A boundary value problem for quasilinear hyperbolic systems with a retarded argument

by Z. Kamont and J. Turo (Gdańsk)

Abstract. A theorem of existence, uniqueness and continuous dependence on boundary data is proved concerning a.e. solutions of a boundary value problem for systems of quasilinear hyperbolic differential equations with retarded argument, including the Cauchy problem as a particular case. The proof is based on recent results due to P. Bassanini for quasilinear hyperbolic systems without a retarded argument.

1. Introduction. We shall use the symbols \( D_x z(x, y) = (D_x z_1(x, y), \ldots, D_x z_n(x, y)) \), and \( D_x z(x, y) = [D_y z_j(x, y)]_i, \ i = 1, \ldots, m, \ j = 1, \ldots, n \), to denote the partial derivatives of a vector function \( z: D_a = I_a \times \mathbb{R}^m \to \mathbb{R}^n \), where \( I_a = [0, a], a \geq 0 \) and \( D_x z_i(x, y) = \partial z_i(x, y)/\partial x \).

We consider quasilinear hyperbolic systems with retarded argument in the second canonical [6] (or bicharacteristic [9]) form

\[
\sum_{j=1}^{n} A_{ij}(x, y, z(x, y))[D_x z_j(x, y) + \sum_{k=1}^{m} \theta_{ik}(x, y, z(x, y), (z \circ \alpha)(x, y)) D_y z_j(x, y)] = f_i(x, y, z(x, y), (z \circ \beta)(x, y)), \quad i = 1, \ldots, n,
\]

\((x, y) \in D_a, \ y = (y_1, \ldots, y_m) \in \mathbb{R}^m, m \geq 1, \ z(x, y) = (z_1(x, y), \ldots, z_n(x, y)), \) and \((z \circ \alpha)(x, y) = z(\alpha(x, y)), \ \alpha(x, y) = (\alpha_0(x), \alpha'(x, y)), \alpha'(x, y) = (\alpha_1(x, y), \ldots, \alpha_m(x, y))\). In a similar way we define \(z \circ \beta\) with \(\beta(x, y) = (\beta_0(x), \beta'(x, y)), \\beta'(x, y) = (\beta_1(x, y), \ldots, \beta_m(x, y))\).

In the present paper we prove, by means of the fixed point theorem in the product of two Banach spaces [5], a theorem of existence, uniqueness and continuous dependence on the data for systems (1) with the general boundary data [11], [1]

\[
\sum_{k=1}^{N} B_k(y)z(a_k, y) = \psi(y),
\]
for an arbitrary given system of numbers \( a_k, 0 \leq a_k \leq a, k = 1, \ldots, N, n \leq N < + \infty \), and given functions \( B_k(y) = [B_{kij}(y)], i, j = 1, \ldots, n, \psi(y) = (\psi_1(y), \ldots, \psi_n(y))^T \), where \( T \) is the transpose symbol. The boundary condition (2) includes the boundary condition "à la Cesari" [2], [6], [7] as a particular case: take \( N = n, B_{kij}(y) = B_{ij}(y) \delta_{ki} \delta_{ij} \) (\( \delta_{ki} \) the Kronecker symbol). If \( B_{kij}(y) = \delta_{ki} \delta_{ij} \), then the boundary condition (2) reduces to the condition considered in [12]. If, furthermore, all \( a_k = 0, k = 1, \ldots, N \), then we have the usual Cauchy condition. Boundary conditions of the form (2) arise, in a suitable version, from problems of mathematical physics [4].

System (1) with non-retarded argument has been investigated by L. Cesari [6], [7], P. Bassanini [1]–[3], M. Cinquini-Cibrario [8], M. Cinquini-Cibrario, S. Cinquini [9], and P. Pucci [11].

The Cauchy problem for system (1) was considered by Z. Kamont, J. Turo [10] by using slightly different methods.

2. Assumptions. We introduce the norms \( |y|_m = \max_{1 \leq k \leq m} |y_k| \) and \( |z|_n = \max_{1 \leq i \leq n} |z_i| \) in \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively. If \( D = [d_{ij}], i = 1, \ldots, n, j = 1, \ldots, m, \) is an \( n \times m \) matrix, then \( D_i = (d_{i1}, \ldots, d_{im}) \). If \( D \) is a square matrix, then \( \|D\| = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |d_{ij}| \). We denote by \( \tilde{\Omega} \) the cube \( [-\Omega, \Omega]^n \subset \mathbb{R}^n, \Omega > 0 \). Let \( a_0 \) be a given positive constant.

Let

\[
A(x, y, z) = E + \tilde{A}(x, y, z),
\]

\[
A^{-1}(x, y, z) = E + \tilde{A}(x, y, z), \quad B_k(y) = E_k + \tilde{B}_k(y),
\]

\[
\sigma_0 = \sup_{y \in \mathbb{R}^n} \sum_{k=1}^{N} ||\tilde{B}_k(y)||, \quad \sigma_1 = \sup_{D_{a_0} \times \mathbb{R}^n} ||\tilde{A}(x, y, z)||, \quad \sigma_2 = \sup_{D_{a_0} \times \mathbb{R}^n} ||\tilde{A}(x, y, z)||,
\]

where \( A^{-1} \) is the inverse matrix to \( A \), \( E = [\delta_{ij}], E_k = [\delta_{ki} \delta_{ij}], i, j = 1, \ldots, n, k = 1, \ldots, N, \) and \( a_0 = D_{a_0} \times \mathbb{R}^m \).

**Assumption H₁.** Suppose that

1° \( A : D_{a_0} \times \tilde{\Omega} \to \mathbb{R}^2 \) is continuous;

2° \( \det A(x, y, z) \geq \kappa > 0 \) in \( D_{a_0} \times \tilde{\Omega} \) for some constant \( \kappa \);

3° there are constants \( H > 0, C > 0 \) and a function \( p : I_{a_0} \to \mathbb{R}_+ = [0, + \infty), \sigma \in L_1[0, a_0], \) such that, for all \( (x, y, z), (\bar{x}, \bar{y}, \bar{z}), (\tilde{x}, y, z) \in D_{a_0} \times \tilde{\Omega}, \) we have

\[
||A(x, y, z)|| \leq H,
\]

\[
||A(x, y, z) - A(x, \bar{y}, \bar{z})|| \leq C \left[ |y - \bar{y}|_m + |z - \bar{z}|_n \right],
\]

\[
||A(x, y, z) - A(\tilde{x}, y, z)|| \leq \int_{\tilde{x}}^{x} p(t) \, dt.
\]
Since \( \text{det } A(x, y, z) \geq \lambda > 0 \) in \( D_{a_0} \times \bar{\Omega} \), it is easily seen that there are constants \( K', C' \) and a function \( p' : I_{a_0} \to R_+ \), \( p' \in L_1 [0, a_0] \), such that, for all \((x, y, z), (\bar{x}, y, \bar{z}) \in D_{a_0} \times \bar{\Omega} \), we have

\[
\|A^{-1}(x, y, z)\| \leq K',
\]

\[
\|A^{-1}(x, y, z) - A^{-1}(x, \bar{y}, \bar{z})\| \leq C' \left[ |y - \bar{y}|_m + |z - \bar{z}|_n \right],
\]

\[
\|A^{-1}(x, y, z) - A^{-1}(\bar{x}, y, z)\| \leq \int x \ p'(t) \ dt.
\]

\textbf{Assumption} \( H_2 \). Suppose that

1° \( q(\cdot, y, z, u) : I_{a_0} \to R^m \) is measurable for every \((y, z, u) \in R^m \times \bar{\Omega} \times \bar{\Omega} \);

2° \( q(x, \cdot) : R^m \times \bar{\Omega} \times \bar{\Omega} \to R^m \) is continuous for a.e. \( x \in I_{a_0} \);

3° there are functions \( b, l : I_{a_0} \to R_+ \), \( b, l \in L_1 [0, a_0] \), such that, for all \((y, z, u), (\bar{y}, \bar{z}, \bar{u}) \in R^m \times \bar{\Omega} \times \bar{\Omega} \), \( i = 1, \ldots, n \), and a.e. \( x \) in \( I_{a_0} \), we have

\[
|q_i(x, y, z, u)|_m \leq b(x),
\]

\[
|q_i(x, y, z, u) - q_i(x, \bar{y}, \bar{z}, \bar{u})|_m \leq l_i(x) \left[ |y - \bar{y}|_m + |z - \bar{z}|_n + |u - \bar{u}|_n \right];
\]

4° \( \alpha_0 : I_{a_0} \to R_+ \) is measurable and \( \alpha_0(x) \leq x \), a.e. in \( I_{a_0} \);

5° \( \alpha'(\cdot, y) : I_{a_0} \to R^m \) is measurable for \( y \in R^m \), and there is a constant \( c \geq 0 \) such that, for all \( y, \bar{y} \in R^m \) and a.e. \( x \) in \( I_{a_0} \), we have

\[
|\alpha'(x, y) - \alpha'(x, \bar{y})|_m \leq c |y - \bar{y}|_m.
\]

\textbf{Assumption} \( H_3 \). Suppose that

1° \( f(\cdot, y, z, u) : I_{a_0} \to R^m \) is measurable for every \((y, z, u) \in R^m \times \bar{\Omega} \times \bar{\Omega} \);

2° \( f(x, \cdot) : R^m \times \bar{\Omega} \times \bar{\Omega} \to R^m \) is continuous for a.e. \( x \in I_{a_0} \);

3° there are functions \( q, l_1 : I_{a_0} \to R_+ \), \( q, l_1 \in L_1 [0, a_0] \), such that, for all \((y, z, u), (\bar{y}, \bar{z}, \bar{u}) \in R^m \times \bar{\Omega} \times \bar{\Omega} \) and a.e. \( x \) in \( I_{a_0} \), we have

\[
|f(x, y, z, u)|_n \leq q(x),
\]

\[
|f(x, y, z, u) - f(x, \bar{y}, \bar{z}, \bar{u})|_n \leq l_1(x) \left[ |y - \bar{y}|_m + |z - \bar{z}|_n + |u - \bar{u}|_n \right];
\]

4° \( \beta_0 : I_{a_0} \to R_+ \) is measurable and \( \beta_0(x) \leq x \) a.e. in \( I_{a_0} \);

5° \( \beta'(\cdot, y) : I_{a_0} \to R^m \) is measurable for every \( y \in R^m \), and there is a constant \( d \geq 0 \) such that, for all \( y, \bar{y} \in R^m \) and a.e. \( x \) in \( I_{a_0} \), we have

\[
|\beta'(x, y) - \beta'(x, \bar{y})|_m \leq d |y - \bar{y}|_m.
\]

\textbf{Assumption} \( H_4 \). Suppose that

1° \( \psi : R^m \to R^* \) is continuous and there are constants \( \omega_0, A_0 \), \( 0 \leq \omega_0 < \Omega, A_0 \geq 0 \), such that, for all \( y, \bar{y} \in R^m \), we have

\[
|\psi(y)|_n \leq \omega_0, \quad |\psi(y) - \psi(\bar{y})|_n \leq A_0 |y - \bar{y}|_m;
\]
2\textsuperscript{o} \textbf{B}_k: \mathbb{R}^n \rightarrow \mathbb{R}^{n^2} is continuous, \text{det} \textbf{B}_k(y) \neq 0, \ k = 1, \ldots, \ N, and there is a constant \( \tau_0 \geq 0 \) such that, for all \( y, \ y \in \mathbb{R}^n \), we have

\[
\sum_{k=1}^{N} ||\textbf{B}_k(y) - \textbf{B}_k(y)|| \leq \tau_0 |y - \tilde{y}|_m;
\]

3\textsuperscript{o} \( \sigma_0 < 1, \ \zeta = (\sigma_0 + \sigma_1)(1 + \sigma_2) < 1, \) and \( \omega_0(1 + \sigma_2) < \Omega(1 - \zeta) \).

3. Choice of classes \( B_1(a) \) and \( B_2(a) \). Let

\[ q_0 = (1 + \sigma_2)(1 - \zeta)^{-1} \omega_0. \]

Then from 3\textsuperscript{o} of \( \text{H}_a \) it follows that there is a constant \( k, 0 < k < 1, \) such that

\[ \zeta = (\sigma_0 + \sigma_1)(1 + \sigma_2) < k < 1 \text{ and } q_0 < \Omega. \]

We assume \( C\omega_0, C'\omega_0 \) to be so small that

\[ C'\omega_0 + C'q_0(\sigma_0 + \sigma_1) + Cq_0(1 + \sigma_2) < k - \zeta, \]

\[ \delta_0 = C'[\omega_0 + q_0(\sigma_0 + \sigma_1)] + Cq_0(1 + \sigma_2) < 1. \]

Then there certainly is a constant \( s, 0 < s < 1, \) such that

\[ \tilde{\epsilon}_0 = (1 + s)\zeta + C'\omega_0 + C'q_0(\sigma_0 + \sigma_1) + Cq_0(1 + \sigma_2) < 1. \]

Let \( Q \) be a positive constant such that

\[ Q > \tilde{\eta}_0(1 - \tilde{\epsilon}_0)^{-1}, \]

where \( \tilde{\eta}_0 = (1 + s)(1 + \sigma_2)(A_0 + q_0 \tau_0) + (1 + \sigma_2)Cq_0 + C'(\omega_0 + q_0(\sigma_0 + \sigma_1)) \).

Let us take

\[ r(x) = R_0 q(x) + R_1 p(x) + R_2 p(x) + R_3 b(x), \]

where \( R_i, \ i = 0, 1, 2, 3, \) are positive constants satisfying

\[ R_0 > (1 + \sigma_2)(1 - \delta_0)^{-1}, \]

\[ R_1 > q_0(1 + \sigma_2)(1 - \delta_0)^{-1}, \]

\[ R_2 > [\omega_0 + q_0(\sigma_0 + \sigma_1)](1 - \delta_0)^{-1}, \]

\[ R_3 > (1 + \sigma_2)(1 - \delta_0)^{-1}[A_0 + q_0 \tau_0 + \sigma_0 Q + 2q_0 C(1 + Q + \sigma_1 Q)]. \]

We define the following constants:

\[ Q_a = \int_0^a q(x)\,dx, \quad B_a = \int_0^a b(x)\,dx, \]

\[ P_a = \int_0^a p(x)\,dx, \quad P'_a = \int_0^a p'(x)\,dx, \quad L_a = \int_0^a l(x)\,dx, \]

\[ L_{1a} = \int_0^a l_1(x)\,dx, \quad R_a = \int_0^a r(x)\,dx, \quad \gamma = (1 + \sigma_2)[P_a + C(1 + Q)B_a + CR_a]. \]
We assume that \( a \) is so small that \( \gamma + \zeta < k < 1 \).

Now, we can define \( \varrho \) as follows:

\[
\varrho = (1 + \sigma_2)(Q_a + \omega_0)[1 - (\gamma + \zeta)]^{-1}.
\]

Then from 3° of \( H_a \), for \( a \) sufficiently small, we have

\[
(1 + \sigma_2)(Q_a + \omega_0) \leq \Omega[1 - (\gamma + \zeta)];
\]

hence \( \varrho \leq \Omega \).

Let us consider the Banach space \( S_1(a) = [C(D_a) \cap L_{\infty}(D_a)]^n \), \( 0 < a \leq a_0 \), of continuous bounded vector functions \( z: D_a \to \mathbb{R}^n \) with norm \( \|z\|_{S_1} = \sup_{(x, y) \in D_a} |z(x, y)| \).

We denote by \( B_1(a) \) the closed (convex) subset of functions in \( S_1(a) \) satisfying the conditions

\[
|z(x, y)|_n \leq \varrho \leq \Omega, |z(x, y) - z(x, y)|_n \leq Q|y - y|_m,
\]

\[
|z(x, y) - z(x, y)|_n \leq \int \xi r(t) dt, (x, y), (x, y), (x, y) \in D_a,
\]

where \( \varrho, Q \) and \( r \) are defined above.

We also consider the Banach space \( S_2(a) = [C(D_a) \cap L_{\infty}(D_a)]^{nm} \), \( D_a = I_a \times D_a \), of continuous bounded matrix functions \( h = [h_{ik}]: D_a \to \mathbb{R}^{nm}, i = 1, \ldots, n, k = 1, \ldots, m \), with norm \( \|h\|_{S_2} = \max_{1 \leq i \leq n} \sup_{(\xi, x, y) \in D_a} |h_i(\xi, x, y)|_m \).

We denote by \( B_2(a) \) the closed (convex) subset of functions in \( S_2(a) \) whose components satisfy

\[
h_i(x, x, y) = 0,
\]

\[
|h_i(\xi, x, y) - h_i(\xi, x, y)|_m \leq s|y - y|_m,
\]

\[
|h_i(\xi, x, y) - h_i(\xi, x, y)|_m \leq \int \xi b(t) dt,
\]

\[
|h_i(\xi, x, y) - h_i(\xi, x, y)|_m \leq \lambda \int \xi b(t) dt, i = 1, \ldots, n,
\]

for all \( (\xi, x, y), (\xi, x, y), (\xi, x, y), (\xi, x, y) \in D_a \), where \( \lambda = [1/L_a(1 + Q + Qc)]^{-1} \). Here we assume \( a \) to be so small that

\[
L_a(1 + Q + Qc) < 1.
\]

Then all functions in \( B_2(a) \) are uniformly bounded by \( B_a \).

Putting for \( h = [h_{ik}], i = 1, \ldots, n, k = 1, \ldots, m, h \in B_2(a), \)

\[
g_i(\xi, x, y) = y + h_i(\xi, x, y),
\]

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we have
\[ g_i(x, x, y) = y, \quad |g_i(\xi, x, y) - g_i(\xi, x, y)|_m \leq (1 + \sigma) |y - y|_m. \]

Further properties of \( h \) and \( g \) are reported in [6], [7].

Let us define the constants
\[ \gamma = (1 + \sigma)^{-1}, \quad M = \omega _o + Q + \gamma (\gamma + \sigma + \gamma). \]

4. Operator \( T \) and its properties. We now consider an operator \( T = (T_1, T_2) \) defined in \( B_1(a) \times B_2(a) \), where \( T_1 \) and \( T_2 \) are defined in \( B_1(a) \times B_2(a) \) as follows:

\[ (T_1)_{ij} (x, y) = A^{-1} (x, y, z(x, y)) \times \]
\[ \{ A^1(x, y; z, h) + A^2(x, y; z, h) + A^3(x, y; z, h) \} \]
\[ (T_2)_{ij} = [(T_2)_{ij}]_{ij} \]

and

\[ (T_1)_{ij} (x, y) = - \frac{1}{\xi} \int g_i(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z \circ \alpha)(t, g_i(t, x, y)))dt, \]
\[ i = 1, \ldots, n, \]
\[ A^k(x, y; z, h) = (A^k(x, y; z, h), \ldots, A^k(x, y; z, h))_T \]
\[ A^k(x, y; z, h) = \psi_i(g_i(a, x, y)) + \]
\[ + \int g_i(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z \circ \beta)(t, g_i(t, x, y)))dt, \]
\[ (T_2)_{ij} (x, y) = \frac{1}{\xi} \int D_i \{A^k_{ij} (t, x, y; z, h) \} z(t, x, y; h)dt + \]
\[ + A^k_{ij}(a, x, y; z, h)z(t, x, y; h) \]
\[ A^k(x, y; z, h) = - \sum_{k=1}^N B_{ki}(g_i(a, x, y))z(a, g_i(a, x, y)), \]
\[ A^k(t, g_i(t, x, y), z(t, g_i(t, x, y))), \]
\[ z(t, x, y; h) = z(t, g_i(t, x, y)), \]
\[ g_i \text{ is defined by (7)}. \]

Note that, because of (3), we may simultaneously replace \( D_i, A^k, A_i \) and \( B_{ki} \) in the above equalities by \( D_i, A^k, A_i \) and \( B_{ki} \), respectively.

**Lemma 1.** Let Assumptions \( H_1 - H_4 \) hold. Then for \( a, 0 < a \leq a_0, C \omega _o \) and \( C \omega _o \) sufficiently small, the operator \( T \) maps \( B_1(a) \times B_2(a) \) into \( B_1(a) \).

**Proof.** By applying the Chain Rule Differentiation Lemma (4) (ii) of [6], we can show that
\[ |D_i A^k(x, y; z, h)|_m \leq p(t) + C(1 + Q) b(t) + Cr(t), \]
and
\[ |D_i z^i(t, x, y; h_i)|_n \leq r(t) + Qb(t), \quad i = 1, \ldots, n. \]

Hence, for all \((x, y) \in D_a\), we have
\[
\|T_{h}^{(1)} z(x, y)\|_n \leq (1 + \sigma_2)(Q_a + \omega_0) + \varepsilon(1 + \sigma_2)(\bar{y} + \sigma_0 + \sigma_1)
= (1 + \sigma_2)(Q_a + \omega_0) + \varepsilon(\gamma + \zeta) = \varepsilon \leq \Omega.
\]

For any two points \((x, y), (x, \bar{y}) \in D_a\), we can write
\[
(T_{h}^{(1)} z)(x, y) - (T_{h}^{(1)} z)(x, \bar{y}) = v_0 + v_1 + v_2 + v_3,
\]
where
\[
v_0 = \left[ A^{-1}(x, y, z(x, y)) - A^{-1}(x, \bar{y}, z(x, \bar{y})) \right] \times
\times [A^1(x, y; z, h) + A^2(x, y; z, h) + A^3(x, y; z, h)],
\]
\[
v_k = A^{-1}(x, \bar{y}, z(x, \bar{y})) [A^k(x, y; z, h) - A^k(x, \bar{y}; z, h)], \quad k = 1, 2, 3,
\]
and
\[
|v_0|_n \leq C' (1 + \Omega) M_a |y - \bar{y}|_m,
\]
\[
|v_1|_n \leq \left\| A^{-1}(x, y, z(x, y)) \right\| \left\{ \max_{1 \leq i \leq n} \left| \psi_i(g_i(a_i, x, y)) - \psi_i(g_i(a_i, x, \bar{y})) \right| + \right.
+ \max_{1 \leq i \leq n} \left| \int f_i(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z \circ \beta)(t, g_i(t, x, y))) -
- f_i(t, g_i(t, x, \bar{y}), z(t, g_i(t, x, \bar{y})), (z \circ \beta)(t, g_i(t, x, \bar{y}))) \right| dt \right\}
\leq (1 + \sigma_2) [A_0(1 + s) + L_{1a}(1 + s)(1 + \Omega + \Omega d)] |y - \bar{y}|_m,
\]
\[
|v_2|_n \leq (1 + \sigma_2) \left\{ \max_{1 \leq i \leq n} \left| \int_{x_i}^{x} \left[ A_i^*(t, x, y; z, h) - A_i^*(t, x, \bar{y}; h) \right] dt \right| + \right.
\]
\[+ \max_{1 \leq i \leq n} \left| D_i A_i^*(t, x, \bar{y}; z, h) \right| z^i(t, x, y; h) - A_i^*(t, x, \bar{y}; h) | dt | + \right.
\]
\[+ \max_{1 \leq i \leq n} \left[ \int_{x_i}^{x} \left[ A_i^*(t, x, y; z, h) - A_i^*(t, x, \bar{y}; h) \right] z(t, x, y) \right| dt | + \right.
\]
\[+ \max_{1 \leq i \leq n} \left[ A_i^*(a_i, x, y; z, h) - A_i^*(a_i, x, \bar{y}; h) \right] z^i(a_i, x, y; h) - A_i^*(a_i, x, \bar{y}; h) | dt | \right\}
\leq (1 + \sigma_2) [C(R_a + QB_a)(1 + s)(1 + \Omega) + \bar{Q} (1 + s) +
+ \varepsilon C(1 + \Omega + \sigma_1 Q (1 + s)) \times |y - \bar{y}|_m,
\]
\[
|v_2|_n \leq (1 + \sigma_2) \left\{ \max_{1 \leq i \leq n} \left| \sum_{k=1}^{N} \left[ B_{ki}(g_i(a_i, x, y)) -
- B_{ki}(g_i(a_i, x, \bar{y}))) \right] z(a_k, g_i(a_i, x, y)) \right| + \right.
\]
\[
+ \max_{1 \leq i \leq n} \left\{ \sum_{k=1}^{N} \mathcal{B}_{k} \left( g_{i}(a_{i}, x, y) \right) \left[ z(a_{k}, g_{i}(a_{i}, x, y)) - z(a_{k}, g_{i}(a_{i}, x, y)) \right] \right\} \\
\leq (1 + \sigma_{2})(1 + s)(\varrho \tau_{0} + \sigma_{0} Q) |y - \bar{y}|_{m},
\]

\[
\bar{A}^{*}(t, x, \gamma; z, h) = \bar{A}_{i}(t, g_{i}(t, x, y), z(t, g_{i}(t, x, y))), \quad i = 1, \ldots, n.
\]

Summarizing, we get

\[
\|(T_{n}^{(1)} z)(x, y) - (T_{n}^{(1)} z)(x, \bar{y})\|_{n} \leq (\varepsilon Q + \bar{\eta}) |y - \bar{y}|_{m},
\]

where

\[
\varepsilon = C' M_{a} + (1 + \sigma_{2}) \varrho C + \gamma (1 + s) +
\]

\[
+ (1 + \sigma_{2}) (1 + s) [(1 + d) L_{1a} + C(R_{a} + QB_{a}) + \sigma_{0} + \sigma_{1}],
\]

\[
\bar{\eta} = C' M_{a} + (1 + \sigma_{2}) \varrho C + (1 + \sigma_{2}) (1 + s) [L_{1a} + A_{0} + C(R_{a} + QB_{a}) + \varrho \tau_{0}].
\]

We now assume

\[
\varepsilon Q + \bar{\eta} \leq Q,
\]

which is certainly satisfied for a sufficiently small; then

\[
\|(T_{n}^{(1)} z)(x, y) - (T_{n}^{(1)} z)(x, \bar{y})\|_{n} \leq Q |y - \bar{y}|_{m}.
\]

For any two points \((x, y), (\bar{x}, \bar{y}) \in D_{a}\), we have

\[
(T_{n}^{(1)} z)(x, y) - (T_{n}^{(1)} z)(\bar{x}, \bar{y}) = \mu_{0} + \mu_{1} + \mu_{2} + \mu_{3},
\]

where

\[
\mu_{0} = \left[ A^{-1}(x, y, z(x, y)) - A^{-1}(\bar{x}, y, z(\bar{x}, y)) \right] \times
\]

\[
\times \left[ A^{1}(x, y; z, h) + A^{2}(x, y; z, h) + A^{3}(x, y; z, h) \right],
\]

\[
\mu_{k} = A^{-1}(\bar{x}, y, z(\bar{x}, y)) \left[ A^{k}(x, y; z, h) - A^{k}(\bar{x}, y; z, h) \right], \quad k = 1, 2, 3,
\]

and

\[
|\mu_{0}|_{n} \leq M_{a} \left[ \int x P'(t) \, dt \right] + C' \left[ \int x R(t) \, dt \right],
\]

\[
|\mu_{1}|_{n} \leq (1 + \sigma_{2}) \left[ A_{0} \lambda \left[ \int x b(t) \, dt \right] + \left[ \int x q(t) \, dt \right] + L_{1a}(1 + Q + Qd) \lambda \left[ \int x b(t) \, dt \right],
\]

\[
|\mu_{2}|_{n} \leq (1 + \sigma_{2}) \left[ C(1 + Q)(R_{a} + QB_{a}) \lambda \left[ \int x b(t) \, dt \right] + \bar{\eta} \lambda \left[ \int x b(t) \, dt \right] +
\]

\[
+ \varrho \left[ \int x p(t) \, dt \right] + C(1 + Q) \left[ \int x b(t) \, dt \right] + C \left[ \int x R(t) \, dt \right] +
\]

\[
+ \mu_{1} \lambda \left[ \int x b(t) \, dt \right] + C(1 + Q) \left[ \int x b(t) \, dt \right] + C \left[ \int x R(t) \, dt \right] +
\]

\[
\]
\[ + qC (1 + Q) \lambda \int_0^x |f(t)\, dt| + \sigma_1 Q \lambda \int_0^x |g(t)\, dt|, \]

\[ |\mu_3|_m \leq (1 + \sigma_2)(\tau_0 \rho + \sigma_0 Q) \lambda \int_0^x |h(t)|. \]

Hence

\[ |(T_h^{(1)} z)(x, y) - (T_h^{(1)} z)(\bar{x}, \bar{y})|_n \]

\[ \leq \gamma_0 \int_0^x |q(t)\, dt| + \gamma_1 \int_0^x |p(t)\, dt| + \gamma_2 \int_0^x |p'(t)\, dt| + \gamma_3 \int_0^x |b(t)\, dt| + \delta \int_0^x |r(t)\, dt|, \]

where

\[ \gamma_0 = 1 + \sigma_2, \quad \gamma_1 = (1 + \sigma_2) \rho, \quad \gamma_2 = M_a, \]

\[ \gamma_3 = (1 + \sigma_2)[A_0 \lambda + L_{1a}(1 + Q + Qd) \lambda + C(1 + Q)(R_0 + QB_a) \lambda + \lambda \gamma_0 + \rho C(1 + Q) + \rho C(1 + Q) \lambda + \sigma_1 Q \lambda + (\tau_0 \rho + \sigma_0 Q) \lambda], \]

\[ \delta = M_a C + \rho C(1 + \sigma_2). \]

We assume that \( a \) is so small that

\[ \gamma'_k \leq (1 - \delta) R_k, \quad k = 0, \ldots, 3, \]

i.e., \( \gamma'_k + \delta R_k \leq R_k, \quad k = 0, \ldots, 3 \), where \( R_k \) are defined by (5). Then (10) yields

\[ |(T_h^{(1)} z)(x, y) - (T_h^{(1)} z)(\bar{x}, \bar{y})|_n \leq \int_0^x |r(t)|. \]

Thus \( T^{(1)}: B_1(a) \times B_2(a) \rightarrow B_1(a) \). This completes the proof.

**Lemma 2.** If Assumption \( H_2 \) is satisfied and \( a, 0 < a \leq a_0 \), is so small that

\[ L_a(1 + s)(1 + Q + Qc) \leq s, \quad L_a(1 + Q + Qc) < 1, \]

then the operator \( T^{(2)} \) maps \( B_1(a) \times B_2(a) \) into \( B_2(a) \).

**Proof.** Note that, for all \( z \in B_1(a), \ h \in B_2(a) \), the function \( T_z^{(2)} h \) is continuous, and that

\[ (T_z^{(2)} h)_k(x, x, y) = 0, \]

\[ |(T_z^{(2)} h)_k(\xi, x, y) - (T_z^{(2)} h)_k(\xi, x, y)|_m \leq |\int_0^\xi |f(t)\, dt|, \]

\[ |(T_z^{(2)} h)_k(\xi, x, y) - (T_z^{(2)} h)_k(\xi, x, \bar{y})|_m \]

\[ \leq |\int_0^\xi |f(t)(1 + Q + Qc)| g_i(t, x, y) - g_i(t, x, \bar{y})|_m \, dt| \]

\[ \leq L_a(1 + s)(1 + Q + Qc) |y - \bar{y}| \leq s |y - \bar{y}|_m. \]
\[ \| (T^{(2)}_2) h_1 \|_{L_p} (\xi, x, y) - (T^{(2)}_2) h_1 \|_{L_p} (\xi, \bar{x}, y) \leq \int_\pi^x \int_0^1 \left[ \phi(t, g(t, x, y), z(t, g(t, x, y)), (z \circ \alpha)(t, g(t, x, y))) \right] dt_m + \]
\[ + \int_\pi^x \left[ \phi(t, g(t, x, y), z(t, g(t, x, y)), (z \circ \alpha)(t, g(t, x, y))) \right] x - \phi(t, g(t, \bar{x}, y), z(t, g(t, \bar{x}, y)), (z \circ \alpha)(t, g(t, \bar{x}, y))) \right] dt_m \]
\[ \leq \left[ \int b(t) dt \right] + L_a (1 + Q + Qo) \lambda \left[ \int b(t) dt \right] = \lambda \left[ \int b(t) dt \right], \quad i = 1, \ldots, n. \]

Hence we conclude that \( T^{(2)}_2 h \) belongs to \( B_2(a) \). Thus the lemma is proved.

**Lemma 3.** If Assumptions \( H_1-H_4 \) are satisfied, then the operators \( T^{(i)} : B_1(a) \times B_1(a) \rightarrow B_1(a) \) are Lipschitz continuous.

**Proof.** Indeed, for all \( z, \bar{z} \in B_1(a) \), \( h, \bar{h} \in B_2(a) \), we have
\[ \| T^{(1)}_1 z - T^{(1)}_1 \bar{z} \|_{S_1} \leq (1 + \sigma_2) \{ \| A_0 + L_1 a_1 (1 + Q + Qd) + \sigma_0 Q + \xi \sigma_0 \| \times \}
\[ \times \| h - \bar{h} \|_{S_2} + \| A^2 (z, h) - A^2 (z, \bar{h}) \|_{S_1} \}. \]

Integrating by parts, we can write
\[ A^2 (z, h) = \int_0^x \left[ A^* (t, x, y; z, h) \right] [z^* (t, x, y; h) - z^* (t, x, y; \bar{h})] dt - \]
\[ \int_0^x \left[ \tilde{A}^* (t, x, y; z, h) - \tilde{A}^* (t, x, y; z, \bar{h}) \right] D_t z_t (t, x, y; h) dt + \]
\[ + \tilde{g}_i (a_i, x, y; z, h) \left[ z^* (a_i, x, y; h) - z^* (a_i, x, y; \bar{h}) \right], \]
with \( \tilde{g}_i (\xi, x, y) = y + \bar{h}_i (\xi, x, y) \). Hence
\[ \| A^2 (z, h) - A^2 (z, \bar{h}) \|_{S_1} \leq \| T^{(1)}_1 z - T^{(1)}_1 \bar{z} \|_{S_1} \leq \lambda_1 \| h - \bar{h} \|_{S_2}, \]
and so
\[ \| T^{(1)}_1 z - T^{(1)}_1 \bar{z} \|_{S_1} \leq \lambda_1 \| h - \bar{h} \|_{S_2}, \]
where
\[ \lambda_1 = (1 + \sigma_2) (A_0 + \xi \sigma_0) + \xi Q + (1 + \sigma_2) \left[ \tilde{Q} + L_1 a_1 (1 + Q + Qd) + \right. \]
\[ \left. + C (1 + Q) (R_a + QB_0) \right]. \]

Similarly, we have
\[ \| T^{(1)}_1 z - T^{(1)}_1 \bar{z} \|_{S_1} \leq [C' M_0 + (1 + \sigma_2) (2L_1 a_1 + \sigma_0)] \| z - \bar{z} \|_{S_1} + \]
\[ + (1 + \sigma_2) \| A^2 (z, h) - A^2 (\bar{z}, h) \|_{S_1}. \]
Integrating by parts, we can write
\[
\Delta^2_t(z, h) - \Delta^2_t(\bar{z}, h) = \int_a^x \left[ D_t A_i^t(t, x, y; \bar{z}, h_i) [z_i^t(t, x, y; h_i) - \bar{z}_i^t(t, x, y; h_i)] dt - \int_a^x \left[ [A_i^t(t, x, y; z, h_i) - \bar{A}^t_i(t, x, y; \bar{z}, h_i)] D_t z_i^t(t, x, y; h_i) dt + \left[ \bar{A}_i(x, y, z(x, y)) \right] z(x, y) + \bar{A}^t_i(a_i, x, y; \bar{z}, h_i) [z_i^t(a_i, x, y; h_i)] - \bar{z}_i^t(a_i, x, y; h_i)] \right].
\]

Hence
\[
||\Delta^2(z, h) - \Delta^2(\bar{z}, h)||_{S_1} \leq [\gamma + C(R_a + QB_a) + C\delta + \sigma_1] ||z - \bar{z}||_{S_1},
\]
and we finally get
\[
||T^{(1)}_a z - T^{(1)}_a \bar{z}||_{S_1} \leq \lambda_3 ||z - \bar{z}||_{S_1},
\]
where
\[
\lambda_3 = [C'\omega_0 + C'\varrho(\sigma_0 + \sigma_1) + C\varrho(1 + \sigma_2) + \zeta] + \{C'\mathcal{Q}_0 + C'\varrho\gamma + \gamma(1 + \sigma_2)[2L_1a + C(R_a + QB_a)]\}.
\]

For the operator $T^{(2)}_a$ we easily obtain
\[
||T^{(2)}_a h - T^{(2)}_a \bar{h}||_{S_2} \leq 2L_a ||z - \bar{z}||_{S_1} + L_a (1 + Q + Qc) ||h - \bar{h}||_{S_2}.
\]
This ends the proof.

5. The main result.

**Theorem.** Let Assumptions H$_1$–H$_4$ hold. Then for $a_0 < a \leq a_0$, $C\omega_0$ and $C'\omega_0$ sufficiently small, and for every system of numbers $a_i$, $0 \leq a_i \leq a_0$, $i = 1, \ldots, N$, there is a vector function $z: \mathbb{R}^a \to \mathbb{R}^n$, $z \in B_1(a)$, satisfying (1) a.e. in $D_a$ and (2) everywhere in $\mathbb{R}^n$. Furthermore, $z$ is unique and depends continuously on $\psi$ in the classes which are described in the proof.

**Proof.** From Lemmas 1–3 we have for $T = (T^{(1)}, T^{(2)}): B_1(a) \times B_2(a) \to B_1(a) \times B_2(a)$
\[
||T^{(1)}_a z - T^{(1)}_a \bar{z}||_{S_1} \leq k_1 ||h - \bar{h}||_{S_2} + k_2 ||z - \bar{z}||_{S_1},
\]
\[
||T^{(2)}_a h - T^{(2)}_a \bar{h}||_{S_2} \leq k_3 ||h - \bar{h}||_{S_2} + k_4 ||z - \bar{z}||_{S_1},
\]
where
\[
k_1 = \lambda_1, \quad k_2 = \lambda_3, \quad k_3 = \lambda_6 = L_a(1 + Q + Qc), \quad k_4 = \lambda_4 = 2L_a,
\]
and we assume (the non-trivial case) $\lambda_1 \lambda_3 \lambda_4 \lambda_6 > 0$.

Thus, the problem is reduced to the general setting considered in [5], with $\lambda_2 = \lambda_5 = 0$, and assumptions (H1), (H2) of [5] are fulfilled.
Accordingly, we choose the weighted 1-norm in $S_1(a) \times S_2(a)$

$$|w|_{S_1 \times S_2} = \alpha |z|_{S_1} + \beta |h|_{S_2}, \quad w = (z, h) \in B_1(a) \times B_2(a),$$

with $\alpha, \beta > 0$ to be specified below, and we require conditions (H3) of [5] to be satisfied:

$$k_2 < k, \quad k_3 < k, \quad k_1 k_4 \leq (k-k_2)(k-k_3),$$

together with

$$k_4(k-k_2)^{-1} \leq \alpha/\beta \leq (k-k_3)/k_1.$$ 

Assumptions (H3) reduce here, because of (6), to the conditions

$$k_2 < k, \quad k_1 k_4 \leq (k-k_2)(k-k_3),$$

which are certainly satisfied for $a$ sufficiently small, by virtue of (4).

Then, as is proved in [5], the map $T: B_1(a) \times B_2(a) \rightarrow B_1(a) \times B_2(a)$ is a contraction with constant $k$, $0 < k < 1$, in the norm (11)

$$||Tw - T\bar{w}||_{S_1 \times S_2} \leq k ||w - \bar{w}||_{S_1 \times S_2}$$

for all $w = (z, h), \bar{w} = (\bar{z}, \bar{h}) \in B_1(a) \times B_2(a)$. Thus $T$ has a unique fixed point $\tilde{w} = T\tilde{w}, \tilde{w} = (\tilde{z}, \tilde{h})$ in $B_1(a) \times B_2(a)$. It is easily seen that $\tilde{z}$ is the (unique) solution of the boundary value problem (1), (2), by means of the same considerations as in the Chain Rule Differentiation Lemma (4.ii) of [7]. Indeed, for this fixed point we find from (8) (with $g = \hat{g}, z = \hat{z}, \hat{g}(\xi, x, y) = y + h(\xi, x, y)$), after integration by parts and simplifications, that

$$0 = A^{-1}(x, y, z(x, y))[A^1(x, y; z, h) + \tilde{A}^2(x, y; z, h) + A^3(x, y; z, h)],$$

where

$$\tilde{A}^2(x, y; z, h) = (\tilde{A}_{1}^2(x, y; z, h_1), ..., \tilde{A}_{n}^2(x, y; z, h_n))^T,$$

$$\tilde{A}^{2^i}(x, y; z, h_i) = \int A^*(t, x, y; z, h)D_tz^i(t, x, y; h_i)dt, \quad i = 1, ..., n,$$

and $A^1, A^3$ are defined by (9).

Since $\det A^{-1} \neq 0$, the expression in brackets in (13) is zero. Thus we see that $z = \tilde{z}$ satisfies the boundary conditions (2), and (13) reduces to

$$\int A_i(t, g_i(t, x, y), z(t, g_i(t, x, y)), (z \circ \beta)(t, g_i(t, x, y)))\left[-A_i(t, g_i(t, x, y), z(t, g_i(t, x, y))D_tz(t, g_i(t, x, y)))dt = 0,\right.$$

$$i = 1, ..., n,$$

whence (1) follows a.e. in $D_{a'}$, by the same considerations as in [6], [7] (in particular, using the group property of $\hat{g}$).

We only need to show that $\tilde{z}[\psi]$ depends continuously on $\psi$. For this
purpose, let us define by
\[
\|\psi\|_{S_0} = \sup_{y \in \mathbb{R}^m} |\psi(y)|_m
\]
the norm in \(S_0 = [C(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)]^r\), and for arbitrary \(\psi, \tilde{\psi}\) satisfying Assumption H, let \(z = \tilde{z} [\psi], \tilde{z} = \tilde{\tilde{z}} [\tilde{\psi}], h = \tilde{h} [\psi], \tilde{h} = \tilde{\tilde{h}} [\tilde{\psi}]\) be the corresponding fixed points. Then we obtain, by the same argument as in the proof of Lemma 3,
\[
\|z - \tilde{z}\|_{S_1} \leq (1 + \sigma_3) \|\psi - \tilde{\psi}\|_{S_0} + \lambda_1 \|h - \tilde{h}\|_{S_2} + \lambda_3 \|z - \tilde{z}\|_{S_1},
\]
\[
\|h - \tilde{h}\|_{S_2} \leq \lambda_5 \|h - \tilde{h}\|_{S_2} + \lambda_4 \|z - \tilde{z}\|_{S_1},
\]
whence
\[
(14) \quad \|h - \tilde{h}\|_{S_2} \leq \lambda_4 (1 - \lambda_5)^{-1} \|z - \tilde{z}\|_{S_1} = \lambda L_u \|z - \tilde{z}\|_{S_1}.
\]
Combining the above relations, we get
\[
(15) \quad \|z - \tilde{z}\|_{S_1} = \|	ilde{z} [\psi] - \tilde{\tilde{z}} [\tilde{\psi}]\|_{S_1} \leq (1 + \sigma_2) (1 - \tilde{k})^{-1} \|\psi - \tilde{\psi}\|_{S_0},
\]
where the constant \(\tilde{k} = \lambda_3 + \lambda_1 \lambda_4 (1 - \lambda_5)^{-1}\) satisfies \(0 < \tilde{k} < k\). Relations (14) and (15) show that the fixed point \(\hat{w} = \hat{w} [\psi]\) depends continuously on \(\psi\). This concludes the proof.

Appendix

System (1) can be written in the matrix form
\[
AD_x z + \sum_{k=1}^m R_k AD_y z = f,
\]
where
\[
A = [A_{ij}(x, y, z(x, y))]_{i,j=1,...,n},
\]
\[
R_k = \text{diag} [e_{1k}(x, y, z(x, y), (z \circ \alpha)(x, y)), \ldots, e_{nk}(x, y, z(x, y), (z \circ \alpha)(x, y))],
\]
\[
f = (f_1(x, y, z(x, y), (z \circ \beta)(x, y)), \ldots, f_m(x, y, z(x, y), (z \circ \beta)(x, y))).
\]
With this notation, it is easy to recognize (as was done in [1]) that a generic first order system of partial differential equations
\[
\sum_{j=0}^m A_j D_{y_j} z = f, \quad y_0 = x,
\]
can be reduced to the bicharacteristic form (1) iff the matrices \(A_0^{-1} A_r (r = 1, \ldots, m)\) commute, i.e., iff "linear shocks" are ruled out (in G. Boillat's terminology).
References


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