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Some properties of solutions of elliptic partial differential equations of the second order

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Let D be a domain in the space of m variables $X(x_1, \ldots, x_m)$. Assume that the boundary $F(D)$ of D or at least a closed part Σ of $F(D)$ is of class C^2, i.e. $F(D)$ or Σ is given by the equation $G(x_1, \ldots, x_m) = 0$, where the function G is of class C^2 in an m-dimensional domain containing $F(D)$ (Σ) and, moreover, $\operatorname{grad}^2 G > 0$. Let

$$\mathcal{E}(u) \equiv \sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{m} b_k(X)\frac{\partial u}{\partial x_k} + c(X)u = f(X), \tag{1}$$

where the coefficients $a_{ij}(X) = a_{ji}(X)$, $b_k(X)$ $(i, j, k = 1, \ldots, m)$, $c(X)$ and the function $f(X)$ are defined and bounded in the closure $\bar{D}$ of a bounded domain D. Let the characteristic form $\sum_{i,j=1}^{m} a_{ij}(X)\lambda_i\lambda_j$ of equation (1) be uniformly positive definite in D. We shall prove

LEMMA 1. *If in the domain $\bar{D}$ 1° $f(X) \geqslant 0$ (or $f(X) \leqslant 0$), 2° a part of $F(D)$ contained in a neighbourhood of a point X_0 of Σ belongs to Σ, 3° $u(X)$ is a regular solution of* (1) (for the definition of the regular solution of (1), see [3], p. 118) *in the closure $\bar{D}$ of D such that $u(X_0) = 0$ and $u(X) < 0$ (or $u(X) > 0$), then*

$$\limsup_{X \to X_0} \frac{u(X) - u(X_0)}{|X - X_0|} < 0 \qquad \left(or \quad \liminf_{X \to X_0} \frac{u(X) - u(X_0)}{|X - X_0|} > 0\right),$$

where X belongs to an open rectilinear segment contained in D such that the segment ends at the point X_0 and is not tangent to $F(D)$ at X_0 ($X \to X_0$ along a half-straightline "penetrating" into the interior of D).

Proof. Take the transformation

$$u(X) = v(X)W(X), \tag{2}$$

where

$$W(X) = \prod_{i=1}^{m} \cos\mu(x_i - \mathring{x}_i), \qquad X_0(\mathring{x}_1, \ldots, \mathring{x}_m).$$

Then (1) takes the form (for $W(X) \neq 0$)

$$(3) \qquad \bar{\mathcal{E}}(v) = \sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{k=1}^{m} \bar{b}_k(X)\frac{\partial v}{\partial x_k} + \bar{c}(X)v = \bar{f}(X),$$

where $\bar{c}(X) = \frac{1}{W(X)}\mathcal{E}(W)$, $\bar{f}(X) = \frac{1}{W(X)}f(X)$.

We shall show that there exist a number μ and a neighbourhood $O(X_0)$ of X_0 such that $\bar{c}(X) < 0$ and $W(X) > 0$ in $O(X_0) \cap \bar{D}$. Indeed, obvious computations show that

$$(4) \qquad \frac{1}{W(X)}\mathcal{E}(W) = -\mu^2 g(X,\mu) - \mu h(X,\mu) + c(X),$$

where

$$g(X,\mu) = \sum_{i=1}^{m} a_{ii}(X) - \sum_{\substack{i,j=1\\(i\neq j)}}^{m} a_{ij}(X)\operatorname{tg}\mu(x_i - \mathring{x}_i)\operatorname{tg}\mu(x_j - \mathring{x}_j),$$

$$h(X,\mu) = \sum_{k=1}^{m} b_k(X)\operatorname{tg}\mu(x_k - \mathring{x}_k).$$

By our assumptions $\sup_{X\in D}|c(X)| = M < \infty$. Observe that if the inequality

$$(5) \qquad -\mu^2 g(X,\mu) - \mu h(X,\mu) + M < 0$$

holds, then also $\mathcal{E}(W) < 0$ holds. For $\mu \neq 0$ (5) is equivalent to

$$(6) \qquad -g(X,\mu) - \frac{h(X,\mu)}{\mu} + \frac{M}{\mu^2} < 0.$$

Take μ such that

$$\frac{4M}{\mu^2} < \inf_{X\in D}\Big[\sum_{i=1}^{m} a_{ii}(X)\Big] = A \qquad (A > 0),$$

and a neighbourhood $O(X_0)$ such that

$$\frac{h(X,\mu)}{\mu} > -\frac{A}{4} \quad \text{and} \quad g(X,\mu) > \frac{3}{4}A, \quad W(X) > 0.$$

Then (6) is satisfied, which implies that $\bar{c}(X) < 0$ and $\bar{f}(X) \geqslant 0$ (or $\bar{f}(X) \leqslant 0$) for $X \in O(X_0) \cap \bar{D}$.

Since the boundary $F(D)$ of D is of class C^2 at X_0, then, as we know (see [1]), there exists a ball K whose boundary $F(K)$ is tangent to $F(D)$ at X_0 and whose other points all belong to D. In the common part Ω

of $O(X_0)$ and K the function v satisfies all the assumptions of a lemma of O. A. Olejnik ([5]). Therefore

$$\text{(7)}\qquad \limsup_{X\to X_0}\frac{v(X)-v(X_0)}{|X-X_0|}<0 \qquad \Big(\text{or}\quad \liminf_{X\to X_0}\frac{v(X)-v(X_0)}{|X-X_0|}>0\Big),$$

where $X\to X_0$ along a rectilinear segment contained in Ω and non-tangent to $F(\Omega)$.

On the other hand, since $u(X_0)=v(X_0)=0$, so

$$\text{(8)}\qquad \frac{u(X)-u(X_0)}{|X-X_0|}=\frac{v(X)-v(X_0)}{|X-X_0|}\,W(X).$$

Observe that

$$\text{(9)}\qquad \lim_{X\to X_0} W(X)=1.$$

Lemma 1 now follows from (7), (8) and (9).

THEOREM 1. *If in equation* (1) $f(X)\geqslant 0$ (*or* $f(X)\leqslant 0$) *for* $X\in D$, *and if* $u(X)$ *is an arbitrary non-positive* (*or non-neativeg*) *solution of* (1) *regular in the closure* $\bar D$ *of* D *such that* $\sup u(\bar D)=0$ (*or* $\inf u(\bar D)=0$), *then* $u(X)$ *vanishes at a point of* $F(D)$. *Moreover, if* $u(X)$ *also vanishes at a point in the interior of* D, *then* $u(X)$ *vanishes identically in* D.

Proof. By Lemma 1 the proof of Theorem 1 is analogous to the proof of E. Hopf's theorem (see [2], [4]).

THEOREM 2. *If* 1° *the boundary* $F(D)$ *of* D *is of class* C^1_σ (¹), 2° *there is no tangent plane to* $F(D)$ *at* $X_0\in F(D)$, 3° *there is a ball* K *belonging to* D *such that boundary* $F(K)$ *of* K *contains the point* X_0, 4° $u(X)$ *is a biregular* (²) *solution of* (1) *with* $f(X)\equiv 0$ *such that* $u(X)\neq 0$ *in* D *and* $u(X)=0$ *on* $F(D)$, *then* $u(X)$ *changes its sign in every neighbourhood of* X_0.

Proof. Suppose there is a neighbourhood $O(X_0)$ of X_0 contained in D such that $u(X)\neq 0$ in $O(X_0)\cap D$. Then, by Lemma 1, the derivative of $u(X)$ in an arbitrary direction non-tangent to the boundary of K at X_0 is different from zero.

On the other hand, since there is no tangent plane to $F(D)$ at X_0, then there are m linearly independent vectors tangent to $F(D)$ at X_0. It is obvious that the derivative of $u(X)$ in each direction tangent to $F(D)$ at X_0 is equal to zero. This, however, implies (comp. [3], p. 44) that $\partial u/\partial x_i=0$ $(i=1,\dots,m)$ at X_0. Thus all the derivatives of $u(X)$ in directions non-tangent to $F(D)$ at X_0 are equal to zero. We have got a contradiction. Thus the proof is complete.

(¹) For a definition of a surface of class C^1_σ, see [3], p. 132.

(²) For a definition of a biregular solution of (1), see [3], p. 118.

The following corollaries are simple consequences of Theorems 1 and 2.

COROLLARY 1. *If the boundary $F(D)$ of D satisfies 1°, 2° and 3° of Theorem 2, then a function $u(X)$ satisfying (1) with $f(X) \equiv 0$ such that $u(X) = 0$ on $F(D)$ and $u(X)$ does not change its sign in D cannot be of class C^1 in the closure $\bar{D}$ of D.*

COROLLARY 2. *If 1° $u(X_0) = 0$ and $du/dl = 0$ at $X_0 \in F(D)$, l denoting a non-tangent direction to $F(D)$ at X_0 "penetrating" into the interior of D, 2° there exists a ball K contained in D such that its boundary $F(K)$ contains X_0, 3° the function $u(X)$ is a non-zero solution of equation (1) with $f(X) \equiv 0$, 4° the $u(X)$ is of class C^1 in the closure $\bar{D}$ of D, then the function $u(X)$ changes its sign in every component of the common part of D and an arbitrary neighbourhood of X_0.*

COROLLARY 3. *If Z denotes the set of zeros of $u(X)$, where $u(X)$ is a non-zero solution of (1) with $f(X) \equiv 0$, then 1° Z contains no isolated points, 2° the function $u(X)$ changes its sign in every neighbourhood of an arbitrary point of Z, 3° the set Z divides the domain D, i.e. in every neighbourhood of an arbitrary point of Z there exist points belonging to at least two different components of $D-Z$.*

COROLLARY 4. *If at a point X_0 belonging to the interior of D we have $u(X_0) = 0$ and $\mathrm{grad}^2 u(X_0) = 0$, then the point X_0 is a "ramification" point of the surface $u(X) = 0$ in the sense that every ball contained in D such that its surface touches X_0 contains points of the surface $u(X) = 0$.*

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