On differentiable solutions of a functional equation

by B. Choczewski (Gliwice)

Dedicated to Professor Stanisław Golab
to celebrate his 60-th birthday

In the present paper I deal with the problem of the existence of solutions \( \varphi(x) \) of the functional equation

\[
\varphi[f(x)] = G(x, \varphi(x)),
\]

which are of class \( C^r \) in an open interval \((a, b)\). \( f(x) \) and \( G(x, y) \) denote here known, real-valued functions of real variables.

This problem has been solved for \( r = 0 \) by J. Kordylewski and M. Kuczma in paper [3]. The authors have proved that there exist an infinite number of solutions of equation (1) that are continuous in the interval \((a, b)\).

I shall prove in § 2 that under suitable assumptions equation (1) possesses also infinitely many solutions of class \( C^r \) in the interval \((a, b)\), where \( r \) may be infinite.

U. T. Bödewadt has considered in [1] the Abel equation

\[
\varphi[f(x)] = \varphi(x) + 1
\]

and has proved, the assumptions on the function \( f(x) \) being similar to those formulated in (I) below, that there exist infinitely many solutions of the Abel equation of class \( C^r \) \( (r \leq \infty) \).

§ 1. Definition. We shall denote by \( C^r[E] \) a class of functions defined and of class \( C^r \) in a set \( E \).

We suppose (cf. [5]) that

(I) The function \( f(x) \) belongs to class \( C^r[(a, b)] \), \( 1 \leq r \leq \infty \), and fulfills the following conditions:

\[
\lim_{x \to a^+} f(x) = a, \quad \lim_{x \to b^-} f(x) = b, \quad f(x) > x \quad \text{for} \quad x \in (a, b),
\]

(2) \( f'(x) > 0 \quad \text{for} \quad x \in (a, b) \).

(The values \( a \) and \( b \) may be infinite.)
(II) The function \( G(x, y) \) belongs to class \( C^r[\Omega] \) in an open region \( \Omega \), normal with respect to the \( x \)-axis. Moreover, the inequality

\[
\frac{\partial G(x, y)}{\partial y} = 0
\]

holds for \((x, y) \in \Omega\).

Let us denote by \( \Omega_x \) the \( x \)-section of the region \( \Omega \), i.e.

\[
\Omega_x \triangleq \{ y; (x, y) \in \Omega \},
\]

and by \( \Gamma_x \) the set of values assumed by the function \( G(x, y) \) for \((x, y) \in \{x\} \times \Omega_x \), i.e.

\[
\Gamma_x \triangleq \{ z; \sum_{y}^{\gamma} [y \in \Omega_x, z = G(x, y)] \}.
\]

We suppose that

(III) \( \Omega_x \neq 0 \), \( \Gamma_x = \Omega_{(x)} \) for \( x \in (a, b) \).

Inequality (3) guarantees the existence of the function \( H(x, z) \) inverse to the function \( G(x, y) \) with respect to the variable \( y \), i.e. we have

\[
z = G(x, y) \iff y = H(x, z).
\]

The function \( H(x, z) \in C^r[\Omega'] \), where

\[
\Omega' \triangleq \{(x, z); x \in (a, b), z \in \Gamma_x \}.
\]

Let us introduce the following notation:

\[
G_1(s, \varphi, \varphi') \triangleq \frac{1}{f'(s)} \left( \frac{G_2(s, \varphi(s)) + G'_2(s, \varphi(s))\varphi'(s)}{f'(s)} \right),
\]

\[
G_{k+1}(s, \varphi, \ldots, \varphi^{(k+1)}) \triangleq \frac{1}{f'(s)} \frac{d}{ds} G_k(s, \varphi, \ldots, \varphi^{(k)}) ,
\]

\[
k = 1, \ldots, r-1.
\]

Finally, we note the following

**Lemma.** Suppose that the sequence \( \{a_n\}_{n=0}^\infty \) is strictly increasing, \( g_n(x) \in C^r[\langle a_n, a_{n+1} \rangle] \), \( n \geq 0 \), \( 0 \leq r \leq \infty \), and that we have

\[
\lim_{x \to a_{n+1}-0} g_n^{(k)}(x) = g_n^{(k)}(a_{n+1}), \quad k = 0, 1, \ldots, r.
\]

Then the function

\[
g(x) \triangleq g_n(x) \quad \text{for} \quad x \in \langle a_n, a_{n+1} \rangle
\]

belongs to class \( C^r[\bigcup_{n=0}^{\infty} \langle a_n, a_{n+1} \rangle] \).

We omit the simple inductive proof of this lemma.
§ 2. Let us take an arbitrary \( x_0 \in (a, b) \). Denoting by \( f^{-1}(x) \) the function inverse to the function \( f(x) \), let us put

\[
\begin{align*}
x_1 &= f(x_0), & x_{n+1} &= f(x_n), & n &> 0, \\
x_{-1} &= f^{-1}(x_0), & x_{n-1} &= f^{-1}(x_n), & n &> 0.
\end{align*}
\]

The sequence \( \{x_n\} \) is strictly increasing and converges to \( b \), and the sequence \( \{x_{-n}\} \) is strictly decreasing and converges to \( a \).

We shall prove the following

**Theorem 1.** Suppose that hypotheses (I)-(III) are fulfilled and a function \( \psi(x) \) is defined in the interval \( \langle x_0, x_1 \rangle \), \( \psi(x) \in C'[\langle x_0, x_1 \rangle] \quad (1 \leq r \leq \infty) \), and fulfills the following conditions:

\[
\begin{align*}
\psi(x) &\in \Omega_x \quad \text{for} \quad x \in \langle x_0, x_1 \rangle, \\
\psi(x_1) &= G(x_0, \psi(x_0)), \\
\psi^{(k)}(x_1) &= G_k(x_0, \psi, \psi', \ldots, \psi^{(k)}), \quad k = 1, \ldots, r.
\end{align*}
\]

Then there exists exactly one solution \( \varphi(x) \in C'[\langle a, b \rangle] \) of equation (1) which is an extension of the function \( \psi \). This solution is given by the formulae

\[
\varphi(x) = \begin{cases} 
\varphi_n(x) & \text{for} \quad x \in \langle x_n, x_{n+1} \rangle, \\
\varphi_{-n}(x) & \text{for} \quad x \in \langle x_{-n}, x_{-n+1} \rangle,
\end{cases} \quad n \geq 0,
\]

where the functions \( \varphi_n(x) \) and \( \varphi_{-n}(x) \) are defined in the intervals \( \langle x_n, x_{n+1} \rangle \) and \( \langle x_{-n}, x_{-n+1} \rangle \) respectively (cf. (6)) by the formulae

\[
\begin{align*}
\varphi_n(x) &= \psi(x), & \varphi_n(x) &= G[f^{-1}(x), \varphi_n[f^{-1}(x)]] \quad (n \geq 1), \\
\varphi_{-n}(x) &= H(x, \varphi_{n+1}(f(x))) \quad (n \geq 1),
\end{align*}
\]

and the function \( H \) is defined by (4).

**Proof** (1). From a theorem proved in [3] it follows in particular that the function \( \varphi(x) \) defined by formulae (10)-(12) is a continuous solution of equation (1) for \( x \in (a, b) \) already when the function \( \psi(x) \) is continuous in the interval \( \langle x_0, x_1 \rangle \) and fulfills conditions (7) and (8). Consequently it remains to prove that, if the function \( \psi(x) \) fulfills the \( r \) of conditions (9), then formulae (10)-(12) define the function of class \( C^r \) in the whole interval \( (a, b) \).

At first we shall prove that

\[
\begin{align*}
\varphi_n(x) &\in C'[\langle x_n, x_{n+1} \rangle], \quad n \geq 0, \\
\lim_{x \to x_{n+1}^-} \varphi_n^{(k)}(x) &= \varphi_n^{(k)}(x_{n+1}), \quad n \geq 0, \quad k = 0, 1, \ldots, r.
\end{align*}
\]

(1) I should like to express my thanks to Dr St. Balcerzyk for his valuable remarks concerning the proof of this theorem.
We shall prove assertion (*) by induction. Let \( n = 0 \). Then, according to (11), \( \varphi(x) = \psi(x) \) for \( x \in \langle a, x_1 \rangle \) and (*) holds as a consequence of the assumptions regarding the function \( \psi(x) \). Further, if \( \varphi_m(x) \in \mathcal{O}^r[\langle x_m, x_{m+1} \rangle] \), \( m \geq 0 \), then by (11) we have
\[
\varphi_{m+1}(x) = G_0 J f^{-1}(x), \; \varphi_m[f^{-1}(x)] \), \; x \in \langle x_{m+1}, x_{m+2} \rangle.
\]
But now \( f^{-1}(x) \in \langle x_m, x_{m+1} \rangle \) and \( \varphi_m[f^{-1}(x)] \in \mathcal{O}^r[\langle x_m, x_{m+2} \rangle] \), and consequently, on account of hypotheses (I) and (II) we have \( \varphi_{m+1}(x) \in \mathcal{O}^r[\langle x_{m+1}, x_{m+2} \rangle] \).

Equalities (**) follow for \( k = 0 \) from the fact that the function \( \varphi(x) \) defined by formulae (10)-(12) is continuous in \((a, b)\). Let us fix a number \( k, 1 \leq k \leq r \). Let us note that from definitions (5) we have
\[
\frac{d^k}{dx^k} G(f^{-1}(x), \varphi[f^{-1}(x)]) = G_k(s, \varphi, \ldots, \varphi^{(k)}), \quad s = f^{-1}(x).
\]
Since we have \( \varphi_{n+1}(x) \in \mathcal{O}^r[\langle x_{n+1}, x_{n+2} \rangle] \), the above formula yields for every natural \( n \) (cf. (11))
\[
\varphi_{n+1}^{(k)}(x) = G(s, \varphi_n, \ldots, \varphi_n^{(k)}), \quad s = f^{-1}(x).
\]
In particular, we have for \( x = x_{n+1} \)
\[
\varphi_{n+1}^{(k)}(x_{n+1}) = G_k(s, \varphi_n, \ldots, \varphi_n^{(k)}).
\]

It follows from (9), (11) and (14) that (**) holds for \( n = 0 \). We now assume that
\[
\lim_{x \to x_{m+1}} \varphi_{m+1}^{(l)}(x) = \varphi_{m+1}^{(l)}(x_{m+1}), \quad m \geq 0, \quad l = 0, 1, \ldots, k.
\]
We have by (11) and (13)
\[
\lim_{x \to x_{m+1}} \varphi_{m+1}^{(k)}(x) = \lim_{x \to x_{m+1} - 0} \frac{d^k}{dx^k} G(f^{-1}(x), \varphi_m[f^{-1}(x)])
\]
\[
= \lim_{x \to x_{m+1} - 0} G_k(s, \varphi, \ldots, \varphi^{(k)}).
\]
Finally, we get on account of (15) and (14) (for \( n = m + 1 \))
\[
\lim_{x \to x_{m+1}} \varphi_{m+1}^{(k)}(x) = \varphi_{m+2}^{(k)}(x_{m+2}), \quad k = 1, \ldots, r.
\]

Assertions (*) and (**) are valid for every \( n \). Now let us put in lemma \( \alpha_n = x_n, \; \gamma_n(x) = \varphi_n(x), \; n \geq 0 \). We infer that the function
\[
\varphi^*(x) \overset{df}{=} \varphi_n(x) \quad \text{for} \; \quad x \in \langle a, x_{n+1} \rangle, \quad n \geq 0
\]
belongs to class \( \mathcal{O}^r[\langle x_0, b \rangle] \), since \( \bigcup_{n=0}^{\infty} \langle x_n, x_{n+1} \rangle = \langle x_0, b \rangle \) (cf. (6)).
Now we define the functions

$$
\bar{\varphi}_0(x) = \varphi^*(x) \quad \text{for} \quad x \in \langle x_0, b \rangle,
$$

$$
\bar{\varphi}_{n-1}(x) = H\{x, \bar{\varphi}_{n-1}(f(x))\} \quad \text{for} \quad x \in \langle x_{n-1}, b \rangle.
$$

We have $$\bar{\varphi}_{n-1}(x) = \bar{\varphi}_n(x)$$ for $$x \in \langle x_{n-1}, b \rangle$$, since $$\varphi_n(x)$$ fulfill equation (1) in $$\langle x_{n-1}, b \rangle$$. If $$\bar{\varphi}_n(x) \in C^r(\langle x_{n-1}, b \rangle)$$ then $$\bar{\varphi}_{n-1}(x) \in C^r(\langle x_{n-1}, b \rangle)$$, because for $$x \in \langle x_{n-1}, b \rangle$$, $$\varphi_n[f(x)] \in C^r[\langle x_n, b \rangle]$$. Consequently

$$
\bar{\varphi}(x) = \frac{d}{dx} \bar{\varphi}_{n}(x) \quad \text{for} \quad x \in \langle x_{n-1}, b \rangle, \quad n \geq 0
$$

is a function of class $$C^r[\langle a, b \rangle]$$. But we have

$$
\bar{\varphi}_n(x) = \begin{cases} 
\varphi^*(x) & \text{for } x \in \langle x_0, b \rangle, \\
\varphi_n(x) & \text{for } x \in \langle x_{n-1}, x_{n+1} \rangle, \quad n \geq 1
\end{cases}
$$

whence, according to (16), we see that

$$
\bar{\varphi}(x) = \varphi(x) \quad \text{for} \quad x \in (a, b).
$$

This completes the proof of the theorem.

**Theorem 2.** Suppose that (I)-(III) are fulfilled. Then equation (1) possesses an infinite number of solutions of class $$C^r[\langle a, b \rangle]$$ ($$1 \leq r \leq \infty$$). These solutions are given by formulae (10)-(12), where $$\varphi(x) \in C^r[\langle x_0, x_1 \rangle]$$ is an arbitrary function which fulfills conditions (7)-(9).

**Proof.** For $$r < \infty$$ this theorem is an immediate consequence of theorem 1. For $$r = \infty$$ the question arises whether one can find a function $$\varphi(x) \in C^\infty(\langle x_0, x_1 \rangle)$$ which will satisfy an infinite number of conditions (9).

Let us take an arbitrary $$\bar{x} \in (x_0, x_1)$$. Let $$\alpha(x)$$ be an arbitrary function of class $$C^\infty(\langle x_0, \bar{x} \rangle)$$ which fulfills the condition

$$
\alpha(x) \in \Omega_x \quad \text{for} \quad x \in \langle x_0, \bar{x} \rangle.
$$

Then the function $$G(f^{-1}(x), \alpha[f^{-1}(x)])$$ belongs to class $$C^\infty[\langle x_1, f(\bar{x}) \rangle]$$.

It is known (cf. [6]) that there exists a function $$\varphi(x) \in C^\infty[\langle x_0, f(\bar{x}) \rangle]$$ such that

$$
\varphi(x) = \begin{cases} 
\alpha(x) & \text{for } x \in \langle x_0, \bar{x} \rangle, \\
G(f^{-1}(x), \alpha[f^{-1}(x)]) & \text{for } x \in \langle x_1, f(\bar{x}) \rangle,
\end{cases}
$$

and

$$
\varphi(x) \in \Omega_x \quad \text{for} \quad x \in \langle x_0, f(\bar{x}) \rangle.
$$

Of course, this function fulfills an infinite number of conditions (9).

The function $$\alpha(x)$$ may be chosen in infinitely many ways, and thus we obtain an infinite number of functions $$\varphi(x) \in C^\infty[\langle x_0, x_1 \rangle]$$ (since we have $$\langle x_0, x_1 \rangle \subset \langle x_0, f(\bar{x}) \rangle$$) which fulfill conditions (9) for $$r = 1, 2, \ldots$$
From theorem 1 we infer that there exist infinitely many solutions \( \varphi(x) \) of equation (1) given by formulae (10)-(12), belonging to class \( C^\infty([a, b]) \). This completes the proof.

**Remark.** Theorems analogous to theorem 2 may be proved for more general equations

\[
F(x, \varphi(x), \varphi[f(x)], \ldots, \varphi[f^r(x)]) = 0 ,
\]

(17)

\[
F(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) = 0 ,
\]

(18)

which have been considered in [2] and [4]. In these equations \( \varphi(x) \) denotes the required function, and the remaining ones are known.

Namely if the assumptions stated in [2] (resp. [4]) are fulfilled, the known functions belong to class \( C^r \) in suitable sets, and the function \( f(x) \) (resp. \( f_n(x) \)) in equation (17) (resp. (18)) fulfils condition (2), then equation (17) (resp. (18)) possesses an infinite number of solutions of class \( C^r \) \( (1 \leq r \leq \infty) \) in the interval \([a, b]\).

**References**


[4] — *On the functional equation \( F(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) = 0 \)*, Ann. Pol. Math. 8 (1960), p. 55-60.


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