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Extension of a result of Ladyženskaja and Ural'ceva concerning second-order parabolic equations of arbitrary order

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Abstract. In this paper we give conditions under which every weak solution for the initial boundary value problem for semilinear parabolic equations $u'+A(t)u+f(t, x, u, \nabla u, \ldots, \nabla^m u) = 0$ is a regular one. $2m$ denotes the order of the elliptic operators $A(t)$. The result carries over a theorem of Ladyženskaja and Ural'ceva for second order equations to equations of arbitrary order.

0. Introduction and notation. In this paper we deal with the question of finding conditions under which every weak solution on $(0, T)\times\Omega \subset \mathbf{R}^{n+1}$ for the initial boundary value problem for

$$u'+A(t)\,u+f(t, u, \nabla u, \ldots, \nabla^m u) = 0 \tag{0.1}$$

is a regular one. Here $A(t)$ denotes a time dependent elliptic operator of order $2m$ having divergence structure. The answer is affirmative if $n \geqslant 2m-2$ [1] and

$$|f(t, x, u, \nabla u, \ldots, \nabla^m u)| \leqslant c\Big(\sum_{\nu=0}^{m} |\nabla^\nu u|^{1+(4m-2\nu)/(n+2\nu)}+1\Big),$$

$$uf(t, x, u, \nabla u, \ldots, \nabla^m u) \geqslant c' u^2+c'' u.$$

The constant C must be positive, c', c'' may be arbitrary. The growth condition on f is the weakest possible in order to guarantee the existence of the integrals occurring in the definition of a weak solution to (0.1). In [5] we treated the same equation but the non-linearity was not allowed to contain any derivatives of u.

In their book [2] Ladyženskaja and Ural'ceva proved the same result for $m = 1$, i.e. for the case of second order equations. Since no uniqueness theorem is known, the authors had to insert suitable testing functions to achieve the desired result [2], p. 423–425. Our method is necessarily com-

[1] Actually we deal with the case $n \geqslant 2m$ extensively. $n \geqslant 2m-2$ is treated in the remark after Lemma I.1.

pletely different from that of Ladyženskaja and Ural'ceva. It is based on solving an auxiliary equation. It is for this purpose that we need some stronger conditions on $A(t)$, on the initial value and the boundary values.

We introduce some notation. Let X be a Banach space. Then $C^\nu([0, T], X)$ is the Banach space of all ν times continuously differentiable mappings from $[0, T]$ into X. The Banach space $C^{\nu+\alpha}([0, T], X)$, $0 < \alpha < 1$, consists of all ν times continuously differentiable mappings from $[0, T]$ into X with the finite norm

$$\sup_{t,s\in[0,T]} \frac{1}{|t-s|^\alpha} \left\| \frac{d^\nu u}{dt^\nu}(t) - \frac{d^\nu u}{dt^\nu}(s) \right\|_X + \sum_{\lambda=0}^{\nu} \sup_{0\leq t\leq T} \left\| \frac{d^\lambda u}{dt^\lambda}(t) \right\|_X .$$

Further, $L^p((0, T), X)$, $1 \leqslant p \leqslant \infty$, is the Banach space of all measurable mappings $u\colon (0, T) \to X$ with the norm

$$\Big(\int_0^T \|u(t)\|_X^p \, dt\Big)^{1/p}, \qquad \text{respectively,} \qquad \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|u(t)\|_X .$$

If Ω is a bounded open subset of $\boldsymbol{R}^n$, then $H^{\nu,p}(\Omega)$ is the Banach space of all complex-valued functions defined on Ω with distributional derivatives in $L^p(\Omega)$ up to the order ν. Its norm is denoted by $\|\cdot\|_{\nu,p}$ or $\|\cdot\|_{H^{\nu,p}(\Omega)}$. As usual, $\mathring{H}^{\nu,p}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the norm of $H^{\nu,p}(\Omega)$. The norm of $C^0([0, T], H^{\nu,p}(\Omega))$ is denoted by $|||\cdot|||_{\nu,p}$.

Frequently we will consider functions which are continuous on $Q_T := [0, t] \times \bar{\Omega}$. For such functions we introduce the notation

$$\mathbf{l}f\mathbf{l}(r) := \sup_{\substack{t,s\in[0,T]\\ |t-s|\leqslant r\\ x\in\bar{\Omega}}} |f(t, x) - f(s, x)| + \sup_{\substack{t\in[0,T]\\ x,y\in\bar{\Omega}, |x-y|\leqslant r}} |f(t, x) - f(t, y)|,$$

$$\mathbf{l}f\mathbf{l}_x(r) := \sup_{\substack{t\in[0,T]\\ x,y\in\bar{\Omega}\\ |x-y|\leqslant r}} |f(t, x) - f(t, y)| .$$

For $\sup_{(t,x)\in Q_T} |f(t, x)|$ we simply write $|||f|||$.

The Banach spaces $C^\nu(\bar{\Omega})$, $C^{\nu+\alpha}(\bar{\Omega})$ consist of all complex-valued functions defined on $\bar{\Omega}$ which are continuously differentiable up to the order ν in Ω and have derivatives of order ν continuous in $\bar{\Omega}$ or Hölder continuous with exponent α in $\bar{\Omega}$. The norms of $C^\nu(\bar{\Omega})$, $C^{\nu+\alpha}(\bar{\Omega})$ are denoted by $\|\cdot\|_\nu$, $\|\cdot\|_{\nu+\alpha}$, respectively. Finally, we use the notation

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)} := \|\cdot\|_{0,p}, \qquad H^\nu(\Omega) := H^{\nu,2}(\Omega), \qquad \mathring{H}^\nu(\Omega) := \mathring{H}^{\nu,2}(\Omega),$$

$$\|\cdot\| := \|\cdot\|_{0,2},$$

$$D_j := \frac{1}{i}\frac{\partial}{\partial x_j}, \qquad D^\alpha := \prod_{j=1}^{n} D_j^{\alpha_j},$$

where $\tilde{\alpha}$ is a multi-index with components $\alpha_j \in N \cup \{0\}$. N denotes the set of positive integers.

We say that Ω is of class C^ν or $C^{\nu+\alpha}$ if $\partial\Omega$ locally admits a representation

$$x_k = \Phi(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$$

with $\Phi \in C^\nu$ or $C^{\nu+\alpha}$, respectively, and if Ω locally is on one side of $\partial\Omega$.

I. Parabolic equations. Let $\Omega \subset R^n$ be bounded and of class C^{4m}, $m \in N$. Suppose that for each pair of multi-indices $(\tilde{\alpha}, \tilde{\beta}) \in R^n \times R^n$ with $|\tilde{\alpha}| \leqslant m$, $|\tilde{\beta}| \leqslant m$ there are given functions $A_{\tilde{\alpha}\tilde{\beta}}\colon R^+ \times \bar{\Omega} \to C$ with

$$A_{\tilde{\alpha}\tilde{\beta}} \in C^0([0, \tilde{T}], C^m(\bar{\Omega})), \qquad \tilde{T} \geqslant 0, \qquad (1/i)^{|\tilde{\alpha}|+|\tilde{\beta}|} A_{\tilde{\alpha}\tilde{\beta}}(t, x) \in R,$$

$$M|\xi|^{2m} \geqslant \sum_{\substack{|\tilde{\alpha}|=m \\ |\tilde{\beta}|=m}} A_{\tilde{\alpha}\tilde{\beta}}(t, x)\xi^{\tilde{\alpha}}\xi^{\tilde{\beta}} \geqslant M^{-1}|\xi|^{2m}, \qquad \xi \in R^n,\ (t, x) \in R^+ \times \bar{\Omega}.$$

Let f be real-valued and in $L^1((0, \tilde{T}), L^2(\Omega))$ or in $L^2((0, \tilde{T}), L^{2n/(n+2m)}(\Omega))$ for some $\tilde{T} > 0$.

We say that a real-valued element

$$u \in L^2((0, \tilde{T}), H^m(\Omega)) \cap L^\infty((0, \tilde{T}), L^2(\Omega))$$

is a weak solution of the equation

$$u' + \sum_{\substack{|\tilde{\alpha}|\leqslant m \\ |\tilde{\beta}|\leqslant m}} D^{\tilde{\alpha}}(A_{\tilde{\alpha}\tilde{\beta}}(t, x) D^{\tilde{\beta}} u) + f(t) = 0 \tag{I.1}$$

on $(0, \tilde{T}) \times \Omega$ with initial value $\Phi \in L^2(\Omega)$ if and only if the relation

$$-\int_0^{\tilde{T}} (u(t), \psi'(t))\,dt + \sum_{\substack{|\tilde{\alpha}|\leqslant m \\ |\tilde{\beta}|\leqslant m}} \int_0^{\tilde{T}} (A_{\tilde{\alpha}\tilde{\beta}}(t, x) D^{\tilde{\beta}} u(t), D^{\tilde{\alpha}}\psi(t))\,dt + \int_0^{\tilde{T}} (f(t), u(t))\,dt = -(\Phi, \psi(0))$$

holds for all real testing functions ψ with

$$\psi' \in L^2((0, \tilde{T}), L^2(\Omega)), \qquad \psi \in L^2((0, \tilde{T}), \mathring{H}^m(\Omega)), \qquad \psi(\tilde{T}) = 0.$$

Observe that $\psi(0)$, $\psi(\tilde{T})$ make sense since $\psi \in C^0([0, \tilde{T}], L^2(\Omega))$.

We set

$$A(t)u = \sum_{\substack{|\tilde{\alpha}|\leqslant m \\ |\tilde{\beta}|\leqslant m}} D^{\tilde{\alpha}}(A_{\tilde{\alpha}\tilde{\beta}}(t, x) D^{\tilde{\beta}} u), \qquad u \in H^{2m,1}(\Omega),$$

in the distributional sense and often we write instead of (I.1) $u' + A(t)u + f(t) = 0$.

Let n_s be the number of multi-indices α of R^n with $|\alpha| = s$. Let $n \geqslant 2m$. If $h\colon R^+ \times \Omega \times R \times R^{n_1} \times R^{n_2} \times \ldots \times R^{n_m} \to R$ is a measurable function fulfilling the growth condition

$$\text{(I.2)}\qquad |h(t, x, p_0, p_1, p_2, \ldots, p_m)| \leqslant c_1 + c_2 \sum_{\nu=0}^{m} |p_\nu|^{1+(4m-2\nu)/(n+2\nu)},$$

$$p_0 \in \boldsymbol{R},\ p_\nu \in \boldsymbol{R}^{n^\nu},\ 1 \leqslant \nu \leqslant m,$$

with positive constants c_1, c_2, and if, moreover, h is continuous in $(p_0, \ldots, p_m)$, then

$$\|h(t, u, \nabla u, \ldots, \nabla^m u)\|_{L^{2n/(n+2m)}(\Omega)} \leqslant c(n, m, \Omega, c_1, c_2) \left(\|u\|_{m,2} \cdot \sum_{\nu=0}^{m} \|u\|_{L^2(\Omega)}^{(4m-2\nu)/(n+2\nu)} + 1\right),$$

$u \in H^m(\Omega)$ (see [1], p. 27). Thus we see that for a function

$$u \in L^2((0, \tilde{T}), H^m(\Omega)) \cap L^\infty((0, \tilde{T}), L^2(\Omega))$$

the expression $h(t, u, \nabla u, \ldots, \nabla^m u)$ fulfils

$$h(t, u, \nabla u, \ldots, \nabla^m u) \in L^2((0, \tilde{T}), L^{2n/(n+2m)}(\Omega)).$$

Therefore it makes sense to speak of weak solutions for the equation

$$u' + A(t)u + h(t, u, \nabla u, \ldots, \nabla^m u) = 0$$

on $(0, \tilde{T}) \times \Omega$ with initial values from $L^2(\Omega)$ (see also [2], p. 423, for $m = 1$).

Finally we assume that Gårding's inequality

$$\text{(I.3)}\qquad \operatorname{Re} \sum_{\substack{|\bar{\alpha}| \leqslant m \\ |\bar{\beta}| \leqslant m}} (A_{\bar{\alpha}\bar{\beta}}(t) D^{\bar{\alpha}} u, D^{\bar{\beta}} u) \geqslant c(n, m, \Omega, M, |A_{\bar{\alpha}\bar{\beta}}|_x) \|u\|_{m,2}^2,$$

$$u \in \mathring{H}^m(\Omega),$$

holds with a positive constant $c(n, m, \ldots)$. As regards the parabolic equations treated here this yields no loss of generality. Moreover, let $T > 0$ be arbitrary but fixed.

Let f be a measurable function

$$f\colon \boldsymbol{R}^+ \times \Omega \times \boldsymbol{R} \times \boldsymbol{R}^{n_1} \times \ldots \times \boldsymbol{R}^{n_m} \to \boldsymbol{R}$$

with the following properties: f is continuous in $(p_0, \ldots, p_m)$,

$$\text{(I.4)}\qquad |f(t, x, p_0, \ldots, p_m)| \leqslant \hat{K} + \hat{M} \sum_{\nu=0}^{m} |p_\nu|^{1+(4m-2\nu)/(n+2\nu)},$$

$$\text{(I.5)}\qquad uf(t, x, u, p_1, p_2, \ldots, p_m) \geqslant L'u^2 + L''u,$$

with positive constants $\hat{M}$, $\hat{K}$ and arbitrary constants L', L''.

Now we introduce a new nonlinearity, namely

$$F(t, x, \omega, p_1, \ldots, p_m) = q(t, x)\tilde{q}(t, x)\,\{|L''|+\hat{K}\}+\tilde{F}(t, x, \omega, p_1, \ldots, p_m),$$

$$\tilde{F}(t, x, \omega, p_1, \ldots, p_m) = q(t, x)\frac{\omega}{\delta(|\omega|)}\,\Big\{|L'|\,|\omega|+\hat{M}\sum_{\nu=1}^{m}|p_\nu|^{1+(4m-2\nu)/(n+2\nu)}+\hat{M}\,|\omega|^{1+(4m/n)}\Big\}.$$

Here $\delta(|\omega|) = 1$ if $\omega = 0$ and $\delta(|\omega|) = |\omega|$ if $\omega \neq 0$. The functions $F, q, \tilde{q}$ depend on the weak solution u under consideration. Namely, if u is an arbitrary but fixed weak solution for the equation

$$u'+A(t)u+f(t, u, \nabla u, \ldots, \nabla^m u) = 0$$

on $(0, T)\times\Omega$ with initial value $\Phi\in C^{2m}(\bar{\Omega})$, $D^{\bar{\alpha}}\Phi|\partial\Omega = 0$, $|\bar{\alpha}|\leqslant m-1$, and with $u(t)\in \mathring{H}^m(\Omega)$ for almost every $t\in(0, T)$, then

$$\text{(I.6)}\quad q(t, x) = \begin{cases} \dfrac{f(t, x, u, \nabla u, \ldots, \nabla^m u)-L'u-L''}{\dfrac{u}{\delta(|u|)}\,\Big\{\hat{M}\sum\limits_{\nu=0}^{m}|\nabla^\nu u|^{1+(4m-2\nu)/(n+2\nu)}+\hat{K}+|L''|+|L'|\,|u|\Big\}}, & u(t, x)\neq 0,\\ 0, \quad u(t, x) = 0, \end{cases}$$

$$\tilde{q}(t, x) = u(t, x)/\delta\big(|u(t, x)|\big).$$

Since $u\big(f(t, x, u, \nabla u, \ldots, \nabla^m u)-L'u-L''\big)\geqslant 0$, q has the following property: $q(t, x)\geqslant 0$ a.e. in $(0, T)\times\Omega$. It is evident that $|q(t, x)|\leqslant 1$ a.e. $(0, t)\times\Omega$.

LEMMA I.1. *We have*

$$\big(\tilde{F}(t, \omega_2, \nabla\omega_2, \ldots, \nabla^m\omega_2)-\tilde{F}(t, \omega_1, \nabla\omega_1, \ldots, \nabla^m\omega_1), \omega_2-\omega_1\big) \geqslant -\varepsilon\,\|\omega_2-\omega_1\|_m^2-c(\varepsilon, n, m, \Omega, \|\omega_1\|_{L^2(\Omega)}+\|\omega_2\|_{L^2(\Omega)})\times \times(\|\omega_2\|_m^2+\|\omega_1\|_m^2)\|\omega_2-\omega_1\|_{L^2(\Omega)}^2, \quad \omega_1, \omega_2\in H^m(\Omega),\ 1>\varepsilon>0.$$

Proof. We have

$$\big(\tilde{F}(t, \omega_2, \nabla\omega_2, \ldots, \nabla^m\omega_2)-\tilde{F}(t, \omega_1, \nabla\omega_1, \ldots, \nabla^m\omega_1)\big)(\omega_2-\omega_1)$$

$$= q(t, x)\left(\frac{\omega_2}{\delta(|\omega_2|)}-\frac{\omega_1}{\delta(|\omega_1|)}\right)\{|L'|\,|\omega_2|+ +\hat{M}\sum_{\nu=0}^{m}|\nabla^\nu\omega_2|^{1+(4m-2\nu)/(n+2\nu)}\}\,(\omega_2-\omega_1)+$$

$$+\,q(t, x)\frac{\omega_1}{\delta(|\omega_1|)}\,\{|L'|\,(|\omega_2|-|\omega_1|)+ +\hat{M}\sum_{\nu=0}^{m}(|\nabla^\nu\omega_2|^{1+(4m-2\nu)/(n+2\nu)}-|\nabla^\nu\omega_1|^{1+(4m-2\nu)/(n+2\nu)})\}\cdot(\omega_2-\omega_1).$$

Since $\omega/\delta(|\omega|)$ is monotonically non-decreasing and bounded, the first term on the right is $\geqslant 0$ and integrable. As regards the second term we may restrict ourselves to

$$(|\nabla^\nu \omega_2|^{1+(4m-2\nu)/(n+2\nu)} - |\nabla^\nu \omega_1|^{1+(4m-2\nu)/(n+2\nu)})\,(\omega_2-\omega_1).$$

We have

$$\int_\Omega \big||\nabla^\nu \omega_2|^{1+(4m-2\nu)/(n+2\nu)} - |\nabla^\nu \omega_1|^{1+(4m-2\nu)/(n+2\nu)}\big|\,|\omega_2-\omega_1|\,dx$$

$$\leqslant c(n,m)\int_\Omega \big||\nabla^\nu \omega_2|^{(4m-2\nu)/(n+2\nu)} + |\nabla^\nu \omega_1|\,|\nabla^\nu(\omega_2-\omega_1)|\big|\,\omega_2-\omega_1\,|\,dx$$

$$\leqslant c(n,m)\,\|\omega_2-\omega_1\|_\nu\,(\|\nabla^\nu \omega_2\|_{L^2(\Omega)}^{(4m-2\nu)/(n+2\nu)} +$$

$$+\|\nabla^\nu \omega_1\|_{L^2(\Omega)}^{(4m-2\nu)/(n+2\nu)})\,\|\omega_2-\omega_1\|_{L^{2(n+2\nu)/(n+4\nu-4m)}(\Omega)}, \qquad n>4m-4\nu,$$

$$\leqslant c(n,m,\Omega)\,\|\omega_2-\omega_1\|_m^{\nu/m}\,\|\omega_2-\omega_1\|_{L^2(\Omega)}^{1-\nu/m}\,\big(\|\omega_2\|_m^{(\nu/m)(4m-2\nu)/(n+2\nu)}\times$$

$$\times\|\omega_2\|_{L^2(\Omega)}^{(1-\nu/m)(4m-2\nu)/(n+2\nu)} + \|\omega_1\|_m^{(\nu/m)(4m-2\nu)/(n+2\nu)}\,\|\omega_1\|_{L^2(\Omega)}^{(1-\nu/m)(4m-2\nu)/(n+2\nu)}\big)\times$$

$$\times\|\omega_2-\omega_1\|_m^{(n/m)(4m-2\nu)/2(n+2\nu)}\,\|\omega_2-\omega_1\|_{L^2(\Omega)}^{1-(n/m)(4m-2\nu)/2(n+2\nu)}$$

$$\leqslant c(n,m,\Omega)\,\|\omega_2-\omega_1\|_m^{(\nu/m)+(n/m)(4m-2\nu)/2(n+2\nu)}\times$$

$$\times\|\omega_2-\omega_1\|_{L^2(\Omega)}^{2-(\nu/m)-(n/m)(4m-2\nu)/2(n+2\nu)}\times$$

$$\times(\|\omega_2\|_m^{(\nu/m)(4m-2\nu)/(n+2\nu)}\,\|\omega_2\|_{L^2(\Omega)}^{(1-\nu/m)(4m-2\nu)/(n+2\nu)} +$$

$$+\|\omega_1\|_m^{(\nu/m)(4m-2\nu)/(n+2\nu)}\,\|\omega_1\|_{L^2(\Omega)}^{(1-\nu/m)(4m-2\nu)/(n+2\nu)})$$

$$\leqslant c(n,m,\Omega)\cdot\varepsilon\,\|\omega_2-\omega_1\|_m^2 + c(n,m,\Omega,\varepsilon)\,\|\omega_2-\omega_1\|_{L^2(\Omega)}^2\times$$

$$\times(\|\omega_2\|_m^2\,\|\omega_2\|_{L^2(\Omega)}^{[(1-\nu/m)(4m-2\nu)/(n+2\nu)]\cdot 2/[2-\nu/m)-(n/m)(4m-2\nu)/2(n+2\nu)]} +$$

$$+\|\omega_1\|_m^2\,\|\omega_1\|_{L^2(\Omega)}^{[(1-\nu/m)(4m-2\nu)/(n+2\nu)]\cdot 2/[2-\nu/m)-(n/m)(4m-2\nu)/2(n+2\nu)]}),$$

see [1]; p. 27.

If $2m \leqslant n \leqslant 4m-4\nu$ we have

$$I_\nu = \int_\Omega \big||\nabla^\nu \omega_2|^{(4m-2\nu)/(n+2\nu)} + |\nabla^\nu \omega_1|^{(4m-2\nu)/(n+2\nu)}\big|\,|\nabla^\nu(\omega_2-\omega_1)|\cdot|\omega_2-\omega_1|\,dx$$

$$\leqslant \|\omega_2-\omega_1\|_{L^2(\Omega)}\,\|\nabla^\nu(\omega_2-\omega_1)\|_{L^{2q_1}(\Omega)}\times$$

$$\times(\|\nabla^\nu \omega_1\|_{L^{2q_2(4m-2\nu)/(n+2\nu)}(\Omega)}^{(4m-2\nu)/(n+2\nu)} + \|\nabla^\nu \omega_2\|_{L^{2q_2(4m-2\nu)/(n+2\nu)}(\Omega)}^{(4m-2\nu)/(n+2\nu)}),$$

where $1/q_1+1/q_2=1$. If $(m-\nu)/n<\frac{1}{2}$ we choose

$$1/2q_1 = \tfrac{1}{2}-(m-\nu)/n, \qquad 1/2q_2 = (m-\nu)/n.$$

Then

$$\frac{1}{[2(4m-2\nu)/(n+2\nu)]\,q_2}=\frac{\nu}{n}+a(\tfrac{1}{2}-m/n)+(1-a)\tfrac{1}{2}$$

with

$$a=\frac{1}{(4m-2\nu)/(n+2\nu)}.$$

Because of $n\leqslant 4m-4\nu$ we have $a<1$ and because of $n>2(m-\nu)$ we have $a\geqslant \nu/m$. Thus we arrive at

$$I_\nu\leqslant c(n,m,\Omega)\,\|\omega_2-\omega_1\|_{L^2(\Omega)}\|\omega_2-\omega_1\|_{m,2}\times$$
$$\times\left(\|\omega_1\|_{m,2}\,\|\omega_1\|_{L^2(\Omega)}^{(4m-2\nu)/(n+2\nu)-1}+\|\omega_2\|_{m,2}\,\|\omega_2\|_{L^2(\Omega)}^{(4m-2\nu)/(n+2\nu)-1}\right)$$

(see [1], p. 27). The case $(m-\nu)/n>\frac{1}{2}$, $n\geqslant 2m$, is not possible. Now let $(m-\nu)/n=\frac{1}{2}$. Since $n\geqslant 2m$, we then have $\nu=0$. The term

$$I(t,x)=q(t,x)\hat{M}\left(\frac{\omega_2(t,x)}{\delta(|\omega_2(t,x)|)}|\omega_2(t,x)|^{1+4m/n}-\right.$$
$$\left.-\frac{\omega_1(t,x)}{\delta(|\omega_1(t,x)|)}|\omega_1(t,x)|^{1+4m/n}\right)(\omega_2-\omega_1)(t,x)$$

is $\geqslant 0$ a.e. in $(0,T)\times\Omega$, and, moreover,

$$\|\omega\|_{L^{2+4m/n}(\Omega)}^{2+4m/n}\leqslant c(n,m,\Omega)\,\|\omega\|_{L^{2(1+4m/n)}(\Omega)}^{1+4m/n}\,\|\omega\|_{L^2(\Omega)}$$
$$\leqslant c(n,m,\Omega)\,\|\omega\|_{m,2}^2\,\|\omega\|_{L^2(\Omega)}^{4m/n},$$

by [1], p. 27. Therefore I is integrable over $(0,t)\times\Omega$. Thus the lemma is proved.

Remark. The restriction $n\geqslant 2m$ is in fact needed only for Lemma I.1. If we assume

$$n/2>m-\nu\quad\text{or}\quad\nu=0$$

the second part of the proof of Lemma I.1 goes through as well. This would mean that instead of (I.4) one must have

$$|f(t,x,p_0,\ldots,p_m)|\leqslant\hat{K}+\hat{M}\sum_{\substack{\nu=0\text{ or}\\ n>2(m-\nu)}}^{m}|p_\nu|^{1+(4m-2\nu)/(n+2\nu)}.$$

Next we prove

LEMMA I.2. *Let f be real valued and in $L^2((0,T),L^{2n/(n+2m)}(\Omega))$. Let $\Phi\in L^2(\Omega)$. Then every weak solution on $(0,T)\times\Omega$ of the equation $u'+A(t)u=f(t)$ with initial value Φ and with $u(t)\in\mathring{H}^m(\Omega)$ for a.e. $t\in(0,T)$*

fulfils the a priori estimate

$$\tfrac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \sum_{\substack{|\bar\alpha| \leqslant m \\ |\bar\beta| \leqslant m}} \big(A_{\bar\alpha\bar\beta}(\sigma) D^{\bar\alpha} u(\sigma), D^{\bar\beta} u(\sigma)\big) d\sigma$$

$$\leqslant \tfrac{1}{2}\|\Phi\|^2_{L^2(\Omega)} - \int_0^t \big(f(\sigma), u(\sigma)\big) d\sigma, \qquad t \in (0, T).$$

Proof. This follows from Lions [3], p. 55, by an ordinary approximation argument.

Lemma I.1, Lemma I.2 and Gronwall's inequality yield

LEMMA I.3. *Let $\Phi \in L^2(\Omega)$. Then the equation*

$$\text{(I.7)} \qquad \omega' + A(t)\omega + F(t, \omega, \nabla\omega, \ldots, \nabla^m\omega) = -L'\omega - L''$$

has at most one weak solution on $(0, T) \times \Omega$ with initial value Φ and with $\omega(t) \in \mathring{H}^m(\Omega)$ for a.e. $t \in (0, T)$.

From [6], proof of Theorem II.1, we get with the aid of Lemma I.1

LEMMA I.4. *Let $p > n+1$. Let $\Phi \in C^{2m}(\bar\Omega)$, $D^{\bar\alpha}\Phi | \partial\Omega = 0$, $|\bar\alpha| \leqslant m-1$. Then there exists a unique*

$$\omega \in L^p\big((0, T), H^{2m,p}(\Omega)\big) \cap L^p\big((0, T), \mathring{H}^{m,p}(\Omega)\big)$$

with

$$\omega' \in L^p\big((0, T), L^p(\Omega)\big), \qquad \omega(0) = \varphi,$$

$$\omega' + A(t)\omega + F(t, \omega, \nabla\omega, \ldots, \nabla^m\omega) = -L'\omega - L''.$$

Moreover, the a priori estimate

$$\int_0^T \|\omega'(t)\|^p_{L^p(\Omega)} dt + \int_0^T \|\omega(t)\|^p_{2m,p} dt$$

$$\leqslant c\big(n, m, p, \Omega, T, M, \hat M, \hat K, L', L'', |A_{\bar\alpha\bar\beta}|, |||\nabla^{|\bar\alpha|} A_{\bar\alpha\bar\beta}|||, \|\Phi\|_{2m,p}\big)$$

holds.

Proof. The assumptions of Theorem II.1 in [6] are all fulfilled with the exceptions that F is not Lipschitz continuous with respect to ω. The a priori estimates in the proof of Theorem II.1 in [6] are not affected but the application of Schaefer's fixed point theorem – as it was done in [6], proof of Theorem I.1 – needs a little additional consideration:

If $\{\omega_\nu\}$ is a sequence from

$$\Big\{\omega \,|\, \omega \in L^p\big((0, T), \dot H^{2m,p}(\Omega)\big),$$

$$\omega' \in L^p\big((0, T), L^p(\Omega)\big), \int_0^T \|\omega'(t)\|_{L^p(\Omega)} dt + \int_0^T \|\omega(t)\|_{2m,p} dt \leqslant 1\Big\}$$

with

$$\omega_\nu \rightharpoonup \omega' \quad \text{in} \quad L^p((0, T), L^p(\Omega)), \quad \omega_\nu \rightharpoonup \omega \quad \text{in} \quad L^p((0, T), H^{2m,p}(\Omega))$$

for $\nu \to \infty$ we must show that $F(\omega_\nu, \nabla\omega_\nu, \ldots, \nabla^m \omega_\nu) \to F(\omega, \nabla\omega, \ldots, \nabla^m \omega)$ in $L^p((0, T), L^p(\Omega))$. We have ([1], p. 27)

$$\int_\Omega \left|\frac{\omega_\nu}{\delta(|\omega_\nu|)}\right|^p \Big| \sum_{\mu=0}^m |\nabla^\mu \omega_\nu|^{1+(4m-2\mu)/(n+2\mu)} - \sum_{\mu=0}^m |\nabla^\mu \omega|^{1+(4m-2\mu)/(n+2\mu)}\Big|^p dx$$

$$\leqslant c(n, m, p, \Omega) \sum_{\mu=0}^m \|\omega_\nu - \omega\|_{2m,p}^{p/(1+q_\mu)} \|\omega_\nu - \omega\|_{L^2(\Omega)}^{pq_\mu/(1+q_\mu)} \times$$

$$\times \left(\|\omega_\nu\|_{2m,p}^{pq_\mu/(1+q_\mu)} \cdot \|\omega_\nu\|_{L^2(\Omega)}^{pq_\mu^2/(1+q_\mu)} + \|\omega\|_{2m,p}^{pq_\mu/(1+q_\mu)} \cdot \|\omega\|_{L^2(\Omega)}^{pq_\mu^2/(1+q_\mu)}\right)$$

with $q_\mu = (4m-2\mu)/(n+2\mu)$. From $p > n+1$ it follows that $\omega_\nu \to \omega$ in $C^0([0, T] \times \bar{\Omega})$. Moreover,

$$\lim_{\nu\to\infty} \int_\Omega \left|\frac{\omega_\nu}{\delta(|\omega_\nu|)} - \frac{\omega}{\delta(|\omega|)}\right| \sum_{\mu=0}^m |\nabla^\mu \omega|^{1+(4m-2\mu)/(n+2\mu)} dx$$

$$= \lim_{\nu\to\infty} \int_{\substack{\Omega \\ \omega=0}} \left|\frac{\omega_\nu}{\delta(|\omega_\nu|)} - \frac{\omega}{\delta(|\omega|)}\right| \sum_{\mu=0}^m |\nabla^\mu \omega|^{1+(4m-2\mu)/(n+2\mu)} dx.$$

Since $\nabla^\mu \omega = 0$ a.e. on $\omega = 0$ we see that the last limes is $= 0$. Thus we have in fact $F(\omega_\nu, \nabla\omega_\nu, \ldots, \nabla^m \omega_\nu) \to F(\omega, \nabla\omega, \ldots, \nabla^m \omega)$ in $L^p((0, T), L^p(\Omega))$.

Thus we see that for sufficiently regular initial data Φ with boundary values 0, equation (I.7) has in fact a uniquely determined weak solution ω. Moreover, this solution is a regular one as it follows from Lemma I.4. It is evident that u solves (I.7) weakly. Thus $u = \omega$ and we arrive at our final result, namely

THEOREM I.5. *Let $p > n+1$. Let $\Phi \in C^{2m}(\bar{\Omega})$, $D^{\bar{\alpha}} \Phi | \partial\Omega = 0$, $|\bar{\alpha}| \leqslant m-1$. Let u be a weak solution for the equation*

$$u' + A(t)u + f(t, u, \nabla u, \ldots, \nabla^m u) = 0$$

on $(0, T) \times \Omega$ with initial value Φ and with $u(t) \in \mathring{H}^m(\Omega)$ for almost every $t \in (0, T)$. Then

$$u \in L^p((0, T), H^{2m,p}(\Omega)) \cap L^p((0, T), \mathring{H}^{m,p}(\Omega)), \quad u' \in L^p((0, T), L^p(\Omega)).$$

Moreover, the following a priori estimate holds:

$$\int_0^T \|u'(t)\|_{L^p(\Omega)}^p dt + \int_0^T \|u(t)\|_{2m,p}^p dt$$

$$\leqslant c(n, m, p, \Omega, T, M, \hat{M}, \hat{K}, L', L'', |A_{\bar{\alpha}\bar{\beta}}|, \||\nabla^{|\bar{\alpha}|} A_{\bar{\alpha}\bar{\beta}}|\|, \|\Phi\|_{2m,p}).$$

If we impose suitable additional regularity assumptions on the $A_{\bar{\alpha}\bar{\beta}}$ and f, we can prove that

$$u' \in \bigcap_{\delta, 0<\delta<T} C^{\varepsilon}([\delta, T], C^{\varepsilon}(\bar{\Omega})), \quad u \in \bigcap_{\delta, 0<\delta<T} C^{\varepsilon}([\delta, T], C^{2m+\varepsilon}(\bar{\Omega}))$$

for a sufficiently small $\varepsilon > 0$. For this purpose one has to use the theory of C^{α}-semigroups developed by the author (cf. [4]).

Also non-vanishing boundary values $g(t, x)$, $g \in L^p((0, T), H^{2m,p}(\Omega))$, $g' \in L^p((0, T), L^p(\Omega))$ are admissible.

Moreover, it should be remarked that from our a priori estimate an existence theorem for classical solutions for all t can easily be derived. Our results will remain valid for systems with a one sided condition for the non-linearity, e.g. $u^l \cdot f^l(x, u, \nabla u, \ldots, \nabla^m u) \geqslant 0$, where the u^l are the components of u. The proof of Lemma I.4 being no longer correct in the case of a system, the growth condition (I.4) must be fulfilled also for f^l/u^l, $|u^l| \leqslant 1$.

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