineations*, Mathematical Journal, Okayama University, 10 (1960), p. 75-85.
[22] T. Y. Thomas, *The fundamental theorem on quadratic integrals*, Proceedings of the National Academy of Sciences of the U.S.A. 32 (1946), p. 10-15.
[23] M. A. Walker, *Some global aspects of space-time structures*, Ph. D. Thesis, University of London, Birkbeck College (1969).
[24] K. Yano, *The theory of Lie derivatives and its applications*, Amsterdam 1957.

NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA, U.S.A.

Reçu par la Rédaction le 7. 9. 1971