COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 2

CONCERNING HEREDITARILY LOCALLY CONNECTED CONTINUA

BY

JOSEPH N. SIMONE (KANSAS CITY, MISSOURI)

A *continuum* is a compact connected Hausdorff space. A continuum is *hereditarily locally connected* if each of its subcontinua is locally connected. This paper* is concerned with the characterization of hereditarily locally connected continua. Several of the theorems presented are generalizations of theorems due to Whyburn [3] to the non-metric setting. If S is a net whose domain is the directed set D, then we will use the notation $\{S_a, a \in D\}$ for S. In the special case of a sequence, we will use the notation $\{S_n, n \in N\}$ for $S_1, S_2, \ldots$, where N always denotes the set of natural numbers. Furthermore, the expression $\limsup S_n$ is used in place of $\limsup \{S_n, n \in N\}$ in the case where the net in question is in fact a sequence. Similar remarks hold for $\liminf S_n$ and $\lim S_n$. Throughout this paper, the results that are used to manipulate nets of point sets are due to Frolík [1]. Let X be a continuum and let K_0 be a non-degenerate subcontinuum of X. Then K_0 is called a *continuum of convergence in X* if there exists a net $K = \{K_a, a \in D\}$ of subcontinua of X converging to K_0 such that for all a and β in D either $K_a = K_\beta$ or $K_a \cap K_\beta = \emptyset$ and $K_a \cap K_0 = \emptyset$.

THEOREM 1. *If X is a compact Hausdorff space and $K = \{K_a, a \in D\}$ is a convergent net of closed subsets of X such that $\lim K$ is non-degenerate and such that for all a and β in D either $K_a = K_\beta$ or $K_a \cap K_\beta = \emptyset$ and $K_a \cap \lim K = \emptyset$, then there exist a sequence $\{K_n, n \in N\}$ of elements of $\{K_a, a \in D\}$ and open sets U and V such that $\overline{U} \cap \overline{V} = \emptyset$, $\limsup K_n$ is non-degenerate,*

$$K_i \cap K_j = \emptyset \quad \textit{for all } i \neq j,$$

and

$$K_i \cap \limsup K_n = \emptyset, \quad K_i \cap U \neq \emptyset \quad \textit{and} \quad K_i \cap V \neq \emptyset \quad \textit{for all } i.$$

Proof. Let a and b be distinct elements of $\lim K$, and let U and V be open sets such that $a \in U$, $b \in V$, and $\overline{U} \cap \overline{V} = \emptyset$. Let $W_0 = \emptyset$. There

* This paper is a part of the author's doctoral dissertation under the direction of Professor B. J. Pearson.

exists an a_1 in D such that if $a \geqslant a_1$, then $K_a \cap U \neq \emptyset$ and $K_a \cap V \neq \emptyset$. Let $K_1 = K_{a_1}$. Now, K_1 and $\lim K$ are disjoint closed sets. Let W_1 be an open set such that $K_1 \subseteq W_1$ and $\overline{W}_1 \subseteq X - \lim K$.

Let k be a positive integer. Suppose that we have constructed $K_1, K_2, \ldots, K_k$ and $W_0, W_1, \ldots, W_k$ such that

$$K_i \cap K_j = \emptyset \quad \text{for } i \neq j,$$

and for $i = 1, 2, \ldots, k$

$$K_i \cap U \neq \emptyset, \quad K_i \cap V \neq \emptyset, \quad K_i \cap \overline{W}_{i-1} = \emptyset,$$
$$\overline{W}_{i-1} \cup K_i \subseteq W_i, \quad \text{and} \quad \overline{W}_i \cap \lim K = \emptyset.$$

We claim that there exists an $a \geqslant a_1$ such that $K_a \cap \overline{W}_k = \emptyset$. Suppose that, for every $a \geqslant a_1$, $K_a \cap \overline{W}_k \neq \emptyset$. For $a \geqslant a_1$, let $x_a \in K_a \cap \overline{W}_k$. Then $\{x_a, a \geqslant a_1\}$ is a net of points in $\overline{W}_k$, and since $\overline{W}_k$ is compact, $\{x_a, a \geqslant a_1\}$ has a cluster point x in $\overline{W}_k$. However, it is clear that $x \in \limsup K = \lim K$ and, therefore, $\overline{W}_k \cap \lim K \neq \emptyset$. Hence there exists an $a_{k+1} \geqslant a_1$ such that $K_{a_{k+1}} \cap \overline{W}_k = \emptyset$. Let $K_{k+1} = K_{a_{k+1}}$. Since $K_j \subseteq W_k$ for all $j \leqslant k$, we have $K_{k+1} \cap K_j = \emptyset$ for all $j \leqslant k$. Since $a_{k+1} \geqslant a_1$, we also have $K_{k+1} \cap U \neq \emptyset$ and $K_{k+1} \cap V \neq \emptyset$. Now, $\overline{W}_k \cup K_{k+1}$ is a closed set and is disjoint from $\lim K$. Thus, there exists an open set W_{k+1} such that

$$\overline{W}_k \cup K_{k+1} \subseteq W_{k+1} \quad \text{and} \quad \overline{W}_{k+1} \subseteq X - \lim K.$$

It follows now, by induction, that a sequence $\{K_n, n \in N\}$ of elements of $\{K_a, a \in D\}$ and a sequence $\{W_n, n \in J\}$ of open sets exist, where J is the set of non-negative integers, such that

$$K_i \cap K_j = \emptyset \quad \text{for } i \neq j,$$

and for all i

$$K_i \cap U \neq \emptyset, \quad K_i \cap V \neq \emptyset, \quad K_i \cap \overline{W}_{i-1} = \emptyset,$$
$$\overline{W}_{i-1} \cup K_i \subseteq W_i, \quad \text{and} \quad \overline{W}_i \cap \lim K = \emptyset.$$

For each i, let $y_i \in K_i \cap U$ and let $z_i \in K_i \cap V$. Since $\overline{U}$ and $\overline{V}$ are compact, $\{y_i, i \in N\}$ has a cluster point y in $\overline{U}$, and $\{z_i, i \in N\}$ has a cluster point z in $\overline{V}$. Since $\overline{U} \cap \overline{V} = \emptyset$, $y \neq z$. Now, clearly, $y, z \in \limsup K_n$, so that $\limsup K_n$ is non-degenerate. Finally, notice that, for each i, $W_i - \overline{W}_{i-1}$ is an open set containing K_i which is disjoint from each K_j for $j \neq i$ and, therefore,

$$K_i \cap \limsup K_n = \emptyset \quad \text{for every } i.$$

THEOREM 2. *If X is a compact Hausdorff space, A and B are disjoint closed subsets of X, and $K = \{K_n, n \in D\}$ is a net of closed subsets of X such that $K_n \cap A \neq \emptyset$ and $K_n \cap B \neq \emptyset$ for all n in D, then there exists a subnet*

$K' = \{K_{n_\alpha}, \alpha \in E\}$ *of* K *such that* $\liminf K' \neq \emptyset$ *and* $\limsup K'$ *is non-degenerate.*

Proof. For each n in D, let $x_n \in K_n \cap A$ and $y_n \in K_n \cap B$. Since A is compact, $\{x_n, n \in D\}$ has a cluster point x in A. Let $\{x_{n_\alpha}, \alpha \in E\}$ be a subnet of $\{x_n, n \in D\}$ converging to x. Consider the subnet $K' = \{K_{n_\alpha}, \alpha \in E\}$ of $\{K_n, n \in D\}$. Clearly, $x \in \liminf K'$ so that $\liminf K' \neq \emptyset$. Since B is compact, the net $\{y_{n_\alpha}, \alpha \in E\}$ has a cluster point y in B. Then, clearly, x and y are distinct points of $\limsup K'$.

Frolík [1] has shown that a continuum which contains no continuum of convergence is hereditarily locally connected. However, the following theorem proves that this condition, in fact, characterizes hereditarily locally connected continua.

THEOREM 3. *The continuum* X *is hereditarily locally connected if and only if it contains no continuum of convergence.*

Proof. Suppose that X is hereditarily locally connected and that X contains a continuum of convergence K_0. Let $K = \{K_\alpha, \alpha \in D\}$ be a net of continua converging to K_0. By Theorems 1 and 2, there exist a sequence $\{K_n, n \in N\}$ of disjoint elements of $\{K_\alpha, \alpha \in D\}$ such that $K_i \cap \limsup K_n = \emptyset$ for all i and a subnet $K' = \{K_{n_\alpha}, \alpha \in E\}$ of $\{K_n, n \in N\}$ such that $\liminf K' \neq \emptyset$ and $\limsup K'$ is non-degenerate. Furthermore, $\limsup K'$ is a continuum and

$$\limsup K' \subseteq \limsup K_n.$$

Let $a \in \liminf K'$ and $b \in \limsup K'$ be such that $a \neq b$. Now, X is locally connected and, therefore, there exists a connected open set U such that $a \in U$ and $b \notin \overline{U}$. Since $a \in \liminf K'$, there exists an α_0 in E such that $K_{n_\alpha} \cap U \neq \emptyset$ for all $\alpha \geqslant \alpha_0$. Let

$$A = \limsup K_n \cup \bigcup \{K_{n_\alpha}, \alpha \geqslant \alpha_0\} \cup \overline{U}.$$

A is a closed set. Let H be the component of A containing $\limsup K'$. Then H is a continuum. Clearly,

$$H \supseteq \limsup K' \cup \bigcup \{K_{n_\alpha}, \alpha \geqslant \alpha_0\} \cup \overline{U},$$

since this latter set is a connected subset of A and contains $\limsup K'$. The continuum H is locally connected since X is hereditarily locally connected. Since $b \in H - \overline{U}$, there exists a connected H-open set V such that $b \in V$ and $\overline{V} \subseteq H - \overline{U}$. Also, $b \in \limsup K'$, and hence there exists $\beta \geqslant \alpha_0$ such that $K_{n_\beta} \cap V \neq \emptyset$. Now, $V \subseteq A$ and $\overline{V} \cap \overline{U} = \emptyset$, and hence

$$\overline{V} = (\overline{V} \cap \limsup K_n) \cup \bigcup \{\overline{V} \cap K_{n_\alpha}, \alpha \geqslant \alpha_0\}.$$

Since $K_{n_\alpha} \cap \limsup K_n = \emptyset$ for all $\alpha \in E$ and since $\{K_{n_\alpha}, \alpha \in E\}$ is countable, we have expressed $\overline{V}$ as the union of a countable collection of disjoint compact sets. Furthermore, since

$$b \in \bar{V} \cap \limsup K_n \quad \text{and} \quad K_{n_\beta} \cap V \neq \emptyset,$$

this collection is non-degenerate. However, no continuum is the union of a non-degenerate countable collection of disjoint compact sets. Therefore, X contains no continuum of convergence.

Let X be a space and let F be a family of subsets of X. F is called a *G-null family* if, for each two open sets U and V in X with $\bar{U} \cap \bar{V} = \emptyset$, not more than a finite number of elements of F meet both U and V. Let $\{F_a, a \in A\}$ be a family of disjoint subsets of X. Then $\{F_a, a \in A\}$ is said to have *property* D if

$$F_a \cap \mathrm{Cl} \bigcup \{F_\beta, \beta \neq a\} = \emptyset \quad \text{for all } a \text{ in } A.$$

THEOREM 4. *The continuum X is hereditarily locally connected if and only if every family of disjoint continua in X with property* D *is a G-null family.*

Proof. Suppose that there exists a family $\{K_a, a \in A\}$ of disjoint continua in X with property D which is not a G-null family. Then there exist open sets U and V and an infinite sequence $\{K_n, n \in N\}$ of distinct members of $\{K_a, a \in A\}$ such that $\bar{U} \cap \bar{V} = \emptyset$ and, for all n, $K_n \cap U \neq \emptyset$ and $K_n \cap V \neq \emptyset$. Since $\bar{U}$ is compact and since each K_n meets $\bar{U}$, it follows easily that there is a p in $\bar{U}$ such that $p \in \limsup K_n$. Let $K' = \{K_{n_a}, a \in E\}$ be a convergent subnet of $\{K_n, n \in N\}$ such that $p \in \lim K'$. Each K_{n_a} meets $\bar{V}$, and $\bar{V}$ is compact, and hence there exists a q in $\bar{V}$ such that $q \in \lim K'$. Since $\bar{U} \cap \bar{V} = \emptyset$, we have $p \neq q$ and, therefore, $\lim K'$ is non-degenerate. Let $K_0 = \lim K'$. Then K_0 is a continuum. Now, since $K_n \cap K_m = \emptyset$ for $n \neq m$ and $K' = \{K_{n_a}, a \in E\}$ is a subnet of $\{K_n, n \in N\}$, for all a and β in E, either $K_{n_a} \cap K_{n_\beta} = \emptyset$ or $K_{n_a} = K_{n_\beta}$. Finally, notice that $K_{n_a} \cap K_0 = \emptyset$ for every a in E. For, suppose that there exists a β in E such that $K_{n_\beta} \cap K_0 \neq \emptyset$. Let $y \in K_{n_\beta} \cap K_0$, and let W be an open neighborhood of y. Since $\{K_{n_a}, a \in E\}$ is a subnet of $\{K_n, n \in N\}$, there exists an a_0 in E such that if $a \geqslant a_0$, then $n_a \geqslant n_\beta + 1$. Now, $K_i \cap K_j = \emptyset$ for $i \neq j$, and hence if $a \geqslant a_0$, then $K_{n_a} \cap K_{n_\beta} = \emptyset$. However, $y \in \lim K'$, and hence there is an $a_1 \geqslant a_0$ such that $K_{n_{a_1}} \cap W \neq \emptyset$. Since $K_{n_{a_1}} \cap K_{n_\beta} = \emptyset$, we have

$$K_{n_\beta} \cap \mathrm{Cl} \bigcup \{K_{n_a}, K_{n_a} \neq K_{n_\beta}\} \neq \emptyset$$

and, therefore,

$$K_{n_\beta} \cap \mathrm{Cl} \bigcup \{K_a, K_a \neq K_{n_\beta}\} \neq \emptyset,$$

so that $\{K_a, a \in A\}$ does not have property D. Thus,

$$K_{n_a} \cap \lim K' = \emptyset \quad \text{for all } a \text{ in } E.$$

Hence, K_0 is a continuum of convergence in X and, therefore, X is not hereditarily locally connected.

Suppose now that X is not hereditarily locally connected. By Theorem 3, X contains a continuum of convergence K_0. Hence, there is a net $K = \{K_a, a \in D\}$ of subcontinua of X such that $\lim K = K_0$. By Theorems 1 and 2, there exist a sequence $\{K_n, n \in N\}$ of disjoint elements of $\{K_a, a \in D\}$ such that

$$K_i \cap \limsup K_n = \emptyset \quad \text{for all } i,$$

and a subnet $K' = \{K_{n_a}, a \in E\}$ of $\{K_n, n \in N\}$ such that $\liminf K' \neq \emptyset$ and $\limsup K'$ is a non-degenerate continuum. Let $a \in \liminf K'$ and $b \in \limsup K'$ be such that $a \neq b$. Let U and V be open sets such that $a \in U$, $b \in V$, and $\overline{U} \cap \overline{V} = \emptyset$. Since $a \in \liminf K'$, there exists an a_0 in E such that $K_{n_a} \cap U \neq \emptyset$ for all $a \geqslant a_0$. Since

$$b \in \limsup K' \quad \text{and} \quad K_{n_a} \cap \limsup K' = \emptyset \text{ for all } a \text{ in } E,$$

there exist infinitely many K_{n_a} with $a \geqslant a_0$ which meet V. Let $\{K'_j, j \in N\}$ be an infinite family of distinct K_{n_a} such that $a \geqslant a_0$ and $K_{n_a} \cap V \neq \emptyset$. Thus, $K'_j \cap U \neq \emptyset$ and $K'_j \cap V \neq \emptyset$ for each j. Suppose, now, that

$$K'_i \cap \mathrm{Cl} \bigcup \{K'_j, j \neq i\} \neq \emptyset \quad \text{for some } i.$$

Then there exists a y in K'_i which is a limit point of $\bigcup \{K'_j, j \neq i\}$. Let W be an open set such that $y \in W$. Now, since the K'_j are all disjoint, we have $y \notin K'_j$ for all $j \neq i$. Since each K'_j is closed, infinitely many K'_j meet W. Therefore, $y \in \limsup K_n$, so that

$$K'_i \cap \limsup K_n \neq \emptyset,$$

which contradicts Theorem 1. Thus, $\{K'_j, j \in N\}$ is a family of disjoint continua in X with property D which is not a G-null family.

If X is a metric space and F is a collection of subsets of X, then F is called a *null family* if for every $\varepsilon > 0$ not more than a finite number of elements of F have diameter greater than ε. Let X be a compact metric space. Then it follows immediately that a collection of subsets of X is a null family if and only if it is a G-null family. Whyburn [3] has shown that a metric continuum X is hereditarily locally connected if and only if the components of every subset of X form a null family. In the non-metric case, we obtain the following characterization:

THEOREM 5. *The continuum X is hereditarily locally connected if and only if the components of every closed subset of X form a G-null family.*

Proof. Suppose that X is not hereditarily locally connected. By Theorem 4, there exist an infinite collection $\{K_n, n \in N\}$ of disjoint continua in X with property D and open sets U and V such that $\overline{U} \cap \overline{V} = \emptyset$

and, for all i, $K_i \cap U \neq \emptyset$ and $K_i \cap V \neq \emptyset$. Let

$$E = \mathrm{Cl}\bigcup\{K_n,\ n \in N\}.$$

Since $\{K_n, n \in N\}$ has property D, each K_i is a component of E. Therefore, E is a closed set whose components do not form a G-null family.

Suppose now that X contains a closed set E whose components do not form a G-null family. Then there exist disjoint open sets U and V such that $\bar{U} \cap \bar{V} = \emptyset$ and an infinite family $\{K_n, n \in N\}$ of distinct components of E such that $K_n \cap U \neq \emptyset$ and $K_n \cap V \neq \emptyset$ for all n. It is clear that each K_n is a non-degenerate subcontinuum of X.

Let A_1 and B_1 be two distinct elements of $\{K_n, n \in N\}$ and let $L_0 = E$. Now, L_0 is a compact set containing A_1 and B_1 but containing no continuum intersecting both A_1 and B_1. It follows that L_0 is the union of two disjoint closed sets M_1 and L_1 such that $A_1 \subseteq M_1$ and $B_1 \subseteq L_1$. Let U_1 and V_1 be disjoint open sets such that $M_1 \subseteq U_1$ and $L_1 \subseteq V_1$. Now, each K_n is a connected subset of L_0, so that each K_n is contained in either M_1 or L_1. Thus, at least one of M_1 and L_1, say L_1, contains infinitely many K_n. Let A_2 and B_2 be two distinct elements of $\{K_n, n \in N\}$ contained in L_1. Then L_1 is the union of two disjoint closed sets M_2 and L_2 such that $A_2 \subseteq M_2$ and $B_2 \subseteq L_2$. Let U_2 and V_2 be disjoint open sets contained in V_1 such that $M_2 \subseteq U_2$ and $L_2 \subseteq V_2$. At least one of M_2 and L_2 contains infinitely many K_n. Continue the above process, and assume that, for each j, the sets A_j and B_j are labeled in such a way that L_j contains infinitely many K_n. Now, since $U_j \cap V_j = \emptyset$ and $U_{j+1} \subseteq V_j$ for each j, we have $U_i \cap U_j = \emptyset$ for $i \neq j$. Furthermore, for each j, $A_j \subseteq M_j \subseteq U_j$, so that $\{A_n, n \in N\}$ is a collection of disjoint continua in X. Since each $A_j \subseteq U_j$, we obtain $A_i \cap U_j = \emptyset$ for $i \neq j$, and hence

$$\mathrm{Cl}\bigcup\{A_n,\ n \neq i\} \subseteq X - U_i \subseteq X - A_i \quad \text{for each } i,$$

so that $\{A_n, n \in N\}$ has property D. Finally, recall that each A_n is some K_i and, therefore, $A_n \cap U \neq \emptyset$ and $A_n \cap V \neq \emptyset$ for each n. By Theorem 4, X is not hereditarily locally connected.

Whyburn [3] has shown that hereditarily locally connected metric continua can be characterized as metric continua for which the components and quasicomponents of any subset are identical. In arbitrary continua, we obtain the following theorem:

THEOREM 6. *If the quasicomponents and components of any subset of the continuum X are identical, then X is hereditarily locally connected.*

Proof. Suppose that X is not hereditarily locally connected. By Theorem 3, X contains a continuum of convergence K_0, where $K = \{K_a, a \in D\}$ is a net of subcontinua of X such that $\lim K = K_0$. By Theorems 1 and 2 there exist a sequence $\{K_n, n \in N\}$ of disjoint elements of $\{K_a, a \in D\}$.

such that

$$K_i \cap \limsup K_n = \emptyset \quad \text{for all } i,$$

and a subnet $K' = \{K_{n_a}, a \in E\}$ of $\{K_n, n \in N\}$ such that $\liminf K' \neq \emptyset$ and $\limsup K'$ is a non-degenerate continuum. Let $a \in \liminf K'$ and $b \in \limsup K'$ with $a \neq b$. Let

$$B = \{a, b\} \cup \bigcup \{K_{n_a}, a \in E\}.$$

Suppose that a and b belong to different quasicomponents of B. Then B is the union of two disjoint B-open sets U and V such that $a \in U$ and $b \in V$. Since $a \in \liminf K'$, there exists an a_0 in E such that $K_{n_a} \cap U \neq \emptyset$ for all $a \geqslant a_0$. Since each K_{n_a} is connected and U and V are separated, $K_{n_a} \subseteq U$ for all $a \geqslant a_0$. But this is contradictory to $b \in \limsup K'$. Therefore, a and b belong to the same quasicomponent C of B. By hypothesis, C is a component of B. Now,

$$K_{n_a} \cap \limsup K_n = \emptyset \quad \text{for all } a \text{ in } E,$$

so that for each a in E there exists an open set W with $K_{n_a} \subseteq W$ and $W \cap K_{n_\beta} = \emptyset$ for each β such that $K_{n_\beta} \neq K_{n_a}$. Since

$$B \cap (W - \{a, b\}) = K_{n_a},$$

K_{n_a} is a B-open set. Also, since K_{n_a} is closed, each K_{n_a} is open and closed in B. Since C is a component of B and each K_{n_a} is connected, either $K_{n_a} \subseteq C$ or $K_{n_a} \cap C = \emptyset$. Now, for each a in E, $a, b \in C - K_{n_a}$. If C contains some K_{n_a}, then K_{n_a} is a proper open and closed subset of C, which contradicts the fact that C is connected. Thus, $K_{n_a} \cap C = \emptyset$ for all a in E and, therefore, $C = \{a, b\}$, which is impossible since $\{a, b\}$ is not connected. Therefore, X is hereditarily locally connected.

Let X and Y be Hausdorff spaces. A continuous function f from X onto Y is called *weakly confluent* if for each continuum $C \subseteq Y$ there exists a component of $f^{-1}(C)$ which is mapped onto C by f. Maćkowiak [2] has defined a continuous function f from X onto Y to be *locally weakly confluent* if for each point y in Y there exists a closed neighborhood V of y in Y such that $f \mid f^{-1}(V)$ is weakly confluent. In the same paper he shows that a locally weakly confluent image of a hereditarily locally connected metric continuum is hereditarily locally connected. By applying Theorem 4, we obtain a similar theorem for weakly confluent mappings.

THEOREM 7. *If f is a weakly confluent mapping of the hereditarily locally connected continuum X onto the continuum Y, then Y is hereditarily locally connected.*

Proof. Suppose that Y is not hereditarily locally connected. By Theorem 4, there exist Y-open sets U and V such that $\overline{U} \cap \overline{V} = \emptyset$ and an infinite collection $\{K_n, n \in N\}$ of disjoint continua in Y with property D such that $K_i \cap U \neq \emptyset$ and $K_i \cap V \neq \emptyset$ for all i. Let $U^* = f^{-1}(U)$ and

$V^* = f^{-1}(V)$. Then, clearly, U^* and V^* are X-open sets and $\overline{U^*} \cap \overline{V^*} = \emptyset$. Since f is weakly confluent, there exists a component K_i^* of $f^{-1}(K_i)$ such that $f(K_i^*) = K_i$ for each i. Thus, $\{K_i^*, i \in N\}$ is a family of disjoint continua in X. It follows immediately that

$$K_i^* \cap U^* \neq \emptyset \quad \text{and} \quad K_i^* \cap V^* \neq \emptyset \quad \text{for each } i.$$

Thus, $\{K_i^*, i \in N\}$ is not a G-null family. Finally, we claim that $\{K_i^*, i \in N\}$ has property D. To see this, recall that $\{K_n, n \in N\}$ has property D, let $i \in N$, and consider the following computation:

$$K_i^* \cap \mathrm{Cl} \bigcup \{K_n^*, n \neq i\} \subseteq f^{-1}(K_i) \cap \mathrm{Cl} \bigcup \{f^{-1}(K_n), n \neq i\}$$
$$\subseteq f^{-1}(K_i) \cap f^{-1}\big(\mathrm{Cl} \bigcup \{K_n, n \neq i\}\big) = f^{-1}\big(K_i \cap \mathrm{Cl} \bigcup \{K_n, n \neq i\}\big) = \emptyset.$$

It follows from Theorem 4 that X is not hereditarily locally connected.

THEOREM 8. *If X is a hereditarily locally connected continuum in which each closed set is a G_δ and $K = \{K_a, a \in D\}$ is a net of continua in X such that for all a and β in D either $K_a = K_\beta$ or $K_a \cap K_\beta = \emptyset$, then either $|\liminf K| \leqslant 1$ or there exists an a_0 in D such that $\{K_a, a \geqslant a_0\}$ is countable.*

Proof. Suppose that $\liminf K$ contains two points x and y, and assume that $\{K_a, a \geqslant a_0\}$ is uncountable for every a_0 in D. Let U and V be open sets such that $x \in U$, $y \in V$, and $\overline{U} \cap \overline{V} = \emptyset$. There exists an a_0 in D such that if $a \geqslant a_0$, then $K_a \cap U \neq \emptyset$ and $K_a \cap V \neq \emptyset$. Let $S_0 = \{K_a, a \geqslant a_0\}$. Then S_0 is an uncountable set. Let $K_1 = K_{a_0}$. Now, every closed subset of X is a G_δ and, therefore, if F is a closed subset of X, then X has a countable base at F. Since K_1 is a closed subset of X, X has a countable base at K_1. Also, since S_0 is an uncountable set of disjoint closed sets, it follows easily that there exist an open set $V_1 \supseteq K_1$ and an uncountable subset S_1 of S_0 such that if $K \in S_1$, then $K \cap V_1 = \emptyset$.

Let $k \geqslant 1$, and suppose that $S_0, S_1, \ldots, S_k$, $K_1, K_2, \ldots, K_k$ and $V_1, V_2, \ldots, V_k$ have been defined so that, for each $i \leqslant k$, S_i is an uncountable subset of S_{i-1}, $K_i \in S_{i-1}$, V_i is an open set containing K_i such that if $K \in S_i$, then $K \cap V_i = \emptyset$, and $K_n \cap K_m = \emptyset$ for $n \neq m$. Let $K_{k+1} \in S_k$. Then $K_{k+1} \cap V_n = \emptyset$, so that $K_{k+1} \cap K_n = \emptyset$ for all $n \leqslant k$. Now, K_{k+1} is a closed subset of X, and hence X has a countable base at K_{k+1}. Since S_k is an uncountable set of disjoint closed sets, there exist an open set $V_{k+1} \supseteq K_{k+1}$ and an uncountable subset S_{k+1} of S_k such that if $K \in S_{k+1}$, then $K \cap V_{k+1} = \emptyset$. It follows by induction that there exist a sequence $\{K_n, n \in N\}$ of disjoint members of $\{K_a, a \geqslant a_0\}$ and a sequence $\{V_n, n \in N\}$ of open sets such that, for all n, $K_n \subseteq V_n$ and $K_m \cap V_n = \emptyset$ for $m > n$. Let $K_0 = \emptyset$ and, for each integer $n \geqslant 1$, let

$$U_n = V_n - \bigcup_{i=0}^{n-1} K_i.$$

Then $\{U_n, n \in N\}$ is a sequence of open sets such that $K_n \subseteq U_n$ and $K_m \cap U_n = \emptyset$ for $m \neq n$. Thus, for each integer n we have

$$\mathrm{Cl}\bigcup\{K_i, i \neq n\} \subseteq X - K_n.$$

Since each $K_n \in \{K_a, a \geqslant a_0\}$, it follows that $\{K_n, n \in N\}$ is a family of disjoint continua in X with property D which is not G-null. Therefore, X is not hereditarily locally connected.

Let X be a space and let F be a collection of subsets of X. Then F is said to be a *cover-null family* if for each open cover $\mathcal{U}$ of X there exists a finite subset F^* of F such that $F - F^*$ refines $\mathcal{U}$. If X is a compact Hausdorff space and F is a collection of subsets of X, then it can be seen that F is a cover-null family if and only if F is a G-null family. Thus, a continuum X is hereditarily locally connected if and only if every family of disjoint continua in X with property D is a cover-null family. This result is used to prove the following theorem.

THEOREM 9. *If X is a hereditarily locally connected continuum and S is a decomposition of X into continua such that the set of non-degenerate elements of S has property* D, *then S is upper semi-continuous.*

Proof. Let $K \in S$ and let U be an open set such that $K \subseteq U$. Let W be an open set such that $K \subseteq W$ and $\overline{W} \subseteq U$, and let $\mathcal{U} = \{U, X - \overline{W}\}$. Then $\mathcal{U}$ is an open cover of X. Let T be the set of non-degenerate elements of S. Since T has property D, there exists a finite subset T^* of T such that $T - T^*$ refines $\mathcal{U}$. Let $V = W - \bigcup T^*$. Since K is contained in an element of $\mathcal{U}$, $K \notin T^*$ and, therefore, V is an open set such that $K \subseteq V \subseteq W$. Let $H \in S$ with $H \cap V \neq \emptyset$. If $H = \{x\}$ for some x in X, then, clearly, $H \subseteq U$. Thus, suppose that $H \in T$. Since $H \cap V \neq \emptyset$, we have $H \in T - T^*$ and, therefore, $H \subseteq U$ or $H \subseteq X - \overline{W}$. However, $H \cap W \neq \emptyset$ since $V \subseteq W$ and, therefore, $H \nsubseteq X - \overline{W}$. Thus, $H \subseteq U$, so that S is upper semi-continuous.

REFERENCES

[1] Z. Frolík, *Concerning topological convergence of sets*, Czechoslovak Mathematical Journal 10 (85) (1960), p. 168-180.

[2] T. Maćkowiak, *Locally weakly confluent mappings on hereditarily locally connected continua*, Fundamenta Mathematicae 88 (1975), p. 225-240.

[3] G. T. Whyburn, *Concerning hereditarily locally connected continua*, American Journal of Mathematics 53 (1931), p. 374-384.

[4] — *Analytic topology*, American Mathematical Society Colloquium Publications 28 (1942).

UNIVERSITY OF MISSOURI
KANSAS CITY

Reçu par la Rédaction le 24. 4. 1976