JUST-NON-SINGLY GENERATED VARIETIES
OF LOCALLY CONVEX SPACES

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1. Introduction. In [2] the study of varieties of locally convex Hausdorff real topological vector spaces (LCS's) was initiated. (Selected results were announced in [1].) Of particular interest are those varieties which are singly generated, including, for example, the variety $\mathcal{N}$ of all nuclear spaces and the variety of all Schwartz spaces. It was shown that every singly generated variety has a universal generator, thus, putting into perspective the work of Kōmura and Kōmura [3] in finding a (concrete) universal generator for $\mathcal{N}$. Various characterizations of singly generated varieties were given which, in particular, imply that a subvariety of a singly generated variety is singly generated.

In this note we investigate the variety generated by the class of all LCS's with the finest locally convex topology. We show that this variety is just-non-singly generated; that is, it is not singly generated but every proper subvariety is singly generated. (This is the first such example found and it raises the question of whether such examples exist in profusion or not.) We also see that this variety does not have a maximal proper subvariety. This is an contradistinction with the situation for varieties generated by a single normed vector space.

2. Preliminaries.

Definitions. A non-empty class of LCS's is said to be a variety if it is closed under the operations of taking subspaces (not necessarily closed), separated quotient spaces, (arbitrary) cartesian products and isomorphic images.

If $\Omega$ is a class of LCS's and $\mathcal{V}(\Omega)$ is the intersection of all varieties containing $\Omega$, then $\mathcal{V}(\Omega)$ is said to be the variety generated by $\Omega$. If $\Omega$ consists of a single LCS $E$, then $\mathcal{V}(\Omega)$ is written as $\mathcal{V}(E)$ and is said to be singly generated.

Notation. Let $\Omega$ be any class of LCS's. Then (a) $\mathcal{S}\Omega$, (b) $\mathcal{Q}\Omega$, (c) $\mathcal{C}\Omega$ and (d) $\mathcal{P}\Omega$ denote the class of all LCS's isomorphic to (a) subspaces of
LCS's in $\Omega$, (b) quotient spaces of LCS's in $\Omega$, (c) cartesian products of families of LCS's in $\Omega$ and (d) products of finite families of LCS's in $\Omega$, respectively.

We will need the following basic theorem which is proved in [2]:

**Theorem 1.** If $\Omega$ is any class of LCS's, then $\mathcal{V}(\Omega) = \mathbf{SCQP}(\Omega)$.

**Notation.** For each cardinal number $m$, let $\varphi_m$ be the $m$-dimensional real-vector space with the finest locally convex topology (see [4], [5] and [2]). The class of all $\varphi_m$-spaces will be denoted by $\Phi$.

We shall use the following properties of $\varphi_m$-spaces:

(A) For any cardinal $m$, $Q(\varphi_m) = S(\varphi_m)$.

(B) For any fixed infinite cardinal $m$, each member of $P(\varphi_m)$ is isomorphic to $\varphi_m$.

(C) For any fixed infinite cardinal $m$, the LCS $E$ is in $S(\varphi_m)$ if and only if $E$ is isomorphic to $\varphi_n$ for some $n \leq m$.

(D) For any cardinal $m$ and any LCS $E$, every continuous linear operator of $E$ onto $\varphi_m$ is an open mapping. (Of course, this property characterizes $\varphi_m$-spaces.)

3. $\mathcal{V}(\Phi)$.

**Theorem 2.** If $\mathcal{V}$ is any proper subvariety of $\mathcal{V}(\Phi)$, then $\mathcal{V} \subset \mathcal{V}(\varphi_m)$ for some cardinal $m$.

**Proof.** Since $\mathcal{V}$ is a proper subvariety, there exists an $m$ such that $\varphi_m \not\subset \mathcal{V}$. Let $E$ be any LCS in $\mathcal{V}$ by Theorem 1, $E \in \mathbf{SCQP}(\Phi)$. Using (A) and (B), we infer that $E \in \mathbf{SCS}(\Phi)$, which implies that $E \in \mathbf{SC}(\Phi)$. Thus $E$ is (isomorphic to) a subspace of a product $\prod_{i \in I} \varphi_{\alpha_i}$ for some index set $I$.

Let $\varphi_{\alpha_i}$ be the projection mapping of $E$ into $\varphi_{\alpha_i}$. Using (C), we see that each $\varphi_{\alpha_i}(E)$ is a $\varphi_n$ for some $n$. Suppose that there exists an $i \in I$ such that $\varphi_{\alpha_i}(E)$ is a $\varphi_k$ for some $k \geq m$. From (D) we infer that $\varphi_k \in Q(E)$. However, using (C), we then conclude that $\varphi_m \in S(E) \subset \mathcal{V}$, which is a contradiction. Therefore, each $\varphi_{\alpha_i}(E)$ is a $\varphi_n$ for some $n < m$. Hence, using (C) again, $E \in \mathbf{SCS}(\varphi_m) = \mathcal{V}(\varphi_m)$. Consequently, $\mathcal{V} \subset \mathcal{V}(\varphi_m)$ and the proof is complete.

**Corollary 1.** $\mathcal{V}(\Phi)$ has no maximal proper subvarieties. (Cf. Theorem 3.4 of [2].)

**Corollary 2.** $\mathcal{V}(\Phi)$ is a just-non-singly generated variety; that is, it is not singly generated but every proper subvariety is singly generated.

**Proof.** Theorem 2.7 of [2] and sundry comments thereabout imply that $\mathcal{V}(\Phi)$ is not singly generated. However, by Corollary 2.8 of [2], every proper subvariety of $\mathcal{V}(\Phi)$ being a subvariety of a singly generated variety, namely, $\mathcal{V}(\varphi_m)$ for some $m$, is singly generated.

**Acknowledgement.** The author thanks S. A. Saxon for his valuable comments.
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Reçu par la Rédaction le 18. 5. 1972