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ACYCLIC DECOMPOSITION OF ACYCLIC SPACES

BY

SRIBATSA NANDA (ST. JOHN'S, NEWFOUNDLAND)

0. Introduction. Given an *acyclic space* X (i.e., a topological space X with $\tilde{H}_*(X) = 0$, where $\tilde{H}_*$ denotes the reduced integral homology functor), Dror [7] has given a general procedure for constructing a tower of acyclic spaces (called by him an *acyclic decomposition* of X) which successively approximate X. An acyclic decomposition of a space is a Postnikov-like decomposition and, as Dror has shown, has many advantages, particularly in analyzing the homotopy structure of an acyclic space. The n-stage of an acyclic decomposition in Dror's construction is obtained by the acyclic functor from the corresponding n-stage of the Postnikov tower. The object of this paper is to show that the acyclic decomposition tower can be obtained by a general categorical completion process which is due to Adams [1] and which has been developed by Deleanu et al. [5] in a much more general context. More precisely, it is shown that each stage of the acyclic tower can be obtained as the generalized Adams completion of a certain set of morphisms in the homotopy category of based CW-complexes. The relation between any stage of the acyclic decomposition and the corresponding stage of the Postnikov decomposition is also clearly demonstrated within the framework of generalized Adams completion. We formulate the results in the homotopy category of based CW-complexes. There is no loss of generality in working in this category; for if X is an arbitrary topological space, then there are a CW-complex Y and a map $f\colon Y \to X$ which is a weak homotopy equivalence, and since the reduced integral homology functor satisfies the weak homotopy equivalence axiom ([9], p. 181), it follows that if X is acyclic, so is Y. The problem then is to construct a tower of acyclic spaces in the homotopy category of based CW-complexes whose inverse limit is Y. This we do using the notion of generalized Adams completion, a brief summary of which is given in Section 1 together with other relevant results.

1. Generalized Adams completion. Let $\mathcal{D}$ be a category and S a given set of morphisms of $\mathcal{D}$. If $\mathcal{D}[S^{-1}]$ denotes the category of fractions, then

for a fixed object Y of $\mathscr{D}$ we have a contravariant functor

$$\mathscr{D}[S^{-1}](-, Y) : \mathscr{D} \to \text{Ens},$$

where Ens denotes the category of sets. If this functor is representable, then the representing object Y_S is called the (*generalized*) *Adams completion* of the object Y with respect to S or, simply, the *S-completion* of Y. This means that there is a natural equivalence of functors

$$\mathscr{D}[S^{-1}](-, Y) \overset{\tau}{\simeq} \mathscr{D}(-, Y_S).$$

The category $\mathscr{D}[S^{-1}]$ takes a simple form when S admits a calculus of left fractions (see Definition 2.3 in [8]). Recall that there is a canonical functor $F: \mathscr{D} \to \mathscr{D}[S^{-1}]$. Theorem 1.2 of [5] describes the exact relation between an object Y and its S-completion Y_S in the case where S is saturated. For our applications, S will not be saturated. Therefore, we give a variant of this result in the following

THEOREM 1.1. *Let S be a family of morphisms of $\mathscr{D}$ and let Y_S be the S-completion of an object Y. Then there exists a morphism $e: Y \to Y_S$ in $\mathscr{D}$ having couniversal property with respect to any morphism in $\mathscr{D}$ which is taken into an isomorphism by the canonical functor F: Given $f: Y \to Z$ such that $F(f)$ is an isomorphism, there exists a $t: Z \to Y_S$ such that $tf = e$.*

The theorem follows from Propositions 4.1 and 4.2 of [8].

It is not, in general, true that under the assumptions of Theorem 1.1 we have $e \in S$. In the case where S is saturated and admits a calculus of left fractions, e is in S. In many concrete situations, however, S is not saturated. For our purpose, it is enough to show that $F(e)$ is an isomorphism in the category $\mathscr{D}[S^{-1}]$. This happens when every object of $\mathscr{D}$ admits an Adams completion with respect to S. To prove this, we need only to collect some relevant results.

THEOREM 1.2. *If every object of $\mathscr{D}$ admits an S-completion and $e: Y \to Y_S$ is the morphism obtained by Theorem* 1.1, *then $F(e)$ is an isomorphism in the category $\mathscr{D}[S^{-1}]$.*

Proof. Since every object of $\mathscr{D}$ admits an S-completion, the canonical functor $F: \mathscr{D} \to \mathscr{D}[S^{-1}]$ has a right adjoint G (Corollary 2.2 of [5]), and in that case the S-completion of an object Y is simply $G(Y)$. Moreover, G is full and faithful (Proposition 2.3 of [5]). Then Proposition 2.4 of [5] implies that the unit of the adjunction $\varepsilon: 1 \to GF$ is rendered invertible by the functor F. Observe that the functor $F: \mathscr{D} \to \mathscr{D}[S^{-1}]$ is such that $F(Y) = Y$ for any object Y of $\mathscr{D}$, and under the assumptions of the theorem we have $GF(Y) = G(Y) = Y_S$. We also recall that, given a pair of categories $\mathscr{E}$ and $\mathscr{F}$ and a pair of functors $H: \mathscr{E} \to \mathscr{F}$ and $K: \mathscr{F} \to \mathscr{E}$ with $H \dashv K$ (i.e., K is the right adjoint of H), we have the natural equivalence

$$\eta : \mathscr{F}(HY, Z) \xrightarrow{\cong} \mathscr{E}(Y, KZ),$$

and the unit of the adjunction is defined by $\varepsilon(Y) = \eta(1_{HY})$. In our case, we take $\mathscr{E} = \mathscr{D}$, $\mathscr{F} = \mathscr{D}[S^{-1}]$, $H = F$, $K = G$ and $\eta = \tau$. Then

$$\tau : \mathscr{D}[S^{-1}](Y, Z) \xrightarrow{\cong} \mathscr{D}(Y, Z_S)$$

and, clearly, $\varepsilon(Y) = \tau({}^1Y) = e$, as defined in Theorem 1.1. This completes the proof.

2. The set of morphisms S_n and the S_n-completion of a space. Let $\mathscr{C}$ be a $\mathscr{U}$-category (where $\mathscr{U}$ denotes a fixed Grothendieck universe which remains fixed during the rest of our discussion) whose objects are CW-complexes and whose morphisms are continuous maps between them. Let $\tilde{\mathscr{C}}_*$ denote the corresponding based homotopy category; clearly, $\tilde{\mathscr{C}}_*$ is a $\mathscr{U}$-category. Let U_n be the set of all $(n+1)$-equivalences of this category, i.e., all maps $f : X \to Y$ in $\tilde{\mathscr{C}}_*$ such that $f_* : \pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m \leqslant n$ and an epimorphism for $m = n+1$. We define V to be the set of all morphisms of this category which induce isomorphisms in reduced integral homology. Let $S_n = U_n \cap V$. The set S_n has many desirable properties for proofs of which we need the following

PROPOSITION 2.1 ([5], Theorem 1.3). *Let T be a closed family of morphisms in a category $\mathscr{D}$ satisfying the following conditions*:

(a) *If $tt' \in T$ and $t' \in T$, then $t \in T$.*

(b) *Every diagram $\cdot \xleftarrow{t} \cdot \xrightarrow{f} \cdot$ with $t \in T$ can be embedded in a weak push-out diagram*

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ \cdot & \xrightarrow[f']{} & \cdot \end{array} \qquad \text{with } t' \in T.$$

Then T admits a calculus of left fractions.

PROPOSITION 2.2 ([5], Proposition 3.1). *The family of morphisms V satisfies* (a) *and* (b) *of Proposition* 2.1 *and, therefore, admits a calculus of left fractions.*

PROPOSITION 2.3 ([8], Proposition 2.1). *The family of morphisms U_n satisfies* (a) *and* (b) *of Proposition* 2.1 *and, therefore, admits a calculus of left fractions.*

PROPOSITION 2.4. *The family of morphisms $S_n = U_n \cap V$ satisfies* (a) *and* (b) *of Proposition* 2.1 *and, therefore, admits a calculus of left fractions.*

Proof. Condition (a) of Proposition 2.1 is obvious. Moreover, since the weak push-out diagrams in Propositions 2.2 and 2.3 are the same, the result follows easily.

PROPOSITION 2.5. *For any object Y of $\tilde{\mathscr{C}}_*$, the set $\{s : Y \to Y' \mid s \in U_n\}$ is an element of the universe $\mathscr{U}$.*

This has been proved in [8].

PROPOSITION 2.6. *For any object Y of $\tilde{\mathscr{C}}_*$, the set $\{s : Y \to Y' \mid s \in S_n\}$ is an element of the universe $\mathscr{U}$.*

Proof. Since the set $\{s : Y \to Y' \mid s \in S_n\}$ is a subset of $\{s : Y \to Y' \mid s \in U_n\}$, the result follows from the properties of a Grothendieck universe.

PROPOSITION 2.7. *The category $\tilde{\mathscr{C}}_*$ and the set S_n satisfy conditions* (A) *and* (B) *of the main Theorem in* [4].

Proof. Let I be an index set belonging to $\mathscr{U}$. If each s_i $(i \in I)$ is in V, then $\coprod s_i$ is in V. This can be seen by Theorem 2 of [4], p. 36, as follows: Observe that V is a saturated family of morphisms and that every object of $\tilde{\mathscr{C}}_*$ has a V-completion and, therefore, Theorem 2 of [4] applies. On the other hand, if each s_i is in U_n, then it has been proved in [8] that $\coprod s_i$ is in U_n. Combining these two results, we see that condition (A) of the Theorem in [4] is satisfied.

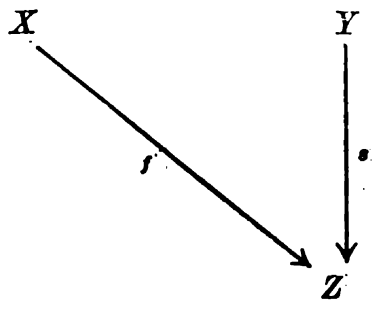
To see that condition (B) is also satisfied, we take $\mathscr{V} = \mathscr{U}$ and, for a fixed object Y of $\tilde{\mathscr{C}}_*$, we let $S_Y = \{s : Y \to Y' \mid s \in S_n\}$ which is an element of $\mathscr{U}$ by Proposition 2.6. Then, a commutative diagram, as in condition (B) of the Theorem in [4], is easily obtained by taking $s' = s$ and $u = 1_{Y'}$. As a consequence, we have the following

THEOREM 2.1. *Every object Y of $\tilde{\mathscr{C}}_*$ has an Adams completion with respect to S_n.*

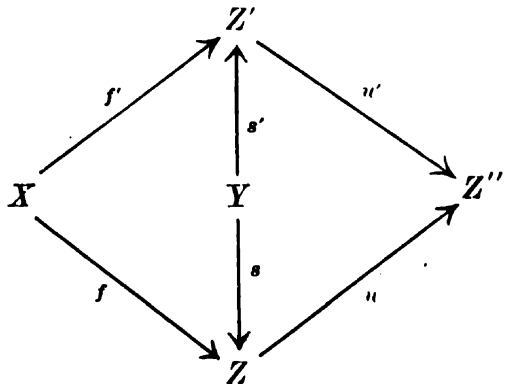
We denote the S_n-completion of a space Y by Y^n and by e_n the map $Y \to Y^n$ which arises by Theorem 1.1. Let $F : \tilde{\mathscr{C}}_* \to \tilde{\mathscr{C}}_*[S_n^{-1}]$ be the canonical functor. As a consequence of Theorems 1.2 and 2.1, we infer that $F(e_n)$ is an isomorphism in $\tilde{\mathscr{C}}_*[S_n^{-1}]$.

PROPOSITION 2.8. *e_n is an element of S_n.*

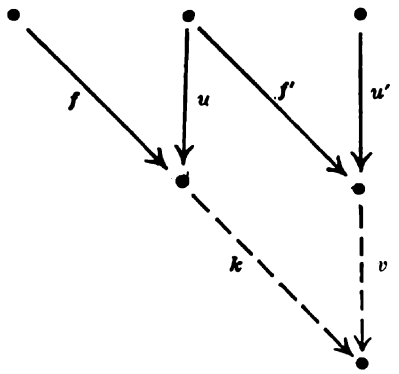
Proof. We observe first of all that since S_n admits a calculus of left fractions, we get a better hold of the category $\tilde{\mathscr{C}}_*[S_n^{-1}]$. The objects of this category are the same as those of $\tilde{\mathscr{C}}_*$ and a morphism in this category between two objects X and Y is represented by a pair of morphisms (f, s) with $s \in S_n$:



Two such pairs (f, s) and (f', s') represent the same morphism from X to Y if there is a diagram

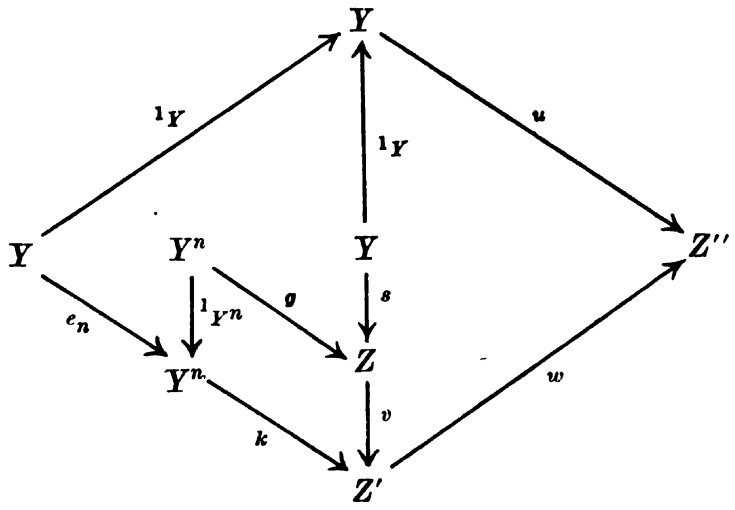


with $uf = u'f'$ and $us = u's' \in S_n$. The composition of two morphisms (f, u) and (f', u') is defined to be (kf, vu'):



The existence of morphisms k and v is guaranteed by the fact that S_n admits a calculus of left fractions. The canonical functor $F : \tilde{\mathscr{C}}_* \to \mathscr{C}_*[S_n^{-1}]$ is then defined as follows: For any object X of $\tilde{\mathscr{C}}_*$, $F(X) = X$ and, for a morphism $f : X \to Y$ in $\tilde{\mathscr{C}}_*$, $F(f) = (f, {}^1Y)$. The identity morphism on X in $\tilde{\mathscr{C}}_*[S_n^{-1}]$ is represented by the pair $({}^1X, {}^1X)$ or, equivalently, by any pair (u, u), where $u : X \to Y$ is an element of S_n.

Now, since $F(e_n) = (e_n, 1_{Y^n})$ is an isomorphism in $\tilde{\mathscr{C}}_*[S_n^{-1}]$, let (g, s) be its inverse (with $s \in S_n$): $(g, s)(e_n, 1_{Y^n}) = (1_Y, 1_Y)$. Therefore, we have



with $u = wke_n$ and $u = wvs \in S_n$. Since s and v are in S_n, so is vs, and it follows easily that $w \in S_n$. This, together with the fact that $u = wke_n$, implies that

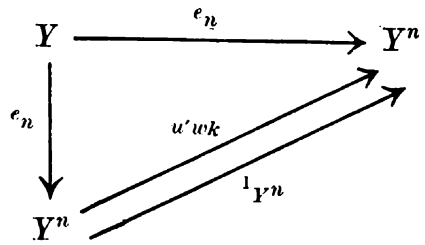
(i) $k_* : \tilde{H}_*(Y^n) \to \tilde{H}_*(Z')$ is an epimorphism (because the mapping $w_* : \tilde{H}_*(Z') \to \tilde{H}_*(Z'')$ is an isomorphism);

(ii) $e_{n*} : \pi_m(Y) \to \pi_m(Y^n)$ is a monomorphism for $m \leqslant n$.

Moreover, $u \in S_n$; therefore, $F(u)$ is an isomorphism in $\tilde{\mathscr{C}}_*[S_n^{-1}]$ and, by Theorem 1.1, there is a unique morphism $u' : Z'' \to Y^n$ such that $u'u = e_n$. Thus, we have

$$e_n = u'u = u'wke_n$$

and a diagram



Since $F(e_n)$ is an isomorphism, Theorem 1.1 applies and we get $u'wk = 1_{Y^n}$. This implies that

(iii) $k_* : \tilde{H}_*(Y^n) \to \tilde{H}_*(Z')$ is a monomorphism;

(iv) $u'_* : \pi_*(Z'') \to \pi_*(Y^n)$ is an epimorphism.

From (i) and (iii) it follows that $k_* : \tilde{H}_*(Y^n) \to \tilde{H}_*(Z')$ is an isomorphism which, together with the facts that w and u are in S_n and that $u = wke_n$, implies that $e_{n*} : \tilde{H}_*(Y) \to \tilde{H}_*(Y^n)$ is an isomorphism. Now (iv) together with the fact that u is an $(n+1)$-equivalence imply that $e_{n*} : \pi_m(Y) \to \pi_m(Y^n)$ is an epimorphism for $m \leqslant n+1$. Consequently, by (ii), e_n is an $(n+1)$-equivalence. Thus $e_n \in V \cap U_n = S_n$.

3. Y^n and the n-stage of the acyclic tower. We show in this section that Y^n is the n-stage of the acyclic decomposition of an acyclic space in the sense of Dror. Recall that by an *acyclic decomposition* of an acyclic space Y belonging to $\tilde{\mathscr{C}}_*$ we mean a tower of fibrations

$$\varprojlim Y_n = Y \to \ldots \to Y_n \to Y_{n-1} \to \ldots \to Y_0 = \text{pt.},$$

where the Y_i's are in $\tilde{\mathscr{C}}_*$ and such that for each $n \geqslant 0$:

(i) the n-stage Y_n is acyclic for each n;

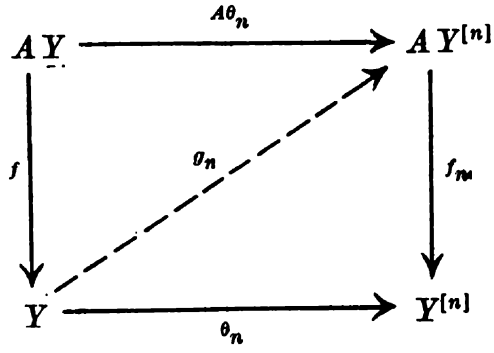
(ii) the n-stage Y_n is j-simple for $j > n$ (i.e., $\pi_1 Y_n$ acts trivially on $\pi_j Y_n$ for $j > n$);

(iii) the fibre $Y_n \to Y_{n-1}$ is $(n-1)$-connected.

It is clear that once we get a tower of acyclic spaces such that, for each n, $Y_n \to Y_{n-1}$ is an n-equivalence, then this can be converted to a fibration with $(n-1)$-connected fibre, thus giving rise to a tower of fibrations. We now recall from [8] that every object Y of $\tilde{\mathscr{C}}_*$ has a U_n-completion — denoted by $Y^{[n]}$ — which is precisely the n-th Postnikov section of Y, i.e., $\pi_j(Y^{[n]}) = \pi_j(Y)$ for $j \leqslant n$ and $\pi_j(Y^{[n]}) = 0$ for $j > n$. Moreover, if $\theta_n : Y \to Y^{[n]}$ denotes the map that arises by Theorem 1.1, then $\theta_n \in U_n$. Since θ_n is also an element of U_{n-1}, it follows from the couniversal property of the map $\theta_{n-1} : Y \to Y^{[n-1]}$ that there is a map $q_n : Y^{[n]} \to Y^{[n-1]}$ such that $q_n \theta_n = \theta_{n-1}$. We thus get a tower of spaces, and we can assume that the maps q_n's are all fibrations.

We have the same sort of situation with respect to the families of morphisms S_n's; for, corresponding to each $n \geqslant 0$, we get a space Y^n, the S_n-completion of Y. Moreover, the map $e_n : Y \to Y^n$, being an $(n+1)$-equivalence, is also an n-equivalence, and hence belongs to S_{n-1}. The couniversal property of e_{n-1} then implies that there is a map $p_n : Y^n \to Y^{n-1}$ such that $p_n e_n = e_{n-1}$. We may assume that the maps p_n's are all fibrations. It follows from Proposition 2.8 that $\tilde{H}_*(Y) = \tilde{H}_*(Y^n)$; therefore, if Y is acyclic, so is Y^n for each n. It is also equally clear that the fibre of $p_n : Y^n \to Y^{n-1}$ is $(n-1)$-connected. Thus, conditions (i) and (iii) of an acyclic tower are satisfied by the spaces $\{Y^n\}$. To show directly that condition (ii) is also satisfied seems to be difficult. We, therefore, take the help of Dror's acyclic functor A to show that Y^n and $AY^{[n]}$ (which is the n-stage of the Dror's acyclic tower) are homotopically equivalent.

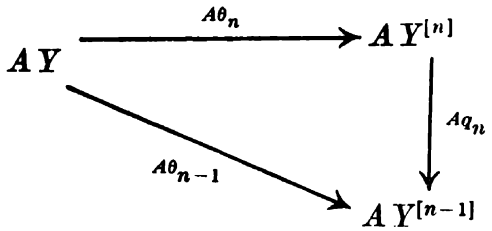
Consider the commutative diagram



Since Y is acyclic and $Y^{[n]}$ is the n-th Postnikov section of Y, $H_i(Y^{[n]}) = 0$ for $i \leqslant n+1$. It follows from Theorem 2.1 (iii) of [7] that f_n is an $(n+1)$-equivalence and that f is a homotopy equivalence, showing that

$$(A\theta_n)_* : \pi_m(AY) \to \pi_m(AY^{[n]})$$

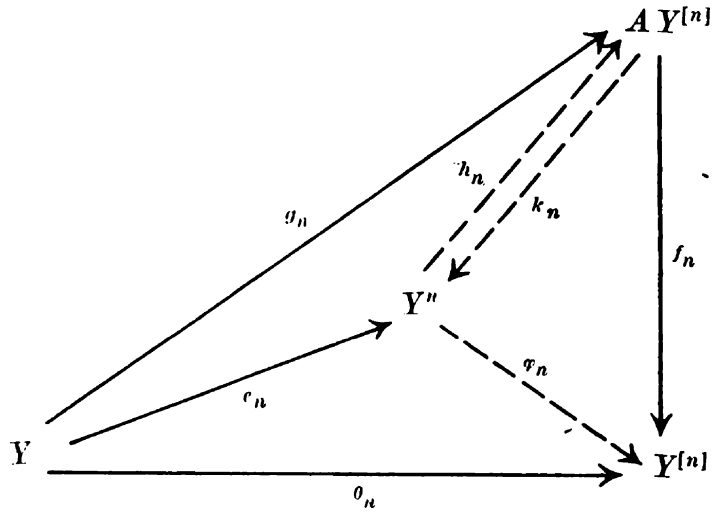
is an isomorphism for $m \leqslant n$. This together with the fact that Aq_n is an n-equivalence in the diagram



implies that $A\theta_{n-1}$ is an n-equivalence. Thus, we have a map

$$g_n = (A\theta_n)f^{-1} : Y \to AY^{[n]}$$

which is an $(n+1)$-equivalence. We can now collect together the relevant spaces and maps:



The map φ_n arises because of the couniversal property of θ_n with respect to all $(n+1)$-equivalences, the map k_n arises due to the couniversal property of e_n with respect to all $(n+1)$-equivalences which induce homology isomorphisms, whereas the map h_n arises due to the universal property of g_n. The maps φ_n, h_n and k_n are all unique and, moreover,

$$\theta_n = \varphi_n e_n = f_n g_n, \qquad e_n = k_n g_n, \qquad g_n = h_n e_n.$$

Thus, we have $\varphi_n k_n g_n = \varphi_n e_n = \theta_n = f_n g_n$; hence, by the couniversal property of θ_n, $\varphi_n k_n = f_n$. Now, $f_n h_n k_n = \varphi_n k_n = f_n$; hence, by the universal property of f_n, $h_n k_n = \mathrm{id}_{AY^{[n]}}$. Moreover, $e_n = k_n g_n = k_n h_n e_n$ implying that $k_n h_n = 1_{Y^n}$. Therefore, h_n is a homotopy equivalence. This completes the proof that Y^n is the n-stage of the acyclic decomposition. The diagram above also brings out the relation between Y^n and $Y^{[n]}$ in the framework of generalized Adams completion.

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