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ON ANALYTIC-DENSE BLACKWELL SETS

BY

WŁODZIMIERZ BZYL (GDAŃSK)

In their monograph on Borel structures K. P. S. Bhaskara Rao and B. V. Rao posed the following problems:

1. whether every Blackwell set is strongly Blackwell;
2. whether the intersection of a Blackwell set with an analytic set is a Blackwell set;
3. whether the union of a Blackwell set and an analytic set is a Blackwell set;
4. whether the Cartesian product of a Blackwell set and the set $\{0, 1\}$ is a Blackwell set.

Shortt answers in [6] problem 1 in the affirmative for Blackwell sets with totally imperfect complement in some Borel set.

In the present note we answer all the above problems in the affirmative for Blackwell sets with totally imperfect complement in some analytic set.

However, in [2] and [3] it is proved under the Martin Axiom and the negation of the Continuum Hypothesis that the answers to questions 1–3 are negative.

1. Introduction. Let S be a metric space. In what follows $\mathscr{B}(S)$ will denote the σ-field of all Borel subsets of the metric space S. If $\mathscr{C}$ is a sub-σ-field of $\mathscr{B}(S)$ and $X \subset S$, then we use the notation

$$\mathscr{C} \cap X = \{C \cap X\colon\ C \in \mathscr{C}\}.$$

A σ-field $\mathscr{C}$ is called *separable* if it is countably generated and contains all singletons.

A subset X of some Polish space is called a *Blackwell* (*strongly Blackwell*) *set* if whenever $\mathscr{C} \subset \mathscr{B}(X)$ is a separable σ-field, then $\mathscr{C} = \mathscr{B}(X)$ (whenever $\mathscr{C} \subset \mathscr{D} \subset \mathscr{B}(X)$ are countably generated σ-fields with the same atoms, then $\mathscr{C} = \mathscr{D}$).

Let A be an analytic set; a subset X of A is *analytic-dense in* A if $A \setminus X$ includes no uncountable analytic set. We say that a set X is *analytic-dense* if there is some analytic set A including the set X such that $A \setminus X$ includes no uncountable analytic set.

Shortt introduced in [5] the concept of Borel-density. We say that a subset X of some Polish space S is *Borel-dense* if there is a set $B \in \mathscr{B}(S)$ including the set X such that $B \setminus X$ includes no uncountable member of $\mathscr{B}(S)$. Observe that every Borel-dense set is analytic-dense. Also, it is known that non-Borel-dense analytic sets exist (see [4]). For example, so-called "universal" analytic sets are of this variety. Thus the class of all Borel-dense sets is a proper subclass of all analytic-dense sets, since the latter contains all analytic sets.

The principal result of this note is the following

THEOREM. *Let A be an analytic set and let X be a Blackwell set analytic-dense in A. Then*

(1) *X is a strongly Blackwell set;*

(2) *if C is an analytic set, then $X \cap C$ is a Blackwell set analytic-dense in $A \cap C$;*

(3) *if C is an analytic set, then $X \cup C$ is a Blackwell set analytic-dense in $A \cup C$;*

(4) *if N is a countable subset of some Polish space, then $X \times N$ is a Blackwell set analytic-dense in $A \times N$.*

2. Reticulate sets. Let S be a nonempty set and $s \in S$. A *slice* of $S \times S$ is a set of the form $\{s\} \times S$ or $S \times \{s\}$. If $E \subset S \times S$, then by a *section* of E we mean the intersection of E with a slice of $S \times S$. A subset R of $S \times S$ is called *reticulate* if it is contained in a countable union of slices. Let A be a subset of some Polish space S. A *thread* T of $A \times A$ is an uncountable Borel subset of $S \times S$ each section of which contains at most one point; equivalently, T is a graph of a Borel isomorphism between uncountable subsets of $\mathscr{B}(S)$.

The following lemma is an easy consequence of Theorem 1 of [7].

LEMMA 1. *Let A be an analytic set and let D be an analytic subset of $A \times A$; then the following statements are equivalent:*

(1) *D contains a thread of $A \times A$;*

(2) *D is not reticulate.*

3. A characterization of analytic-dense Blackwell sets. If A is an analytic set and a set X is a subset of A, then we say that X is *analytic-dense of order* 2 *in A* if all analytic sets included in $A \times A \setminus X \times X$ are reticulate. Observe that if X is analytic-dense of order 2 in A, then X is analytic-dense in A.

Let A be a subset of some Polish space and let $\mathscr{C}$ and $\mathscr{D}$ be countably generated sub-σ-fields of $\mathscr{B}(A)$; we say that $\mathscr{C}$ is *proper* in $\mathscr{D}$ whenever $\mathscr{C} \subset \mathscr{D}$ and there are uncountably many atoms of $\mathscr{C}$ that are not atoms of $\mathscr{D}$. Let X be a subset of the set A; we say that the set X is *(*)-Blackwell in A* if whenever $\mathscr{C}$ is proper in $\mathscr{B}(A)$, then $\mathscr{C} \cap X$ is not separable; and the set X is *strongly (*)-Blackwell in A* if whenever $\mathscr{C}$ is proper in $\mathscr{D}$, then there are some atom C of $\mathscr{C}$ and two distinct points in $C \cap X$ which are contained in distinct atoms of $\mathscr{D}$.

Let A be a strongly Blackwell set. It is not hard to see that the following lattice of implications can be obtained:

$$\begin{array}{ccc} X \text{ strongly } (*)\text{-Blackwell in } A & \Rightarrow & X \text{ strongly Blackwell} \\ \Downarrow & & \Downarrow \\ X\ (*)\text{-Blackwell in } A & \Rightarrow & X \text{ Blackwell} \end{array}$$

(cf. [1], Proposition 9). In particular, these implications hold for an analytic set A, since each analytic set is a strongly Blackwell set ([1], Propositions 6 and 8).

The next lemma can be proved analogously to Lemma 1 of [5].

LEMMA 2. *Let A be an analytic set and X be a subset of A. If X is $(*)$-Blackwell in A, then X is analytic-dense in A.*

The main result in this section is

LEMMA 3. *Let A be an analytic set and X be a subset of A; then the following statements are equivalent:*

(1) *X is analytic-dense of order 2 in A;*

(2) *X is strongly $(*)$-Blackwell in A;*

(3) *X is $(*)$-Blackwell in A;*

(4) *X is a Blackwell set and is analytic-dense in A.*

We shall prove that (3) implies (1). This and other implications can be proved analogously to those of the theorem in [5].

Proof. (3)$\Rightarrow$(1). Since A is isomorphic with some analytic subset of the real line $\boldsymbol{R}$, we may assume that $A \subset \boldsymbol{R}$.

Assume that X is $(*)$-Blackwell in A. If X is not second-order analytic-dense in A, then by Lemma 1 there is an isomorphism g between uncountable Borel subsets of $\boldsymbol{R}$ such that

$$\mathrm{Graph}(g) \subset A \times A \setminus X \times X.$$

Put

$$\Delta_- = \{(x, y) \in \boldsymbol{R} \times \boldsymbol{R}\colon x < y\}, \quad \Delta_+ = \{(x, y) \in \boldsymbol{R} \times \boldsymbol{R}\colon x > y\},$$
$$\Delta = \{(x, y) \in \boldsymbol{R} \times \boldsymbol{R}\colon x = y\}.$$

Define

$$\mathrm{Graph}(g_-) = \mathrm{Graph}(g) \cap \Delta_-, \quad \mathrm{Graph}(g_+) = \mathrm{Graph}(g) \cap \Delta_+,$$
$$\mathrm{Graph}(g_0) = \mathrm{Graph}(g) \cap \Delta,$$

disjoint Borel sets of $\boldsymbol{R}$. Obviously,

$$\mathrm{Graph}(g) = \mathrm{Graph}(g_-) \cup \mathrm{Graph}(g_+) \cup \mathrm{Graph}(g_0).$$

Observe that $\mathrm{Graph}(g_0)$ is countable since, by Lemma 2, X is analytic-dense in A and the projection of $\mathrm{Graph}(g_0)$ onto one coordinate is an analytic subset of $A\setminus X$. Hence Graph(g_-) or Graph(g_+) is uncountable. Suppose that Graph(g_+) is uncountable. Then $g_+(a) < a$ for all $a \in D = \text{Domain}(g_+)$. Hence there is some $\varepsilon > 0$ such that

$$D(\varepsilon) = \{a \in D\colon\ g_+(a) < a - \varepsilon\}$$

is uncountable. Then there is some open interval I of length ε such that $D_0 = D(\varepsilon) \cap I$ is uncountable. If a and b are elements of D_0, then $g_+(a) < b$; so $D_0 \cap g_+(D_0) = \emptyset$. Define a measurable map $h\colon A \to A$ by

$$h(a) = \begin{cases} g_+(a) & \text{for } a \in D_0, \\ a & \text{for } a \in A\setminus D_0. \end{cases}$$

Consider a countably generated σ-field

$$\mathscr{C} = \{h^{-1}(Z)\colon\ Z \in \mathscr{B}(A)\}.$$

The atoms of $\mathscr{C}$ are given by

$$h^{-1}(a) = \begin{cases} \{a\} & \text{for } a \in A\setminus(D_0 \cup h(D_0)), \\ \emptyset & \text{for } a \in D_0, \\ \{a, g_+^{-1}(a)\} & \text{for } a \in h(D_0), \end{cases}$$

so that $\mathscr{C}$ is proper in $\mathscr{B}(A)$, and since $X \times X$ does not meet Graph(g_+), $\mathscr{C} \cap X$ is separable. Therefore, X is not (*)-Blackwell in A. Hence X is second-order analytic-dense in A.

4. Proof of the Theorem.

(1) This follows from Lemma 3 and the diagram.

(2) By Lemma 3, it suffices to check that $X \cap C$ is analytic-dense of order 2 in $A \cap C$.

Let E be an analytic subset of the set

$$(A \cap C \times A \cap C)\setminus(X \cap C \times X \cap C).$$

Then E is a subset of $A \times A\setminus X \times X$. Hence E is reticulate. Thus $X \cap C$ is analytic-dense of order 2 in $A \cap C$.

(3) By Lemma 3, it suffices to check that $X \cup C$ is analytic-dense of order 2 in $A \cup C$.

Let E be an analytic subset of the set

$$(A \cup C \times A \cup C)\setminus(X \cup C \times X \cup C).$$

Define analytic sets

$$E_1 = E \cap A \times A, \quad E_2 = E \cap A \times C,$$

$$E_3 = E \cap C \times A, \quad E_4 = E \cap C \times C.$$

Observe that

$$E = E_1 \cup E_2 \cup E_3 \cup E_4$$

and

$E_1 \subset A \times A \setminus X \times X$;
the projection of E_2 onto the first factor is contained in $A \setminus X$;
the projection of E_3 onto the second factor is contained in $A \setminus X$;
$E_4 = \emptyset$.

Since X is analytic-dense of order 2 in A, E is reticulate. Thus $X \cup C$ is analytic-dense of order 2 in $A \cup C$.

(4) By Lemma 3, it suffices to check that $X \times N$ is analytic-dense of order 2 in $A \times N$.

Let E be an analytic subset of the set

$$(A \times N \times A \times N) \setminus (X \times N \times X \times N).$$

For $i, j \in N$, define analytic sets

$$E_{ij} = E \cap (A \times \{i\} \times A \times \{j\}).$$

Observe that

$$E = \bigcup_{i,j \in N} E_{ij}$$

and that $p(E_{ij})$, the projection of E_{ij} onto the first and third factor, is an analytic subset of $A \times A \setminus X \times X$. Hence $p(E_{ij})$ is reticulate. Hence, easily, E is reticulate. Thus $X \times N$ is analytic-dense of order 2 in $A \times N$.

5. Remark on the definition of analytic-density of order 2. Let A be an analytic subset of some Polish space P and let X be a subset of A; then the following statements are equivalent:

(1) X is analytic-dense of order 2 in A;

(2) all members of $\mathscr{B}(A \times A)$ included in $A \times A \setminus X \times X$ are reticulate;

(3) all members of $\mathscr{B}(P \times P)$ included in $A \times A \setminus X \times X$ are reticulate.

The only nontrivial implication (3) $\Rightarrow$ (1) follows easily from Lemma 1.

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