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JUST-NON-SINGLY GENERATED VARIETIES OF LOCALLY CONVEX SPACES

BY

SIDNEY A. MORRIS (SYDNEY)

1. Introduction. In [2] the study of varieties of locally convex Hausdorff real topological vector spaces (LCS's) was initiated. (Selected results were announced in [1].) Of particular interest are those varieties which are singly generated, including, for example, the variety $\mathscr{N}$ of all nuclear spaces and the variety of all Schwartz spaces. It was shown that every singly generated variety has a universal generator, thus, putting into perspective the work of Kōmura and Kōmura [3] in finding a (concrete) universal generator for $\mathscr{N}$. Various characterizations of singly generated varieties were given which, in particular, imply that a subvariety of a singly generated variety is singly generated.

In this note we investigate the variety generated by the class of all LCS's with the finest locally convex topology. We show that this variety is just-non-singly generated; that is, it is not singly generated but every proper subvariety is singly generated. (This is the first such example found and it raises the question of whether such examples exist in profusion or not.) We also see that this variety does not have a maximal proper subvariety. This is an contradistinction with the situation for varieties generated by a single normed vector space.

2. Preliminaries.

Definitions. A non-empty class of LCS's is said to be a *variety* if it is closed under the operations of taking subspaces (not necessarily closed), separated quotient spaces, (arbitrary) cartesian products and isomorphic images.

If Ω is a class of LCS's and $\mathscr{V}(\Omega)$ is the intersection of all varieties containing Ω, then $\mathscr{V}(\Omega)$ is said to be the *variety generated by* Ω. If Ω consists of a single LCS E, then $\mathscr{V}(\Omega)$ is written as $\mathscr{V}(E)$ and is said to be *singly generated.*

Notation. Let Ω be any class of LCS's. Then (a) $\mathbf{S}\Omega$, (b) $\mathbf{Q}\Omega$, (c) $\mathbf{C}\Omega$ and (d) $\mathbf{P}\Omega$ denote the class of all LCS's isomorphic to (a) subspaces of

LCS's in Ω, (b) quotient spaces of LCS's in Ω, (c) cartesian products of families of LCS's in Ω and (d) products of finite families of LCS's in Ω, respectively.

We will need the following basic theorem which is proved in [2]:

THEOREM 1. *If Ω is any class of LCS's, then* $\mathscr{V}(\Omega) = \mathbf{SCQP}(\Omega)$.

Notation. For each cardinal number m, let φ_m be the m-dimensional real-vector space with the finest locally convex topology (see [4], [5] and [2]). The class of all φ_m-spaces will be denoted by Φ.

We shall use the following properties of φ_m-spaces:

(A) For any cardinal m, $\mathbf{Q}(\varphi_m) = \mathbf{S}(\varphi_m)$.

(B) For any fixed infinite cardinal m, each member of $\mathbf{P}(\varphi_m)$ is isomorphic to φ_m.

(C) For any fixed infinite cardinal m, the LCS E is in $\mathbf{S}(\varphi_m)$ if and only if E is isomorphic to φ_n for some $n \leqslant m$.

(D) For any cardinal m and any LCS E, every continuous linear operator of E onto φ_m is an open mapping. (Of course, this property characterizes φ_m-spaces.)

3. $\mathscr{V}(\Phi)$.

THEOREM 2. *If $\mathscr{V}$ is any proper subvariety of $\mathscr{V}(\Phi)$, then $\mathscr{V} \subset \mathscr{V}(\varphi_m)$ for some cardinal m.*

Proof. Since $\mathscr{V}$ is a proper subvariety, there exists an m such that $\varphi_m \notin \mathscr{V}$. Let E be any LCS in $\mathscr{V}$. By Theorem 1, $E \in \mathbf{SCQP}(\Phi)$. Using (A) and (B), we infer that $E \in \mathbf{SCS}(\Phi)$, which implies that $E \in \mathbf{SC}(\Phi)$. Thus E is (isomorphic to) a subspace of a product $\prod_{i \in I} \varphi_{a_i}$ for some index set I.

Let ϱ_{a_i} be the projection mapping of E into φ_{a_i}. Using (C), we see that each $\varrho_{a_i}(E)$ is a φ_n for some n. Suppose that there exists an $i \in I$ such that $\varrho_{a_i}(E)$ is a φ_k for some $k \geqslant m$. From (D) we infer that $\varphi_k \in \mathbf{Q}(E)$. However, using (C), we then conclude that $\varphi_m \in \mathbf{SQ}(E) \subset \mathscr{V}$, which is a contradiction. Therefore, each $\varrho_{a_i}(E)$ is a φ_n for some $n < m$. Hence, using (C) again, $E \in \mathbf{SCS}(\varphi_m) = \mathscr{V}(\varphi_m)$. Consequently, $\mathscr{V} \subset \mathscr{V}(\varphi_m)$ and the proof is complete.

COROLLARY 1. *$\mathscr{V}(\Phi)$ has no maximal proper subvarieties.* (Cf. Theorem 3.4 of [2].)

COROLLARY 2. *$\mathscr{V}(\Phi)$ is a just-non-singly generated variety; that is, it is not singly generated but every proper subvariety is singly generated.*

Proof. Theorem 2.7 of [2] and sundry comments thereabout imply that $\mathscr{V}(\Phi)$ is not singly generated. However, by Corollary 2.8 of [2], every proper subvariety of $\mathscr{V}(\Phi)$ being a subvariety of a singly generated variety, namely, $\mathscr{V}(\varphi_m)$ for some m, is singly generated.

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UNIVERSITY OF NEW SOUTH WALES
KENSINGTON, AUSTRALIA

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