On a generalization of Hille's functional equation

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Abstract. The purpose of this paper is to solve a generalization of Hille's functional equation using a lemma which is obtained from Nevanlinna-Pölya Theorem.

1. Hille (see [5], [4], [1], p. 19 and p.31) solved the following functional equation:

\[ |f(s+it)|^2 = |f(s)|^2 + |f(it)|^2, \]

where \( f = f(z) \) is an entire function of a complex variable \( z \) and \( s, t \) are real variables.

A simple calculation shows that (1) implies the following functional equation:

\[ |f(x+y)|^2 + |f(x-y)|^2 = |f(x+\bar{y})|^2 + |f(x-\bar{y})|^2, \]

where \( f = f(z) \) is an entire function of \( z \) and \( x, y \) are complex variables.

The following theorem was proved in [1], p. 30-31, [2]:

Theorem A. If \( f = f(z) \) is an entire function of \( z \), then the only solutions of (2) are \( f(z) = az + b \) and \( f(z) = a\sin az + b\cos az \), where \( a, b \) are arbitrary complex constants and \( a \) is a real or purely imaginary constant.

Now we consider the following functional equation:

\[ \sum_{k=0}^{n-1} |f(x + \omega^k y)|^2 = \sum_{k=0}^{n-1} |f(x + \omega^k \bar{y})|^2, \]

where \( x, y \) are complex variables, \( f = f(z) \) is an entire function of \( z \) and \( \omega \) denotes the complex number \( \exp(2\pi i/n) \). Here \( n \) is a positive integer greater than 1.

It is obvious that (2) results from (3) with \( n = 2 \).

The purpose of this note is to solve (3), i.e., to prove the following

Theorem. If \( f = f(z) \) is an entire function of \( z \), then the only solutions of (3) are \( f(z) = \sum_{k=0}^{n-1} a_k z^k \) and \( f(z) = \sum_{k=0}^{n-1} a_k \exp(\omega^k az) \), where for each \( k = 0, 1, 2, \ldots, n-1, \) \( a_k \) is an arbitrary complex constant and \( a \) is an arbitrary real constant or \( a = c \exp(\pi i/n), \) where \( c \) is an arbitrary real constant.
2. To prove the theorem in Section 1, we shall apply the following four lemmas.

**Lemma 1.** Let \( n \ (>1) \) be an arbitrary positive integer. If \( \omega \) denotes the complex number \( \exp(2\pi i/n) \), the sum \( \sum_{k=0}^{n-1} \omega^{km} \) has the value \( n \) or \( 0 \) according as \( m \) is or is not a multiple of \( n \).

**Proof.** Since the proof is easy, we omit it (cf. [10], p. 924).

**Lemma 2.** If \( f = f(z) \) is regular in a non-empty domain \( D \), then \( \Delta |f(z)|^2 = 4 |f'(z)|^2 \), where \( \Delta \) stands for the Laplacian \( \partial^2/\partial s^2 + \partial^2/\partial t^2 \) \( (z = s + it, s, t \ real) \) holds in \( D \).

**Proof.** See [7], p. 94.

Before we state Lemma 3, we state the following

**Theorem B** (Nevanlinna–Pólya Theorem). Let \( n \ (>1) \) be an arbitrarily fixed positive integer and let \( D \) be a non-empty domain. Suppose that \( f_1 = f_1(z), f_2 = f_2(z), f_3 = f_3(z), \ldots, f_n = f_n(z); \ g_1 = g_1(z), g_2 = g_2(z), g_3 = g_3(z), \ldots, g_n = g_n(z) \) are regular in \( D \). Suppose further that \( f_1 = f_1(z), f_2 = f_2(z), f_3 = f_3(z), \ldots, f_n = f_n(z) \) are linearly independent in \( D \) and that \( f_1 = f_1(z), f_2 = f_2(z), f_3 = f_3(z), \ldots, f_n = f_n(z); \ g_1 = g_1(z), g_2 = g_2(z), g_3 = g_3(z), \ldots, g_n = g_n(z) \) satisfy \( \sum_{k=1}^{n} |f_k(z)|^2 = \sum_{k=1}^{n} |g_k(z)|^2 \) in \( D \). Then there exists an \( n \times n \) unitary matrix

\[
(a_{kl}), \quad k = 1, 2, 3, \ldots, n; \ l = 1, 2, 3, \ldots, n,
\]

such that for each \( k = 1, 2, 3, \ldots, n \), \( g_k(z) = \sum_{l=1}^{n} a_{kl} f_l(z) \) holds in \( D \). Here, for each pair of \( k = 1, 2, 3, \ldots, n \), \( l = 1, 2, 3, \ldots, n \), \( a_{kl} \) is a complex constant.

**Proof.** See [3], [6], [8].

We may now prove the following

**Lemma 3.** Let \( n \ (>1) \) be an arbitrarily fixed positive integer and let \( D \) be a domain including \( 0 \). Suppose that for each \( k = 1, 2, 3, \ldots, n \), \( f_k(z) \) and \( g_k(z) \) are regular in \( D \) and that for each \( k = 1, 2, 3, \ldots, n \), \( f_k(0) = g_k(0) = 0 \). Suppose further that \( f_1 = f_1(z), f_2 = f_2(z), f_3 = f_3(z), \ldots, f_n = f_n(z) \) are linearly independent in \( D \) and that \( f_1 = f_1(z), f_2 = f_2(z), f_3 = f_3(z), \ldots, f_n = f_n(z); \ g_1 = g_1(z), g_2 = g_2(z), g_3 = g_3(z), \ldots, g_n = g_n(z) \) satisfy \( \sum_{k=1}^{n} |f_k(z)|^2 = \sum_{k=1}^{n} |g_k(z)|^2 \) in \( D \). Then \( \sum_{k=1}^{n} |f_k(z)|^2 = \sum_{k=1}^{n} |g_k(z)|^2 \) holds in \( D \).

**Proof.** By hypothesis and by Theorem B there exists a unitary matrix

\[
(a_{kl}), \quad k = 1, 2, 3, \ldots, n; \ l = 1, 2, 3, \ldots, n,
\]
such that for each \( k = 1, 2, 3, \ldots, n \),

\[
g'_k(x) = \sum_{i=1}^{n} a_{ki} f'_i(x)
\]

holds in \( D \). Here, for each pair of \( k = 1, 2, 3, \ldots, n \), \( l = 1, 2, 3, \ldots, n \), \( a_{kl} \) is a complex constant.

Integrating both sides of (4) from 0 to \( x \) along a contour and using the hypothesis that for each \( k = 1, 2, 3, \ldots, n \), \( f_k(0) = g_k(0) = 0 \) yields, for each \( k = 1, 2, 3, \ldots, n \), in \( D \)

\[
g_k(x) = \sum_{i=1}^{n} a_{ki} f_i(x).
\]

Since the matrix \((a_{kl})\) is unitary, by (5) we have \( \sum_{k=1}^{n} |f_k(x)|^2 = \sum_{k=1}^{n} |g_k(x)|^2 \) in \( D \). Q.E.D.

**Lemma 4.** Let \( n \) be an arbitrary positive integer. If for each \( k = 1, 2, 3, \ldots, n \), \( A_k \) is a point set in the finite complex plane which has no accumulation point, then the set \( \bigcup_{k=1}^{n} A_k \) has no accumulation point.

**Proof.** Since the proof is easy, we omit it.

3. We may now prove the theorem in Section 1.

First, we shall prove that an arbitrary polynomial of degree at most \( n-1 \) is a solution of (3). Let \( f = f(x) \) be an arbitrary polynomial of degree at most \( n-1 \). Then we can put

\[
f(x+y) = \sum_{j=0}^{n-1} A_j(x)y^j
\]

for all complex \( x, y \).

Hence, by Lemma 1 we have

\[
\sum_{k=0}^{n-1} |f(x + \omega^ky)|^2 = \sum_{k=0}^{n-1} f(x + \omega^ky)f(x + \omega^k\bar{y})
\]

\[
= \sum_{k=0}^{n-1} \left( \sum_{j=0}^{n-1} A_j(x)(\omega^ky)^j \right) \left( \sum_{l=0}^{n-1} \overline{A_l(x)(\omega^k\bar{y})^l} \right)
\]

\[
= \sum_{k=0}^{n-1} \left( \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} A_j(x)\overline{A_l(x)}\omega^{k(l-j)}y^j\overline{\bar{y}}^l \right)
\]

\[
= \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \left( A_j(x)\overline{A_l(x)}y^j\overline{\bar{y}}^l \right) \sum_{k=0}^{n-1} \omega^{k(l-j)}
\]

\[
= n \sum_{j=0}^{n-1} |A_j(x)|^2 |y|^j.
\]
Replacing $y$ by $\bar{y}$ in (6) yields that $f = f(x)$ is a solution of (3).

We may now assume that $f = f(x)$ is an entire function which is not a polynomial of degree at most $n - 1$.

Keeping $y$ arbitrarily fixed and taking the Laplacians $\partial^2/\partial s^2 + \partial^2/\partial t^2$ of both sides of (3) with respect to $x = s + it$ ($s, t$ real), by Lemma 2 we have

\[ 4 \sum_{k=0}^{n-1} |f'(x + \omega^k y)|^2 = 4 \sum_{k=0}^{n-1} |f'(x + \omega^k \bar{y})|^2, \]

or

\[ \sum_{k=0}^{n-1} |f'(x + \omega^k y)|^2 = \sum_{k=0}^{n-1} |f'(x + \omega^k \bar{y})|^2. \]

(7)

When $x$ is arbitrarily fixed, for each $k = 0, 1, 2, \ldots, n - 1, f(x + \omega^k y) - f(x)$ are entire functions of $y$ with $(f(x + \omega^k y) - f(x))_{y=0} = (f(x + \omega^k y) - f(x))_{y=0} = 0$. Moreover, by (7) we have in $|y| < +\infty$

\[ \sum_{k=0}^{n-1} |(\partial/\partial y) (f(x + \omega^k y) - f(x))|^2 = \sum_{k=0}^{n-1} |(\partial/\partial y) (f(x + \omega^k \bar{y}) - f(x))|^2, \]

since for each $k = 0, 1, 2, \ldots, n - 1, |\omega^k| = |\omega^k| = 1$.

Further, we shall prove that when $x$ is arbitrarily fixed, $f(x + y) - f(x), f(x + \omega y) - f(x), f(x + \omega^2 y) - f(x), \ldots, f(x + \omega^{n-1} y) - f(x)$ are linearly independent in $|y| < +\infty$. To this end, we assume that

\[ \sum_{k=0}^{n-1} C_k(x)(f(x + \omega^k y) - f(x)) = 0, \]

where for each $k = 0, 1, 2, \ldots, n - 1, C_k(x)$ does not depend on $y$ and is a function of $x$ only.

Differentiating both sides of (8) $l$ times with respect to $y$ and putting $y = 0$ in the resulting equality yields for each $l = 1, 2, 3, \ldots, n$

\[ f^{(l)}(x) \sum_{k=0}^{n-1} \omega^{kl} C_k(x) = 0. \]

(9)

Let

\[ D = \{x | \prod_{l=1}^{n} f^{(l)}(x) \neq 0\}. \]

(10)

Since, by our assumption, $f$ is not a polynomial of degree at most $n - 1$, we have $\prod_{l=1}^{n} f^{(l)}(x) \neq 0$. Hence, by (10) $D$ is a non-empty domain.

By (9), (10) we have for each $x$ belonging to $D$ and for each $l = 1, 2, 3, \ldots, n$

\[ \sum_{k=0}^{n-1} \omega^{kl} C_k(x) = 0. \]

(11)
Since, by Vandermonde's Identity in the theory of determinants (see [9], p. 102), the determinant of system (11) of linear equations for \( C_0(x), C_1(x), C_2(x), \ldots, C_{n-1}(x) \) does not vanish, we have \( C_0(x) = 0, C_1(x) = 0, \ldots, C_{n-1}(x) = 0 \) for each \( x \) belonging to \( D \). Hence, when \( x \) is arbitrarily fixed in \( D, f(x + y) - f(x), f(x + \omega y) - f(x), f(x + \omega^2 y) - f(x), \ldots, f(x + \omega^{n-1} y) - f(x) \) are linearly independent in \(|y| < +\infty\).

Since all hypotheses of Lemma 3 are satisfied, by Lemma 3 we have

\[
\sum_{k=0}^{n-1} |f(x + \omega^k y) - f(x)|^2 = \sum_{k=0}^{n-1} |f(x + \omega^k y) - f(x)|^2
\]

in \(|y| < +\infty\) and for each \( x \) belonging to \( D \). Taking into account the fact that, by our assumption, \( f = f(x) \) is not a polynomial of degree at most \( n-1 \), by a famous theorem in analytic function theory for each \( l = 1, 2, 3, \ldots, n \), the set \( \{x | f^{(l)}(x) = 0\} \) has no accumulation point. Hence, by Lemma 4 the set \( S = \bigcup_{l=1}^{n} \{x | f^{(l)}(x) = 0\} \) has no accumulation point. Hence each point of \( S \) is an isolated point. Therefore, by the continuity of \( f \) we see that (12) holds not only for each \( x \) belonging to \( D \), but also holds for each \( x \) belonging to \( \bigcup_{l=1}^{n} \{x | f^{(l)}(x) = 0\} \). Thus, (12) holds for all complex \( x, y \).

Subtracting (12) from (3) side by side and using the formula \(|a - b|^2 = |a|^2 + |b|^2 - 2 \text{Re}(\overline{a}b)\) (\( a, b \) complex) yields

\[
2 \sum_{k=0}^{n-1} \text{Re} \left( f(x + \omega^k y)\overline{f(x)} \right) = 2 \sum_{k=0}^{n-1} \text{Re} \left( f(x + \omega^k y)\overline{f(x)} \right).
\]

Using the formula \( \text{Re}(\gamma) = \text{Re}(\overline{\gamma}) \) (\( \gamma \) complex) in (13) yields

\[
\text{Re} \left( \sum_{k=0}^{n-1} \left( f(x + \omega^k y)\overline{f(x)} - f(x + \omega^k y)\overline{f(x)} \right) \right) = 0.
\]

When \( x \) is arbitrarily fixed, for each \( k = 0, 1, 2, \ldots, n-1, f(x + \omega^k y)\overline{f(x)} - f(x + \omega^k y)\overline{f(x)} \) is an entire function of \( y \). Hence, by (14) and by a famous theorem in analytic function theory we have

\[
\sum_{k=0}^{n-1} \left( f(x + \omega^k y)\overline{f(x)} - f(x + \omega^k y)\overline{f(x)} \right) = A(x),
\]

where \( A(x) \) is a function of \( x \) only.

Putting \( y = 0 \) in (15) yields \( A(x) = 0 \) for each complex \( x \). Hence, by (15) we have for all complex \( x, y \)

\[
\overline{f(x)} \sum_{k=0}^{n-1} f(x + \omega^k y) = f(x) \sum_{k=0}^{n-1} \overline{f(x + \omega^k y)}.
\]
Differentiating both sides of (16) \( n \) times with respect to \( y \), using the formula \( \omega^n = 1 \) and putting \( y = 0 \) in the resulting equality yields for all complex \( x \)

\[
nf^{(n)}(x)f(x) = nf(x)f^{(n)}(x),
\]
or

\[(17) \quad f^{(n)}(x)f(x) = f(x)f^{(n)}(x).\]

By our assumption we have \( f(x) \neq 0 \). If we put \( E = \{ x | f(x) \neq 0 \} \), then \( E \) is a non-empty domain. By (17) we have in \( E \)

\[
f^{(n)}(x)f(x) = \frac{f^{(n)}(x)f(x)}{(f^{(n)}(x)f(x))}.
\]

Hence the regular function \( f^{(n)}(x)f(x) \) in \( E \) is real-valued. Hence, by a famous theorem in analytic function theory we have in \( E \)

\[
f^{(n)}(x)f(x) = A,
\]
where \( A \) is a real constant, and so

\[(18) \quad f^{(n)}(x) = Af(x).\]

Since \( f = f(x) \) is an entire function which is not a polynomial of degree at most \( n - 1 \), by the Identity Theorem (18) holds for all complex \( x \), \( A \) being a non-zero real constant.

Solving (18), we have

\[(19) \quad f(x) = \sum_{k=0}^{n-1} a_k \exp(\omega^k ax),\]
where for each \( k = 0, 1, 2, \ldots, n-1 \), \( a_k \) is a complex constant and \( a \) is a real constant or \( a = c \exp(\pi i/n) \), where \( c \) is a real constant.

We shall prove that (19) is a solution of (3). By (19) we have

\[(20) \quad \sum_{k=0}^{n-1} |f(x + \omega^k y)|^2 = \sum_{k=0}^{n-1} f(x + \omega^k y)f(x + \omega^k y) \]

\[= \sum_{k=0}^{n-1} \left( \sum_{j=0}^{n-1} a_j \exp(\omega^j a(x + \omega^k y)) \sum_{l=0}^{n-1} \bar{a}_l \exp(\omega^l a(x + \omega^k y)) \right) \]

\[= \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \left( a_j \bar{a}_l \sum_{k=0}^{n-1} \exp(\omega^j ax + \omega^{-l} \bar{a} \bar{a} + \omega^{j+k} ay + \omega^{-l-k} a \bar{a}) \right).\]

We discuss two cases.

Case 1. \( a \) is a real constant.
Replacing \( y \) by \( \bar{y} \) in (20) and observing that \( \bar{a} = a \) together with

\[
\sum_{k=0}^{n-1} \exp(\omega^k ax + \omega^{-l}a\bar{x} + \omega^{l+k}ay + \omega^{-l-k}a\bar{y})
\]

\[
= \sum_{k=0}^{n-1} \exp(\omega^k ax + \omega^{-l}a\bar{x} + \omega^{l+k}ay + \omega^{-l-k}ay)
\]

\[
\{\omega^{l+k} | k \in \mathbb{Z}, 0 \leq k \leq n-1\} = \{\omega^{-l-k} | k \in \mathbb{Z}, 0 \leq k \leq n-1\}
\]

\[
= \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}
\]

yields that (19) is a solution of (3).

Case 2. \( a = c\exp(i\pi/n) \), where \( c \) is a real constant.

Since \( \bar{a} = c\exp(-i\pi/n) = a/\omega \), by a similar calculation to that in Case 1 (19) is a solution of (3). Q.E.D.

Remark. It is obvious that Theorem A in Section 1 results from the theorem proved in this section with \( n = 2 \).

References