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A boundary value problem for quasilinear hyperbolic systems with a retarded argument

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Abstract. A theorem of existence, uniqueness and continuous dependence on boundary data is proved concerning a.e. solutions of a boundary value problem for systems of quasilinear hyperbolic differential equations with retarded argument, including the Cauchy problem as a particular case. The proof is based on recent results due to P. Bassanini for quasilinear hyperbolic systems without a retarded argument.

1. Introduction. We shall use the symbols $D_x z(x, y) = (D_x z_1(x, y), \ldots$ $\ldots, D_x z_n(x, y))$, and $D_y z(x, y) = [D_{y_i} z_j(x, y)]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, to denote the partial derivatives of a vector function $z\colon D_a = I_a \times \boldsymbol{R}^m \to \boldsymbol{R}^n$, where $I_a = [0, a]$, $a \geqslant 0$ and $D_x z_i(x, y) = \partial z_i(x, y)/\partial x$.

We consider quasilinear hyperbolic systems with retarded argument in the *second canonic* [6] (or *bicharacteristic* [9]) form

$$(1)\quad \sum_{j=1}^{n} A_{ij}(x, y, z(x, y))\Big[D_x z_j(x, y) + \sum_{k=1}^{m} \varrho_{ik}(x, y, z(x, y), (z\circ\alpha)(x, y)) D_{y_k} z_j(x, y)\Big] = f_i(x, y, z(x, y), (z\circ\beta)(x, y)), \qquad i = 1, \ldots, n,$$

$(x, y) \in D_a$, where $y = (y_1, \ldots, y_m) \in \boldsymbol{R}^m$, $m \geqslant 1$, $z(x, y) = (z_1(x, y), \ldots, z_n(x, y))$, and $(z\circ\alpha)(x, y) = z(\alpha(x, y))$, $\alpha(x, y) = (\alpha_0(x), \alpha'(x, y))$, $\alpha'(x, y) = (\alpha_1(x, y), \ldots$ $\ldots, \alpha_m(x, y))$. In a similar way we define $z\circ\beta$ with $\beta(x, y) = (\beta_0(x), \beta'(x, y))$, $\beta'(x, y) = (\beta_1(x, y), \ldots, \beta_m(x, y))$.

In the present paper we prove, by means of the fixed point theorem in the product of two Banach spaces [5], a theorem of existence, uniqueness and continuous dependence on the data for systems (1) with the general boundary data [11], [1]

$$(2)\qquad \sum_{k=1}^{N} B_k(y) z(a_k, y) = \psi(y),$$

for an arbitrary given system of numbers a_k, $0 \leqslant a_k \leqslant a$, $k = 1, \ldots, N$, $n \leqslant N < +\infty$, and given functions $B_k(y) = [B_{kij}(y)]$, $i, j = 1, \ldots, n$, $\psi(y) = (\psi_1(y), \ldots, \psi_n(y))^T$, where T is the transpose symbol. The boundary condition (2) includes the boundary condition "à la Cesari" [2], [6], [7] as a particular case: take $N = n$, $B_{kij}(y) = B_{ij}(y)\delta_{ki}$ (δ_{ki} the Kronecker symbol). If $B_{kij}(y) = \delta_{ki}\delta_{ij}$, then the boundary condition (2) reduces to the condition considered in [12]. If, furthermore, all $a_k = 0$, $k = 1, \ldots, N$, then we have the usual Cauchy condition. Boundary conditions of the form (2) arise, in a suitable version, from problems of mathematical physics [4].

System (1) with non-retarded argument has been investigated by L. Cesari [6], [7], P. Bassanini [1]–[3], M. Cinquini-Cibrario [8], M. Cinquini-Cibrario, S. Cinquini [9], and P. Pucci [11].

The Cauchy problem for system (1) was considered by Z. Kamont, J. Turo [10] by using slightly different methods.

2. Assumptions. We introduce the norms $|y|_m = \max_{1 \leqslant k \leqslant m} |y_k|$ and $|z|_n = \max_{1 \leqslant i \leqslant n} |z_i|$ in R^m and R^n, respectively. If $D = [d_{ij}]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, is an $n \times m$ matrix, then $D_i = (d_{i1}, \ldots, d_{im})$. If D is a square matrix, then $\|D\| = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |d_{ij}|$. We denote by $\bar{\Omega}$ the cube $[-\Omega, \Omega]^n \subset R^n$, $\Omega > 0$. Let a_0 be a given positive constant.

Let

$$(3) \qquad A(x, y, z) = E + \tilde{A}(x, y, z),$$

$$A^{-1}(x, y, z) = E + \bar{A}(x, y, z), \qquad B_k(y) = E_k + \tilde{B}_k(y),$$

$$\sigma_0 = \sup_{y \in R^m} \sum_{k=1}^{N} \|\tilde{B}_k(y)\|, \qquad \sigma_1 = \sup_{D_{a_0} \times R^m} \|\tilde{A}(x, y, z)\|, \qquad \sigma_2 = \sup_{D_{a_0} \times R^m} \|\bar{A}(x, y, z)\|,$$

where A^{-1} is the inverse matrix to A, $E = [\delta_{ij}]$, $E_k = [\delta_{ki}\delta_{ij}]$, $i, j = 1, \ldots, n$, $k = 1 \ldots, N$, and $D_{a_0} = I_{a_0} \times R^m$.

Assumption H_1. Suppose that

1° A: $D_{a_0} \times \bar{\Omega} \to R^{n^2}$ is continuous;

2° $\det A(x, y, z) \geqslant \varkappa > 0$ in $D_{a_0} \times \bar{\Omega}$ for some constant $\varkappa$;

3° there are constants $H > 0$, $C \geqslant 0$ and a function p: $I_{a_0} \to R_+ = [0, +\infty)$, $p \in L_1[0, a_0]$, such that, for all (x, y, z), $(x, \bar{y}, \bar{z})$, $(\bar{x}, y, z) \in D_{a_0} \times \bar{\Omega}$, we have

$$\|A(x, y, z)\| \leqslant H,$$

$$\|A(x, y, z) - A(x, \bar{y}, \bar{z})\| \leqslant C[|y - \bar{y}|_m + |z - \bar{z}|_n],$$

$$\|A(x, y, z) - A(\bar{x}, y, z)\| \leqslant \left|\int_x^{\bar{x}} p(t)\,dt\right|.$$

Since $\det A(x, y, z) \geqslant \varkappa > 0$ in $D_{a_0} \times \bar{\Omega}$, it is easily seen that there are constants H', C' and a function p': $I_{a_0} \to \boldsymbol{R}_+$, $p' \in L_1[0, a_0]$, such that, for all (x, y, z), $(x, \bar{y}, \bar{z})$, $(\bar{x}, y, z) \in D_{a_0} \times \bar{\Omega}$, we have

$$\|A^{-1}(x, y, z)\| \leqslant H',$$

$$\|A^{-1}(x, y, z) - A^{-1}(x, \bar{y}, \bar{z})\| \leqslant C'[|y-\bar{y}|_m + |z-\bar{z}|_n],$$

$$\|A^{-1}(x, y, z) - A^{-1}(\bar{x}, y, z)\| \leqslant \Big|\int_x^{\bar{x}} p'(t)\,dt\Big|.$$

ASSUMPTION H_2. Suppose that

1° $\varrho(\cdot, y, z, u)$: $I_{a_0} \to \boldsymbol{R}^{nm}$ is measurable for every $(y, z, u) \in \boldsymbol{R}^m \times \bar{\Omega} \times \bar{\Omega}$;

2° $\varrho(x, \cdot)$: $\boldsymbol{R}^m \times \bar{\Omega} \times \bar{\Omega} \to \boldsymbol{R}^{nm}$ is continuous for a.e. $x \in I_{a_0}$;

3° there are functions b, l: $I_{a_0} \to \boldsymbol{R}_+$, b, $l \in L_1[0, a_0]$, such that, for all (y, z, u), $(\bar{y}, \bar{z}, \bar{u}) \in \boldsymbol{R}^m \times \bar{\Omega} \times \bar{\Omega}$, $i = 1, \ldots, n$, and a.e. x in I_{a_0}, we have

$$|\varrho_i(x, y, z, u)|_m \leqslant b(x),$$

$$|\varrho_i(x, y, z, u) - \varrho_i(x, \bar{y}, \bar{z}, \bar{u})|_m \leqslant l(x)[|y-\bar{y}|_m + |z-\bar{z}|_n + |u-\bar{u}|_n];$$

4° α_0: $I_{a_0} \to \boldsymbol{R}_+$ is measurable and $\alpha_0(x) \leqslant x$, a.e. in I_{a_0};

5° $\alpha'(\cdot, y)$: $I_{a_0} \to \boldsymbol{R}^m$ is measurable for $y \in \boldsymbol{R}^m$, and there is a constant $c \geqslant 0$ such that, for all y, $\bar{y} \in \boldsymbol{R}^m$ and a.e. x in I_{a_0}, we have

$$|\alpha'(x, y) - \alpha'(x, \bar{y})|_m \leqslant c\,|y-\bar{y}|_m.$$

ASSUMPTION H_3. Suppose that

1° $f(\cdot, y, z, u)$: $I_{a_0} \to \boldsymbol{R}^n$ is measurable for every $(y, z, u) \in \boldsymbol{R}^m \times \bar{\Omega} \times \bar{\Omega}$;

2° $f(x, \cdot)$: $\boldsymbol{R}^m \times \bar{\Omega} \times \bar{\Omega} \to \boldsymbol{R}^n$ is continuous for a.e. $x \in I_{a_0}$;

3° there are functions q, l_1: $I_{a_0} \to \boldsymbol{R}_+$, q, $l_1 \in L_1[0, a_0]$, such that, for all (y, z, u), $(\bar{y}, \bar{z}, \bar{u}) \in \boldsymbol{R}^m \times \bar{\Omega} \times \bar{\Omega}$ and a.e. x in I_{a_0}, we have

$$|f(x, y, z, u)|_n \leqslant q(x),$$

$$|f(x, y, z, u) - f(x, \bar{y}, \bar{z}, \bar{u})|_n \leqslant l_1(x)[|y-\bar{y}|_m + |z-\bar{z}|_n + |u-\bar{u}|_n];$$

4° β_0: $I_{a_0} \to \boldsymbol{R}_+$ is measurable and $\beta_0(x) \leqslant x$ a.e. in I_{a_0};

5° $\beta'(\cdot, y)$: $I_{a_0} \to \boldsymbol{R}^m$ is measurable for every $y \in \boldsymbol{R}^m$, and there is a constant $d \geqslant 0$ such that, for all y, $\bar{y} \in \boldsymbol{R}^m$ and a.e. x in I_{a_0}, we have

$$|\beta'(x, y) - \beta'(x, \bar{y})|_m \leqslant d\,|y-\bar{y}|_m.$$

ASSUMPTION H_4. Suppose that

1° ψ: $\boldsymbol{R}^m \to \boldsymbol{R}^n$ is continuous and there are constants ω_0, Λ_0, $0 \leqslant \omega_0 < \Omega$, $\Lambda_0 \geqslant 0$, such that, for all y, $\bar{y} \in \boldsymbol{R}^m$, we have

$$|\psi(y)|_n \leqslant \omega_0, \quad |\psi(y) - \psi(\bar{y})|_n \leqslant \Lambda_0\,|y-\bar{y}|_m;$$

2° $B_k\colon \boldsymbol{R}^m \to \boldsymbol{R}^{n^2}$ is continuous, $\det B_k(y) \neq 0$, $k = 1, \ldots, N$, and there is a constant $\tau_0 \geqslant 0$ such that, for all $y, \bar{y} \in \boldsymbol{R}^m$, we have

$$\sum_{k=1}^{N} \|B_k(y) - B_k(\bar{y})\| \leqslant \tau_0 |y - \bar{y}|_m;$$

3° $\sigma_0 < 1$, $\zeta = (\sigma_0 + \sigma_1)(1 + \sigma_2) < 1$, and $\omega_0(1 + \sigma_2) < \Omega(1 - \zeta)$.

3. Choice of classes $B_1(a)$ and $B_2(a)$. Let

$$\varrho_0 = (1 + \sigma_2)(1 - \zeta)^{-1} \omega_0.$$

Then from 3° of H_4 it follows that there is a constant k, $0 < k < 1$, such that $\zeta = (\sigma_0 + \sigma_1)(1 + \sigma_2) < k < 1$ and $\varrho_0 < \Omega$.

We assume $C\omega_0$, $C'\omega_0$ to be so small that

$$(4) \qquad \begin{gathered} C'\omega_0 + C'\varrho_0(\sigma_0 + \sigma_1) + C\varrho_0(1 + \sigma_2) < k - \zeta, \\ \delta_0 = C'[\omega_0 + \varrho_0(\sigma_0 + \sigma_1)] + C\varrho_0(1 + \sigma_2) < 1. \end{gathered}$$

Then there certainly is a constant s, $0 < s < 1$, such that

$$\tilde{\varepsilon}_0 = (1 + s)\zeta + C'\omega_0 + C'\varrho_0(\sigma_0 + \sigma_1) + C\varrho_0(1 + \sigma_2) < 1.$$

Let Q be a positive constant such that

$$Q > \tilde{\eta}_0 (1 - \tilde{\varepsilon}_0)^{-1},$$

where $\tilde{\eta}_0 = (1 + s)(1 + \sigma_2)(\Lambda_0 + \varrho_0 \tau_0) + (1 + \sigma_2) C\varrho_0 + C'[\omega_0 + \varrho_0(\sigma_0 + \sigma_1)]$.

Let us take

$$r(x) = R_0 q(x) + R_1 p(x) + R_2 p(x) + R_3 b(x),$$

where R_i, $i = 0, 1, 2, 3$, are positive constants satisfying

$$(5) \qquad \begin{aligned} & R_0 > (1 + \sigma_2)(1 - \delta_0)^{-1}, \\ & R_1 > \varrho_0(1 + \sigma_2)(1 - \delta_0)^{-1}, \\ & R_2 > [\omega_0 + \varrho_0(\sigma_0 + \sigma_1)](1 - \delta_0)^{-1}, \\ & R_3 > (1 + \sigma_2)(1 - \delta_0)^{-1}[\Lambda_0 + \varrho_0 \tau_0 + \sigma_0 Q + 2\varrho_0 C(1 + Q) + \sigma_1 Q]. \end{aligned}$$

We define the following constants:

$$Q_a = \int_0^a q(x)\,dx, \qquad B_a = \int_0^a b(x)\,dx,$$

$$P_a = \int_0^a p(x)\,dx, \qquad P'_a = \int_0^a p'(x)\,dx, \qquad L_a = \int_0^a l(x)\,dx,$$

$$L_{1a} = \int_0^a l_1(x)\,dx, \qquad R_a = \int_0^a r(x)\,dx, \qquad \gamma = (1 + \sigma_2)[P_a + C(1 + Q)B_a + CR_a].$$

We assume that a is so small that $\gamma+\zeta < k < 1$.

Now, we can define ϱ as follows:

$$\varrho = (1+\sigma_2)(Q_a+\omega_0)[1-(\gamma+\zeta)]^{-1}.$$

Then from 3° of H_4, for a sufficiently small, we have

$$(1+\sigma_2)(Q_a+\omega_0) \leqslant \Omega[1-(\gamma+\zeta)];$$

hence $\varrho \leqslant \Omega$.

Let us consider the Banach space $S_1(a) = [C(D_a) \cap L_\infty(D_a)]^n$, $0 < a \leqslant a_0$, of continuous bounded vector functions $z\colon D_a \to \boldsymbol{R}^n$ with norm $\|z\|_{S_1} = \sup_{(x,y)\in D_a} |z(x, y)|_n$.

We denote by $B_1(a)$ the closed (convex) subset of functions in $S_1(a)$ satisfying the conditions

$$|z(x, y)|_n \leqslant \varrho \leqslant \Omega, \quad |z(x, y)-z(x, \bar{y})|_n \leqslant Q|y-\bar{y}|_m,$$

$$|z(x, y)-z(\bar{x}, y)|_n \leqslant \Big|\int_x^{\bar{x}} r(t)\,dt\Big|, \quad (x, y), (x, \bar{y}), (\bar{x}, y) \in D_a,$$

where ϱ, Q and r are defined above.

We also consider the Banach space $S_2(a) = [C(\Delta_a) \cap L_\infty(\Delta_a)]^{nm}$, $\Delta_a = I_a \times D_a$, of continuous bounded matrix functions $h = [h_{ik}]\colon \Delta_a \to \boldsymbol{R}^{nm}$, $i = 1, \ldots, n$, $k = 1, \ldots, m$, with norm $\|h\|_{S_2} = \max_{1\leqslant i\leqslant n} \sup_{(\xi,x,y)\in\Delta_a} |h_i(\xi, x, y)|_m$.

We denote by $B_2(a)$ the closed (convex) subset of functions in $S_2(a)$ whose components satisfy

$$h_i(x, x, y) = 0,$$

$$|h_i(\xi, x, y)-h_i(\xi, x, \bar{y})|_m \leqslant s|y-\bar{y}|_m,$$

$$|h_i(\xi, x, y)-h_i(\bar{\xi}, x, y)|_m \leqslant \Big|\int_\xi^{\bar{\xi}} b(t)\,dt\Big|,$$

$$|h_i(\xi, x, y)-h_i(\xi, \bar{x}, y)|_m \leqslant \lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|, \quad i = 1, \ldots, n,$$

for all (ξ, x, y), $(\xi, x, \bar{y})$, $(\bar{\xi}, x, y)$, $(\xi, \bar{x}, y) \in \Delta_a$, where $\lambda = [1-L_a(1+Q+Qc)]^{-1}$. Here we assume a to be so small that

$$L_a(1+Q+Qc) < 1. \tag{6}$$

Then all functions in $B_2(a)$ are uniformly bounded by B_a.

Putting for $h = [h_{ik}]$, $i = 1, \ldots, n$, $k = 1, \ldots, m$, $h \in B_2(a)$,

$$g_i(\xi, x, y) = y+h_i(\xi, x, y), \tag{7}$$

we have

$$g_i(x, x, y) = y, \quad |g_i(\xi, x, y) - g_i(\xi, x, \bar{y})|_m \leqslant (1+s)|y-\bar{y}|_m.$$

Further properties of h and g are reported in [6], [7].

Let us define the constants

$$\tilde{\gamma} = \gamma(1+\sigma_2)^{-1}, \quad M_a = \omega_0 + Q_a + \varrho(\tilde{\gamma}+\sigma_0+\sigma_1).$$

4. Operator T and its properties. We now consider an operator $T = (T^{(1)}, T^{(2)})$ defined in $B_1(a) \times B_2(a)$, where $T^{(1)}$ and $T^{(2)}$ are defined in $B_1(a) \times B_2(a)$ as follows:

$$(8) \quad (T_h^{(1)} z)(x, y) = A^{-1}(x, y, z(x, y)) \times \times [\Delta^1(x, y; z, h) + \Delta^2(x, y; z, h) + \Delta^3(x, y; z, h)],$$

$$(T_z^{(2)} h) = [(T_z^{(2)} h)_{ij}], \quad i = 1, \ldots, n, \; j = 1, \ldots, m,$$

and

$$(T_z^{(2)} h)_i(\xi, x, y) = -\int_\xi^x \varrho_i\big(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z \circ \alpha)(t, g_i(t, x, y))\big)\, dt, \quad i = 1, \ldots, n.$$

$$\Delta^k(x, y; z, h) = \big(\Delta_1^k(x, y; z, h_1), \ldots, \Delta_n^k(x, y; z, h_n)\big)^T, \quad k = 1, 2, 3,$$

$$\Delta_i^1(x, y; z, h_i) = \psi_i(g_i(a_i, x, y)) + \int_{a_i}^x f_i\big(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z \circ \beta)(t, g_i(t, x, y))\big)\, dt,$$

$$(9) \quad \Delta_i^2(x, y; z, h_i) = \int_{a_i}^x D_t[A_i^*(t, x, y; z, h_i)]\, z_i^*(t, x, y; h_i)\, dt + A_i^*(a_i, x, y; z, h_i)\, z_i^*(a_i, x, y; h_i),$$

$$\Delta_i^3(x, y; z, h_i) = -\sum_{k=1}^N B_{ki}(g_i(a_i, x, y))\, z(a_k, g_i(a_i, x, y)),$$

$$A_i^*(t, x, y; z, h_i) = A_i\big(t, g_i(t, x, y), z(t, g_i(t, x, y))\big),$$

$$z_i^*(t, x, y; h_i) = z(t, g_i(t, x, y)),$$

g_i is defined by (7).

Note that, because of (3), we may simultaneously replace $D_t A_i^*$, A_i and B_{ki} in the above equalities by $D_t \tilde{A}_i^*$, $\tilde{A}_i$ and $\tilde{B}_{ki}$, respectively.

LEMMA 1. *Let Assumptions* H_1–H_4 *hold. Then for a, $0 < a \leqslant a_0$, $C\omega_0$ and $C'\omega_0$ sufficiently small, the operator $T^{(1)}$ maps $B_1(a) \times B_2(a)$ into $B_1(a)$.*

Proof. By applying the Chain Rule Differentiation Lemma (4) (ii) of [6], we can show that

$$|D_t A_i^*(t, x, y; z, h_i)|_n \leqslant p(t) + C(1+Q)\, b(t) + Cr(t),$$

and

$$|D_t z_i^*(t, x, y; h_i)|_n \leqslant r(t) + Qb(t), \quad i = 1, \ldots, n.$$

Hence, for all $(x, y) \in D_a$, we have

$$\begin{aligned} |(T_h^{(1)} z)(x, y)|_n &\leqslant (1+\sigma_2)(Q_a + \omega_0) + \varrho(1+\sigma_2)(\tilde{\gamma} + \sigma_0 + \sigma_1) \\ &= (1+\sigma_2)(Q_a + \omega_0) + \varrho(\gamma + \zeta) = \varrho \leqslant \Omega. \end{aligned}$$

For any two points (x, y), $(x, \bar{y}) \in D_a$, we can write

$$(T_h^{(1)} z)(x, y) - (T_h^{(1)} z)(x, \bar{y}) = v_0 + v_1 + v_2 + v_3,$$

where

$$\begin{aligned} v_0 &= \big[A^{-1}\big(x, y, z(x, y)\big) - A^{-1}\big(x, \bar{y}, z(x, \bar{y})\big)\big] \times \\ &\qquad \times \big[\Delta^1(x, y; z, h) + \Delta^2(x, y; z, h) + \Delta^3(x, y; z, h)\big], \\ v_k &= A^{-1}\big(x, \bar{y}, z(x, \bar{y})\big)\big[\Delta^k(x, y; z, h) - \Delta^k(x, \bar{y}; z, h)\big], \quad k = 1, 2, 3, \end{aligned}$$

and

$$|v_0|_n \leqslant C'(1+Q) M_a |y - \bar{y}|_m,$$

$$\begin{aligned} |v_1|_n &\leqslant \big\|A^{-1}\big(x, y, z(x, y)\big)\big\| \Big\{ \max_{1 \leqslant i \leqslant n} \big|\psi_i\big(g_i(a_i, x, y)\big) - \psi_i\big(g_i(a_i, x, \bar{y})\big)\big| + \\ &\quad + \max_{1 \leqslant i \leqslant n} \Big| \int_{a_i}^{x} \big[f_i\big(t, g_i(t, x, y), z\big(t, g_i(t, x, y)\big), (z \circ \beta)\big(t, g_i(t, x, y)\big)\big) - \\ &\qquad - f_i\big(t, g_i(t, x, \bar{y}), z\big(t, g_i(t, x, \bar{y})\big), (z \circ \beta)\big(t, g_i(t, x, \bar{y})\big)\big)\big] \Big| \, dt \Big\} \\ &\leqslant (1+\sigma_2)\big[\Lambda_0(1+s) + L_{1a}(1+s)(1+Q+Qd)\big] |y - \bar{y}|_m, \end{aligned}$$

$$\begin{aligned} |v_2|_n &\leqslant (1+\sigma_2) \Big\{ \max_{1 \leqslant i \leqslant n} \Big| \int_{a_i}^{x} \big[\tilde{A}_i^*(t, x, y; z, h_i) - \\ &\quad - \tilde{A}_i^*(t, x, \bar{y}; z, h_i) D_t z_i^*(t, x, y; h_i) \, dt \Big| + \\ &\quad + \max_{1 \leqslant i \leqslant n} \Big| \int_{a_i}^{x} D_t \tilde{A}_i^*(t, x, \bar{y}; z, h_i)\big[z_i^*(t, x, y; h_i) - z_i^*(t, x, \bar{y}; h_i)\big] \, dt \Big| + \\ &\quad + \max_{1 \leqslant i \leqslant n} \big|\big[\tilde{A}_i\big(x, y, z(x, y)\big) - \tilde{A}_i\big(x, \bar{y}, z(x, \bar{y})\big)\big] z(x, y)\big| + \\ &\quad + \max_{1 \leqslant i \leqslant n} \big|\tilde{A}_i^*(a_i, x, y; z, h_i)\big[z_i^*(a_i, x, y; h_i) - z_i^*(a_i, x, \bar{y}; h_i)\big]\big| \Big\} \\ &\leqslant (1+\sigma_2)\big[C(R_a + QB_a)(1+s)(1+Q) + \tilde{\gamma}Q(1+s) + \\ &\qquad + \varrho C(1+Q) + \sigma_1 Q(1+s)\big] \times |y - \bar{y}|_m, \end{aligned}$$

$$\begin{aligned} |v_3|_n &\leqslant (1+\sigma_2) \Big\{ \max_{1 \leqslant i \leqslant n} \Big| \sum_{k=1}^{N} \big[B_{ki}\big(g_i(a_i, x, y)\big) - \\ &\quad - B_{ki}\big(g_i(a_i, x, \bar{y})\big)\big] z\big(a_k, g_i(a_i, x, y)\big)\Big| + \end{aligned}$$

$$+\max_{1\leqslant i\leqslant n}\Big|\sum_{k=1}^{N} B_{ki}\big(g_i(a_i, x, y)\big)\big[z\big(a_k, g_i(a_i, x, y)\big)-z\big(a_k, g_i(a_i, x, \bar{y})\big)\big]\Big|\Big\}$$

$$\leqslant (1+\sigma_2)(1+s)(\varrho\tau_0+\sigma_0 Q)|y-\bar{y}|_m,$$

$$\tilde{A}_i^*(t, x, y; z, h_i) = \tilde{A}_i\big(t, g_i(t, x, y), z\big(t, g_i(t, x, y)\big)\big), \quad i = 1, \ldots, n.$$

Summarizing, we get

$$|(T_h^{(1)} z)(x, y)-(T_h^{(1)} z)(x, \bar{y})|_n \leqslant (\tilde{\varepsilon}Q+\tilde{\eta})|y-\bar{y}|_m,$$

where

$$\tilde{\varepsilon} = C' M_a+(1+\sigma_2)\varrho C+\gamma(1+s)+$$
$$+(1+\sigma_2)(1+s)[(1+d)L_{1a}+C(R_a+QB_a)+\sigma_0+\sigma_1],$$

$$\tilde{\eta} = C' M_a+(1+\sigma_2)\varrho C+(1+\sigma_2)(1+s)[L_{1a}+\Lambda_0+C(R_a+QB_a)+\varrho\tau_0].$$

We now assume

$$\tilde{\varepsilon}Q+\tilde{\eta}\leqslant Q,$$

which is certainly satisfied for a sufficiently small; then

$$|(T_h^{(1)} z)(x, y)-(T_h^{(1)} z)(x, \bar{y})|_n \leqslant Q|y-\bar{y}|_m.$$

For any two points (x, y), $(\bar{x}, y)\in D_a$, we have

$$(T_h^{(1)} z)(x, y)-(T_h^{(1)} z)(\bar{x}, y) = \mu_0+\mu_1+\mu_2+\mu_3,$$

where

$$\mu_0 = \big[A^{-1}\big(x, y, z(x, y)\big)-A^{-1}\big(\bar{x}, y, z(\bar{x}, y)\big)\big]\times$$
$$\times[\Delta^1(x, y; z, h)+\Delta^2(x, y; z, h)+\Delta^3(x, y; z, h)],$$

$$\mu_k = A^{-1}\big(\bar{x}, y, z(\bar{x}, y)\big)[\Delta^k(x, y; z, h)-\Delta^k(\bar{x}, y; z, h)], \quad k = 1, 2, 3,$$

and

$$|\mu_0|_n \leqslant M_a\Big[\Big|\int_x^{\bar{x}} p'(t)\,dt\Big|+C'\Big|\int_x^{\bar{x}} r(t)\,dt\Big|\Big],$$

$$|\mu_1|_n \leqslant (1+\sigma_2)\Big[\Lambda_0\lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|+\Big|\int_x^{\bar{x}} q(t)\,dt\Big|+L_{1a}(1+Q+Qd)\lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|\Big],$$

$$|\mu_2|_n \leqslant (1+\sigma_2)\Big\{C(1+Q)(R_a+QB_a)\lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|+\tilde{\gamma}Q\lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|+$$

$$+\varrho\Big[\Big|\int_x^{\bar{x}} p(t)\,dt\Big|+C(1+Q)\Big|\int_x^{\bar{x}} b(t)\,dt\Big|+C\Big|\int_x^{\bar{x}} r(t)\,dt\Big|\Big]+$$

$$+\varrho C(1+Q)\lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|+\sigma_1 Q\lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|\Big\},$$

$$|\mu_3|_n \leqslant (1+\sigma_2)(\tau_0\varrho+\sigma_0 Q)\lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|.$$

Hence

$$(10)\quad |(T_h^{(1)}z)(x, y)-(T_h^{(1)}z)(\bar{x}, y)|_n$$
$$\leqslant \gamma_0'\Big|\int_x^{\bar{x}} q(t)\,dt\Big|+\gamma_1'\Big|\int_x^{\bar{x}} p(t)\,dt\Big|+\gamma_2'\Big|\int_x^{\bar{x}} p'(t)\,dt\Big|+\gamma_3'\Big|\int_x^{\bar{x}} b(t)\,dt\Big|+\delta\Big|\int_x^{\bar{x}} r(t)\,dt\Big|,$$

where

$$\gamma_0' = 1+\sigma_2,\qquad \gamma_1' = (1+\sigma_2)\varrho,\qquad \gamma_2' = M_a,$$
$$\gamma_3' = (1+\sigma_2)[\Lambda_0\lambda+L_{1a}(1+Q+Qd)\lambda+C(1+Q)(R_a+QB_a)\lambda+$$
$$+\tilde{\gamma}Q\lambda+\varrho C(1+Q)+\varrho C(1+Q)\lambda+\sigma_1 Q\lambda+(\tau_0\varrho+\sigma_0 Q)\lambda],$$
$$\delta = M_a C'+\varrho C(1+\sigma_2).$$

We assume that a is so small that

$$\gamma_k' \leqslant (1-\delta)R_k,\qquad k = 0, \ldots, 3,$$

i.e., $\gamma_k'+\delta R_k \leqslant R_k$, $k = 0, \ldots, 3$, where R_k are defined by (5). Then (10) yields

$$|(T_h^{(1)}z)(x, y)-(T_h^{(1)}z)(\bar{x}, y)|_n \leqslant \Big|\int_x^{\bar{x}} r(t)\,dt\Big|.$$

Thus $T^{(1)}$: $B_1(a)\times B_2(a)\to B_1(a)$. This completes the proof.

LEMMA 2. *If Assumption* H_2 *is satisfied and* a, $0 < a \leqslant a_0$, *is so small that*

$$L_a(1+s)(1+Q+Qc) \leqslant s,\qquad L_a(1+Q+Qc) < 1,$$

then the operator $T^{(2)}$ *maps* $B_1(a)\times B_2(a)$ *into* $B_2(a)$.

Proof. Note that, for all $z\in B_1(a)$, $h\in B_2(a)$, the function $T_z^{(2)}h$ is continuous, and that

$$(T_z^{(2)}h)_i(x, x, y) = 0,$$

$$|(T_z^{(2)}h)_i(\xi, x, y)-(T_z^{(2)}h)_i(\bar{\xi}, x, y)|_m \leqslant \Big|\int_\xi^{\bar{\xi}} b(t)\,dt\Big|,$$

$$|(T_z^{(2)}h)_i(\xi, x, y)-(T_z^{(2)}h)_i(\xi, x, \bar{y})|_m$$
$$\leqslant \Big|\int_\xi^{x} l(t)(1+Q+Qc)\,|g_i(t, x, y)-g_i(t, x, \bar{y})|_m\,dt\Big|$$
$$\leqslant L_a(1+s)(1+Q+Qc)\,|y-\bar{y}|_m \leqslant s\,|y-\bar{y}|_m,$$

$$|(T_z^{(2)}h)_i(\xi, x, y)-(T_z^{(2)}h)_i(\xi, \bar{x}, y)|_m$$
$$\leqslant \Big|\int_x^{\bar{x}} \varrho_i\big(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z\circ\alpha)(t, g_i(t, x, y))\big)\,dt\Big|_m+$$
$$+\Big|\int_\xi^{\bar{x}} \big[\varrho_i\big(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z\circ\alpha)(t, g_i(t, x, y))\big)-$$
$$-\varrho_i\big(t, g_i(t, \bar{x}, y), z(t, g_i(t, \bar{x}, y)), (z\circ\alpha)(t, g_i(t, \bar{x}, y))\big)\big]\,dt\Big|_m$$
$$\leqslant \Big|\int_x^{\bar{x}} b(t)\,dt\Big|+L_a(1+Q+Qc)\lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big| = \lambda\Big|\int_x^{\bar{x}} b(t)\,dt\Big|, \quad i = 1, \ldots, n.$$

Hence we conclude that $T_z^{(2)}h$ belongs to $B_2(a)$. Thus the lemma is proved.

LEMMA 3. *If Assumptions* H_1–H_4 *are satisfied, then the operators* $T^{(i)}$: $B_1(a)\times B_2(a)\to B_i(a)$ $(i = 1, 2)$ *are Lipschitz continuous.*

Proof. Indeed, for all z, $\bar{z}\in B_1(a)$, h, $\bar{h}\in B_2(a)$, we have

$$\|T_h^{(1)}z-T_{\bar{h}}^{(1)}z\|_{S_1} \leqslant (1+\sigma_2)\{[\Lambda_0+L_{1a}(1+Q+Qd)+\sigma_0 Q+\varrho\tau_0]\times$$
$$\times\|h-\bar{h}\|_{S_2}+\|\Delta^2(z, h)-\Delta^2(z, \bar{h})\|_{S_1}\}.$$

Integrating by parts, we can write

$$\Delta_i^2(z, h_i)-\Delta_i^2(z, \bar{h}_i)$$
$$= \int_{a_i}^{x} D_t[\tilde{A}_i^*(t, x, y; z, \bar{h}_i)][z_i^*(t, x, y; h_i)-z_i^*(t, x, y; \bar{h}_i)]\,dt-$$
$$-\int_{a_i}^{x}[\tilde{A}_i^*(t, x, y; z, h_i)-\tilde{A}_i^*(t, x, y; z, \bar{h}_i)]\,D_t z_i^*(t, x, y; h_i)\,dt+$$
$$+\tilde{A}_i(a_i, x, y; z, h_i)[z_i^*(a_i, x, y; h_i)-z_i^*(a_i, x, y; \bar{h}_i)],$$

with $\bar{g}_i(\xi, x, y) = y+\bar{h}_i(\xi, x, y)$. Hence

$$\|\Delta^2(z, h)-\Delta^2(z, \bar{h})\|_{S_1} \leqslant [\tilde{\gamma}Q+C(1+Q)(R_a+QB_a)+\sigma_1 Q]\,\|h-\bar{h}\|_{S_2},$$

and so

$$\|T_h^{(1)}z-T_{\bar{h}}^{(1)}z\|_{S_1} \leqslant \lambda_1\,\|h-\bar{h}\|_{S_2},$$

where

$$\lambda_1 = (1+\sigma_2)(\Lambda_0+\varrho\tau_0)+\zeta Q+(1+\sigma_2)[\tilde{\gamma}Q+L_{1a}(1+Q+Qd)+$$
$$+C(1+Q)(R_a+QB_a)].$$

Similarly, we have

$$\|T_h^{(1)}z-T_h^{(1)}\bar{z}\|_{S_1} \leqslant [C' M_a+(1+\sigma_2)(2L_{1a}+\sigma_0)]\,\|z-z\|_{S_1}+$$
$$+(1+\sigma_2)\,\|\Delta^2(z, h)-\Delta^2(\bar{z}, h)\|_{S_1}.$$

Integrating by parts, we can write

$$\begin{aligned}
&\Delta_i^2(z, h_i) - \Delta_i^2(\bar{z}, h_i) \\
&\quad = \int_{a_i}^{x} D_t \tilde{A}_i^*(t, x, y; \bar{z}, h_i)[z_i^*(t, x, y; h_i) - \bar{z}_i^*(t, x, y; h_i)]\,dt - \\
&\quad - \int_{a_i}^{x} [\tilde{A}_i^*(t, x, y; z, h_i) - \tilde{A}_i^*(t, x, y; \bar{z}, h_i)]\, D_t z_i^*(t, x, y; h_i)\,dt + \\
&\quad + [\tilde{A}_i(x, y, z(x, y)) - \tilde{A}_i(x, y, \bar{z}(x, y))]\, z(x, y) + \\
&\quad + \tilde{A}_i^*(a_i, x, y; \bar{z}, h_i)[z_i^*(a_i, x, y; h_i) - \bar{z}_i^*(a_i, x, y; h_i)].
\end{aligned}$$

Hence

$$\|\Delta^2(z, h) - \Delta^2(\bar{z}, h)\|_{S_1} \leqslant [\tilde{\gamma} + C(R_a + QB_a) + C\varrho + \sigma_1]\|z - \bar{z}\|_{S_1},$$

and we finally get

$$\|T_h^{(1)} z - T_h^{(1)} \bar{z}\|_{S_1} \leqslant \lambda_3 \|z - \bar{z}\|_{S_1},$$

where

$$\begin{aligned}
\lambda_3 = [C'\omega_0 + C'\varrho(\sigma_0 + \sigma_1) + C\varrho(1 + \sigma_2) + \zeta] + \\
+ \{C'Q_0 + C'\varrho\tilde{\gamma} + \gamma + (1 + \sigma_2)[2L_{1a} + C(R_a + QB_a)]\}.
\end{aligned}$$

For the operator $T^{(2)}$ we easily obtain

$$\|T_z^{(2)} h - T_{\bar{z}}^{(2)} \bar{h}\|_{S_2} \leqslant 2L_a \|z - \bar{z}\|_{S_1} + L_a(1 + Q + Qc)\|h - \bar{h}\|_{S_2}.$$

This ends the proof.

5. The main result.

THEOREM. *Let Assumptions* H_1–H_4 *hold. Then for* a, $0 < a \leqslant a_0$, $C\omega_0$ *and* $C'\omega_0$ *sufficiently small, and for every system of numbers* a_i, $0 \leqslant a_i \leqslant a_0$, $i = 1, \ldots, N$, *there is a vector function* $z\colon D_a \to \boldsymbol{R}^n$, $z \in B_1(a)$, *satisfying* (1) *a.e. in* D_a *and* (2) *everywhere in* $\boldsymbol{R}^m$. *Furthermore, z is unique and depends continuously on* ψ *in the classes which are described in the proof.*

Proof. From Lemmas 1–3 we have for $T = (T^{(1)}, T^{(2)})\colon B_1(a) \times B_2(a) \to B_1(a) \times B_2(a)$

$$\begin{aligned}
\|T_h^{(1)} z - T_{\bar{h}}^{(1)} \bar{z}\|_{S_1} &\leqslant k_1 \|h - \bar{h}\|_{S_2} + k_2 \|z - \bar{z}\|_{S_1}, \\
\|T_z^{(2)} h - T_{\bar{z}}^{(2)} \bar{h}\|_{S_2} &\leqslant k_3 \|h - \bar{h}\|_{S_2} + k_4 \|z - \bar{z}\|_{S_1},
\end{aligned}$$

where

$$k_1 = \lambda_1, \quad k_2 = \lambda_3, \quad k_3 = \lambda_6 = L_a(1 + Q + Qc), \quad k_4 = \lambda_4 = 2L_a,$$

and we assume (the non-trivial case) $\lambda_1 \lambda_3 \lambda_4 \lambda_6 > 0$.

Thus, the problem is reduced to the general setting considered in [5], with $\lambda_2 = \lambda_5 = 0$, and assumptions (H1), (H2) of [5] are fulfilled.

Accordingly, we choose the weighted 1-norm in $S_1(a) \times S_2(a)$

(11) $$\|w\|_{S_1 \times S_2} = \alpha \|z\|_{S_1} + \beta \|h\|_{S_2}, \quad w = (z, h) \in B_1(a) \times B_2(a),$$

with α, $\beta > 0$ to be specified below, and we require conditions (H3) of [5] to be satisfied:

$$k_2 < k, \quad k_3 < k, \quad k_1 k_4 \leqslant (k - k_2)(k - k_3),$$

together with

(12) $$k_4 (k - k_2)^{-1} \leqslant \alpha/\beta \leqslant (k - k_3)/k_1.$$

Assumptions (H3) reduce here, because of (6), to the conditions

$$k_2 < k, \quad k_1 k_4 \leqslant (k - k_2)(k - k_3),$$

which are certainly satisfied for a sufficiently small, by virtue of (4).

Then, as is proved in [5], the map $T\colon B_1(a) \times B_2(a) \to B_1(a) \times B_2(a)$ is a contraction with constant k, $0 < k < 1$, in the norm (11)

$$\|Tw - T\bar{w}\|_{S_1 \times S_2} \leqslant k \|w - \bar{w}\|_{S_1 \times S_2}$$

for all $w = (z, h)$, $\bar{w} = (\bar{z}, \bar{h}) \in B_1(a) \times B_2(a)$. Thus T has a unique fixed point $\hat{w} = T\hat{w}$, $\hat{w} = (\hat{z}, \hat{h})$ in $B_1(a) \times B_2(a)$. It is easily seen that $\hat{z}$ is the (unique) solution of the boundary value problem (1), (2), by means of the same considerations as in the Chain Rule Differentiation Lemma (4.ii) of [7]. Indeed, for this fixed point we find from (8) (with $g = \hat{g}$, $z = \hat{z}$, $\hat{g}(\xi, x, y) = y + \hat{h}(\xi, x, y)$), after integration by parts and simplifications, that

(13) $$0 = A^{-1}(x, y, z(x, y))[\Delta^1(x, y; z, h) + \tilde{\Delta}^2(x, y; z, h) + \Delta^3(x, y; z, h)],$$

where

$$\tilde{\Delta}^2(x, y; z, h) = (\tilde{\Delta}_1^2(x, y; z, h_1), \ldots, \tilde{\Delta}_n^2(x, y; z, h_n))^T,$$

$$\tilde{\Delta}_i^2(x, y; z, h_i) = \int_{a_i}^{x} A^*(t, x, y; z, h) D_t z_i^*(t, x, y; h_i)\, dt, \quad i = 1, \ldots, n,$$

and Δ^1, Δ^3 are defined by (9).

Since $\det A^{-1} \neq 0$, the expression in brackets in (13) is zero. Thus we see that $z = \hat{z}$ satisfies the boundary conditions (2), and (13) reduces to

$$\int_{a_i}^{x} \Big[f_i\big(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z \circ \beta)(t, g_i(t, x, y))\big) - \\ - A_i\big(t, g_i(t, x, y), z(t, g_i(t, x, y))\big) D_t z(t, g_i(t, x, y)) \Big] dt = 0,$$

$i = 1, \ldots, n$, whence (1) follows a.e. in D_a, by the same considerations as in [6], [7] (in particular, using the group property of $\hat{g}$).

We only need to show that $\hat{z}[\psi]$ depends continuously on ψ. For this purpose, let us define by

$$\|\psi\|_{S_0} = \sup_{y\in\mathbf{R}^m} |\psi(y)|_m$$

the norm in $S_0 = [C(\mathbf{R}^m)\cap L_\infty(\mathbf{R}^m)]^n$, and for arbitrary ψ, $\bar{\psi}$ satisfying Assumption H_1 let $z = \hat{z}[\psi]$, $\bar{z} = \hat{z}[\bar{\psi}]$, $h = \hat{h}[\psi]$, $\bar{h} = \hat{h}[\bar{\psi}]$ be the corresponding fixed points. Then we obtain, by the same argument as in the proof of Lemma 3,

$$\|z-\bar{z}\|_{S_1} \leqslant (1+\sigma_2)\|\psi-\bar{\psi}\|_{S_0} + \lambda_1\|h-\bar{h}\|_{S_2} + \lambda_3\|z-\bar{z}\|_{S_1},$$
$$\|h-\bar{h}\|_{S_2} \leqslant \lambda_6\|h-\bar{h}\|_{S_2} + \lambda_4\|z-\bar{z}\|_{S_1},$$

whence

$$\|h-\bar{h}\|_{S_2} \leqslant \lambda_4(1-\lambda_6)^{-1}\|z-\bar{z}\|_{S_1} = \lambda L_a\|z-\bar{z}\|_{S_1}. \tag{14}$$

Combining the above relations, we get

$$\|z-\bar{z}\|_{S_1} = \|\hat{z}[\psi]-\hat{z}[\bar{\psi}]\|_{S_1} \leqslant (1+\sigma_2)(1-\bar{k})^{-1}\|\psi-\bar{\psi}\|_{S_0}, \tag{15}$$

where the constant $\bar{k} = \lambda_3+\lambda_1\lambda_4(1-\lambda_6)^{-1}$ satisfies $0<\bar{k}<k$. Relations (14) and (15) show that the fixed point $\hat{w} = \hat{w}[\psi]$ depends continuously on ψ. This concludes the proof.

Appendix

System (1) can be written in the matrix form

$$AD_x z + \sum_{k=1}^{m} R_k A D_{y_k} z = f,$$

where

$$A = [A_{ij}(x, y, z(x, y))]_{i,j=1,\ldots,n},$$

$$R_k = \operatorname{diag}[\varrho_{1k}(x, y, z(x, y), (z\circ\alpha)(x, y)), \ldots, \varrho_{nk}(x, y, z(x, y), (z\circ\alpha)(x, y))],$$

$$f = (f_1(x, y, z(x, y), (z\circ\beta)(x, y)), \ldots, f_n(x, y, z(x, y), (z\circ\beta)(x, y))).$$

With this notation, it is easy to recognize (as was done in [1]) that a generic first order system of partial differential equations

$$\sum_{j=0}^{m} A_j D_{y_j} z = f, \qquad y_0 = x,$$

can be reduced to the bicharacteristic form (1) iff the matrices $A_0^{-1}A_r$ $(r = 1, \ldots, m)$ commute, i.e., iff "linear shocks" are ruled out (in G. Boillat's terminology).

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