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On extending the theory of Cesàro summability

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1. Introduction. In 1928 A. F. Andersen, in [2], proved the following theorem:

THEOREM A. *$\sum u_n$ is summable (C, y) if and only if $\sum n^k \Delta^k u_n$ is summable $(C, y+k)$, k a positive integer.*

Here $\Delta^k u_n$ is the k-th difference of $\{u_n\}$ and the sufficiency condition is meant in the sense that there exists a unique series $\sum u_n^*$, satisfying $\Delta^k u_n^* = \Delta^k u_n$, which is summable (C, y). Defining the classes $\{C_y\}$ by

$$C_y = \left\{\sum a_n \middle| \sum a_n \text{ is summable } (C, y)\right\},$$

Andersen's theorem can be stated as

$$\sum u_n \in C_y \text{ if and only if } \sum n^k \Delta^k u_n \in C_{y+k}.$$

On the other hand, from a well-known classical theorem (see Hardy [4]), $\sum u_n \in C_y$ implies $\sum n^{-\gamma} u_n \in C_{y-\gamma}$ for $\gamma \geqslant 0$, but the converse does not hold for any y or $\gamma > 0$. But such transformations are important, especially in trigonometric series. Whereas a trigonometric series transformed by $v_n = n^k \Delta^k u_n$ is no longer a trigonometric series, one transformed by $v_n = n^\gamma u_n$ is again a trigonometric series and represents, in some sense, the fractional integrated or differentiated series, or its conjugate series. Thus, it would be useful to construct a family of classes, $\{S_\lambda\}$ say, which contains the Cesàro summability classes $\{C_y\}$, in which all the theorems about Cesàro summability can be extended to this new family, where all the methods used in the theory of Cesàro summability are equally applicable in the new theory, and where

$$\sum u_n \in S_\lambda \text{ if and only if } \sum n^\gamma u_n \in S_{\lambda'}.$$

It is such a family $\{R_{x,y}\}$, called the *repeated convergence classes*, that is the object of our investigation. The first extension needed is to define Cesàro summability for all real numbers and not simply to consider numbers greater than -1 as is commonly done. As early as 1930, Hausdorff, [6], suggested a definition for orders $\leqslant -1$, and since that time other definitions, equivalent to his, have appeared (see [3], [10]). In our case we shall investigate repeated convergence classes $\{R_{x,y}\}$ for all real values (x, y) of the Euclidean plane and it is crucial that we consider Cesàro summability of all orders. If not, the range of the parameters x, y, would have to be restricted simply because Cesàro summability was not defined there, and this would detract from those natural values of the parameters, where the theorems actually are false.

We define Cesàro summability for arbitrary order as follows:

DEFINITION 1. Let $\sum a_n$ be a series of numbers,

$$A_n^y = \frac{(y+1)(y+2)\dots(y+n)}{n!}, \quad A_0^y = 1$$

and

$$s_n^y = \sum_{\nu=0}^{n} A_{n-\nu}^y a_\nu.$$

The series $\sum a_n$ is said to be *summable* (C, y) to sum $a^{(0)}$, y an arbitrary real number, if

$$s_n^{y+r} = a^{(0)} A_n^{y+r} + o(n^{y+r}) \quad \text{for } r = 0, 1, 2, \dots \tag{1}$$

Although the definition requires a denumerable number of conditions for a series to be summable (C, y), if $y > -1$ the condition for $r = 0$ will imply all the other conditions, and if $y \leqslant -1$, the condition for $r = 0$ and for any other value of r such that $y+r > -1$ will imply all the other conditions. To see this, suppose

$$s_n^y = a^{(0)} A_n^y + o(n^y) \quad \text{for some } y < -1$$

and

$$s_n^{y+r} = a^{(0)} A_n^{y+r} + o(n_n^{y+r}) \quad \text{for some } r \text{ such that } y+r > -1.$$

Subtracting $a^{(0)}$ from a_0 gives us a series which is summable $(C, y+r)$ to 0. Thus, we may assume $a^{(0)} = 0$. Now, $s_n^{y+1} = \sum_{\nu=0}^{n} s_\nu^y$. Since $y < -1$ the series $\sum_{\nu=0}^{\infty} s_\nu^y$ converges to sum c, say. Then

$$\sum_{\nu=0}^{n} s_\nu^y = c - \sum_{\nu=n+1}^{\infty} s_\nu^y = c + o(n^{y+1}).$$

Thus

$$s_n^{y+r} = \sum_{\nu=0}^{n} A_{n-\nu}^{r-2} s_\nu^{y+1} = cA_n^{r-1} + o(n^{r-1}).$$

But $s_n^{y+r} = o(n^{y+r})$ and since $r-1 > y+r$, $c = 0$.

Continuing in this way, we prove that $s_n^{y+p} = o(n^{y+p})$ for $p = 2, 3, \dots \dots, r-1$.

If $y = -1$, then $a_n = o(1/n)$, and by a well-known Tauberian theorem, $\sum a_n$ converges even if it is summable by Abel's method.

Thus two conditions would suffice to define Cesàro summability in general. However, by using the above definition, one can prove all summability theorems in their full generality without having to consider two separate cases, $y > -1$ and $y \leqslant -1$. Thus, with this definition, $y = -1$ plays no special role.

In Section 2 we define the repeated convergence classes. The concept of repeated convergence originated with Rajchman [12], and is concerned with the convergence of a series of remainders of some initial series. It is from his definition that the classes will be constructed. At first glance this appears to have nothing to do with Cesàro summability, but the connection is quickly established in Theorems 1 and 1*.

In Section 3 we consider the transformations $v_n = n^\gamma \Delta^q u_n$, γ a real number, q a non-negative integer. These transformations reduce to the transformations $v_n = n^\gamma u_n$, discussed above, when $q = 0$. We prove the theorem

$$\sum u_n \epsilon R_{x,y} \text{ if and only if } \sum n^\gamma \Delta^q u_n \epsilon R_{x-\gamma+q, y+\gamma}$$

except for certain special values of the parameters.

Finally, at one such place where the theorem fails, namely $\sum n^x a_n \epsilon R_{0,x}$ does not imply $\sum a_n \epsilon R_{x,0}$ for x fractional and non-negative, we show what the connection is, by proving Theorem 7: $\sum n^x a_n \epsilon R_{0,x}$ if and only if $\sum a_n \epsilon R_{x,0}$ and $\sum n^x a_n$ is summable by Abel's method.

In proving this theorem we use fractional differences which are defined by

$$\Delta^x a_n = \sum_{p=0}^{\infty} A_p^{-x-1} a_{n+p}$$

and we use the notation $\Delta_N^x a_n$ for the expression $\sum_{p=0}^{N} A_p^{-x-1} a_{n+p}$.

The author wishes to express his deep gratitude to Professor Anton Zygmund who not only suggested the problem of repeated convergence of series, but who was a source of constant help and encouragement throughout its preparation.

2. Suppose $\sum a_n$ is a convergent series. If we let

$$a_n^{(1)} = \sum_{\nu=n+1}^{\infty} a_\nu,$$

then $a_n^{(1)}$ is defined for each n and we may consider the series $\sum a_n^{(1)}$. If this series converges, we say, following Zygmund [14], p. 373, Vol. 1, that $\sum a_n$ has convergence of order 1. In general a series $\sum a_n$ has convergence of order k, k a non-negative integer, if $\sum a_n^{(k)}$ converges, where

$$a_n^{(0)} = a_n, \quad a_n^{(k)} = \sum_{\nu=n+1}^{\infty} a_\nu^{(k-1)} \quad (k = 1, 2, 3, \ldots).$$

The series $\sum a_n^{(k)}$ is called the *k-th iterate of* $\sum a_n$.

The above definition does not permit us to consider the k-th iterate of a series unless we first assume that the $(k-1)$-st iterate converges. This is an unnecessary restriction that can be overcome by the following procedure; a procedure that will, in addition, suggest a way of extending repeated convergence to fractional orders.

Suppose a series $\sum a_n$ has convergence of order k and let

$$a^{0)} = \sum_0^\infty a_n, \; a^{(1)} = \sum_0^\infty a_n^{(1)}, \ldots, a^{(k)} = \sum_0^\infty a_n^{(k)}.$$

Then

$$a_n^{(0)} = a_n,$$

$$a_n^{(1)} = \sum_{\nu=n+1}^{\infty} a_\nu = a^{(0)} - s_n^{(0)}, \quad \text{where } s_n^{(0)} = \sum_{\nu=0}^{n} a_\nu,$$

$$a_n^{(2)} = a^{(1)} - \sum_{\nu=0}^{n} a_\nu^{(1)} = a^{(1)} - a^{(0)} A_n^{(1)} + s_n^{(1)},$$

where $s_n^{(1)}$ is the n-th Cesàro sum of order 1. In general

$$a_n^{(k)} = (-1)^k [s_n^{k-1} - a^{(0)} A_n^{k-1} + a^{(1)} A_n^{k-2} - \ldots + (-1)^k a^{(k-1)}].$$

It is not necessary for $\sum a_n$ to converge in order to define $a_n^{(1)}$. If $\sum a_n$ is summable by any method of summation to $a^{(0)}$, we may define $a_n^{(1)} = a^{(0)} - s_n^{(0)}$ and likewise for the terms $a_n^{(k)}$, $k > 1$. Also, since both s_n^k and A_n^k are defined for fractional orders we can define $a_n^{(k)}$ for k fractional since it consists of terms involving s_n^k and A_n^k only. Finally, if the x-th iterate is not convergent but is summable (C, y), we express this by saying that the original series is in the repeated convergence class $R_{x,y}$. Formally, we have the following definition:

Definition. A series $\sum a_n$ is said to be in the *repeated convergence class* $R_{x,y}$, $x = 0$ or $x = a + k - 1$, $0 < a \leqslant 1$, k a positive integer, if there exist numbers $a^{(0)}, a^{(1)}, \ldots, a^{(k)}$ such that $\sum a_n^{(x)}$ is summable (C, y) to $a^{(x)}$, where

$$a_n^{(0)} = a_n \quad \text{if } x = 0,$$

$$a_n^{(x)} = (-1)^k [s_n^{a+k-2} - a^{(0)} A_n^{a+k-2} + a^{(1)} A_n^{a+k-3} - \ldots + (-1)^k a^{(k-1)} A_n^{a-1}]$$

$$\text{if } x = a + k - 1.$$

In addition

$$a^{(x)} = \begin{cases} (-1)^k a^{(k)} & \text{if } x = k, \\ 0 & \text{if } x \text{ is not an integer.} \end{cases}$$

If $x = \gamma - k$, where $0 \leqslant \gamma < 1$, k a positive integer, then the series is said to be *in class* $R_{x,y}$ if $\sum \Delta^k a_n \in R_{\gamma,y}$ and $\sum \Delta^k a_n$ is summable (C, y) to $\Delta^{k-1} a_0$. x is called the *order of convergence* and y, the *order of summability*, of the series.

Note. The definition of repeated convergence classes of negative order arises as a consequence of requiring Lemma 3, below, to be valid for all orders of convergence. Also, the requirement that $a^{(x)} = 0$ if x is not an integer is natural. If $\sum a_n \in R_{x,0}$, $x > 0$, then $a^{(x')} = 0$ for $0 < x' < x$ if x' is non-integral. Thus we have defined classes $R_{x,y}$ for all ordered pairs (x, y) of the Euclidean plane with the classes $R_{0,y}$ being the Cesàro summability classes.

The following theorem characterizes repeated convergence classes by means of asymptotic expansions of Cesàro sums, including, as a special case, the notion of Cesàro summability. Theorem 1* below makes the characterization particularly lucid.

THEOREM 1. *A necessary and sufficient condition for a series* $\sum a_n$ *to be in class* $R_{x,y}$, $x = a + k - 1$, k *any integer*, $0 < a \leqslant 1$, y *arbitrary, is that there exist constants* $c_0, c_1, \ldots, c_k$ *such that for all non-negative integers* r,

$$(3) \quad s_n^{x+y+r} = \begin{cases} c_0 A_n^{x+y+r} + c_1 A_n^{x+y+r-1} + \ldots + c_{k-1} A_n^{a+y+r} + c_k A_n^{y+r} + o(n^{y+r}) & \text{if } x \geqslant 0, \\ o(n^{y+r}) & \text{if } x < 0. \end{cases}$$

The $c_j = (-1)^j a^{(j)}$, $j = 0, 1, \ldots, k-1$,

$$c_k = \begin{cases} (-1)^k a^{(k)} & \text{if } x = k, \\ 0 & \text{otherwise.} \end{cases}$$

For $0 < a < 1$ and $y > -1$ we can drop the r and the theorem states $\sum a_n \in R_{a,y}$ if and only if $s_n^{a+y} = a^{(0)} A_n^{a+y} + o(n^y)$.

In order to prove this, observe that $\sum a_n \in R_{a,y}$ means $\sum_{\nu=0}^{n} A_{n-\nu}^{y} a_\nu^{(a)} = o(n^y)$, $\sum a_n^{(a)}$ being summable to 0 by definition. But

$$\sum_{\nu=0}^{n} A_{n-\nu}^{y} a_\nu^{(a)} = \sum_{\nu=0}^{n} A_{n-\nu}^{y} [a^{(0)} A_\nu^{a-1} - s_\nu^{a-1}] = a^{(0)} A_n^{a+y} - s_n^{a+y}.$$

Hence, if $\sum a_n \in R_{a,y}$ the left-hand side of the equation is $o(n^y)$ and so $s_n^{a+y} = a^{(0)} A_n^{a+y} + o(n^y)$.

Conversely, if

$$s_n^{a+y} = a^{(0)} A_n^{a+y} + o(n^y),$$

then

$$s_n^{a+y} - a^{(0)} A_n^{a+y} = \sum_{\nu=0}^{n} A_{n-\nu}^{y} a_\nu^{(a)} = o(n^y),$$

and $\sum a_n \epsilon R_{a,y}$.

For the general proof of the theorem we use the following lemma:

LEMMA 1. *Given a series $\sum a_n$, the sum of whose first k iterates are $a^{(0)}$, $a^{(1)}, \dots, a^{(k)}$. If a_n^* is defined by*

$$(4)\qquad a_\nu^* = a_\nu - \sum_{j=\nu}^{k} A_{j-\nu}^{-j-1} a^{(j)}, \qquad \nu = 0, 1, \dots, k,$$

$$a_n^* = a_n, \qquad n > k,$$

then the first k iterates of $\sum a_n^*$ have sum equal to 0.

The proof of the lemma is straightforward and we return to the proof of the theorem. First let $x \geqslant 0$ and suppose $\sum a_n \epsilon R_{x,y}$. By Lemma 1, the series $\sum a_n^*$ has the property that the sum of its first k iterates are all zero. Letting $b_n = (-1)^k a_n^{*(x)} = s_n^{*(x-1)}$ we see that $\sum a_n^* \epsilon R_{x,y}$ if and only if $\sum b_n$ is summable (C, y) to zero. Thus if $t_n^{(0)} = \sum_{\nu=0}^{n} b_\nu$ if follows that $t_n^{y+r} = o(n^{y+r})$ or $s_n^{*x+y+r} = o(n^{y+r})$. But

$$s_n^{*x+y+r} = \sum_{\nu=0}^{n} A_{n-\nu}^{x+y+r}\Big[a_\nu - \sum_{j=\nu}^{k} A_{j-\nu}^{-j-1} a^{(j)}\Big]$$

$$= s_n^{x+y+r} - \sum_{j=0}^{k} a^{(j)} (-1)^j A_n^{x+y+r-j} = o(n^{y+r}).$$

Hence

$$s_n^{x+y+r} = a^{(0)} A_n^{x+y+r} - a^{(1)} A_n^{x+y+r-1} + \dots + (-1)^k a^{(x)} A_n^{y+r} + o(n^{y+r}).$$

If $x+y+r$ is a negative integer, then $A_n^{x+y+r-j}$ are 0 for n large enough, and $s_n^{x+y+r} = o(n^{y+r})$.

Conversely, if

$$s_n^{x+y+r} = a^{(0)} A_n^{x+y+r} - a^{(1)} A_n^{x+y+r-1} + \dots + (-1)^k a^{(x)} A_n^{y+r} + o(n^{y+r}),$$

then

$$\sum_{\nu=0}^{n} A_{n-\nu}^{y+r} a_\nu^{(a+k-1)} = (-1)^k [s_n^{x+y+r} - a^{(0)} A_n^{x+y+r} + \dots + (-1)^k a^{(k-1)} A_n^{a+y+r}]$$

$$= a^{(x)} A_n^{y+r} + o(n^{y+r}).$$

Hence $a_n^{(x}$ is summable (C, y) to $a^{(x)}$ and $\sum a_n \epsilon R_{x,y}$.

Suppose $x < 0$, say $x = \gamma - k$, where $0 \leqslant \gamma < 1$, k a positive integer. We have

$$(5)\quad \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma+y+r} \Delta^k a_\nu$$

$$= (-1)^k \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma+y+r} s_{\nu+k}^{-k-1}$$

$$= (-1)^k \sum_{\nu=0}^{n+k} A_{n+k-\nu}^{\gamma+y+r} s_\nu^{-k-1} - (-1)^k \sum_{\nu=0}^{k-1} A_{n+k-\nu}^{\gamma+y+r} s_\nu^{-k-1}$$

$$= (-1)^k s_{n+k}^{x+y+r} - (-1)^k A_n^{\gamma+y+r} s_{k-1}^{-k} - (-1)^k \sum_{\nu=0}^{k-1} \Big(\sum_{j=1}^{k-\nu} A_{n+j}^{\gamma+y+r-1} \Big) s_\nu^{-k-1}$$

$$= (-1)^k s_{n+k}^{x+y+r} + \Delta^{k-1} a_0 A_n^{\gamma+y+r} + o(n^{y+r}).$$

If $s_n^{x+y+r} = o(n^{y+r})$, then

$$\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma+y+r} \Delta^k a_\nu = \Delta^{k-1} a_0 A_n^{\gamma+y+r} + o(n^{y+r})$$

and $\sum \Delta^k a_n \in R_{\gamma,y}$, $\sum \Delta^k a_n$ is summable (C, y) to $\Delta^{k-1} a_0$.

Conversely, if $\sum a_n \in R_{x,y}$, then

$$\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma+y+r} \Delta^k a_\nu = \Delta^{k-1} a_0 A_n^{\gamma+y+r} + o(n^{y+r}).$$

Hence by (5)

$$\Delta^{k-1} a_0 A_n^{\gamma+y+r} + o(n^{y+r}) = (-1)^k s_{n+k}^{x+y+r} + \Delta^{k-1} a_0 A_n^{\gamma+y+r} + o(n^{y+r}).$$

and $s_n^{x+y+r} = o(n^{y+r})$ thus proving the theorem.

From Lemma 1 it follows that by changing the first few terms of a series, which does not effect the repeated convergence class to which it belongs, we can then state that

THEOREM 1*. *$\sum a_n \in R_{x,y}$ if and only if $s_n^{x+y+r} = o(n^{y+r})$ for all non-negative integers r.*

3. We turn our attention to the transformations

$$v_n = A_n^p \Delta^q u_n, \qquad v_n = n^p \Delta^q u_n,$$

p, q being non-negative integers. The following comparison theorems are valid.

THEOREM 2. *In order that $\sum v_n \in R_{x+q-p,y+p}$ it is necessary and sufficient that there exist a solution u_n^* of the equation $v_n = n^p \Delta^q u_n$ such that $\sum u_n^* \in R_{x,y}$, where p, q are non-negative integers, y any real number and $x+q \neq 0, 1, 2, \ldots, p-1$.*

Also

THEOREM 2*. *In order that* $\sum v_n \epsilon R_{x+q-p,y+p}$ *it is necessary and sufficient that there exist a solution* u_n^* *of the equation* $v_n = A_n^p \Delta^q u_n$ *such that* $\sum u_n^* \epsilon R_{x,y}$, *where* p, q *are non-negative integers, y any real number, and* $x+q \neq 0, 1, 2, \ldots, p-1$.

Notice that when $p = q$ the order of convergence remains fixed and the order of summability increases by p under the transformations. In particular for $x = 0, y \geqslant -1$, Theorem 2* reduces to Andersen's theorem stated in the introduction. When $p = 0$ the order of summability remains fixed, the order of convergence increases by q. When $q = 0$ we have the transformation $A_n^p u_n$ and $n^p u_n$ discussed in the introduction. In this case the order of convergence decreases by p and the order of summability increases by p. That the condition on x is needed is seen by the following counter example.

We prove that

$$\sum \frac{n}{n\log n} \epsilon R_{-1,0} \quad \text{and} \quad \sum \frac{1}{\log n} \notin R_{0,-1}$$

which corresponds to the special case of Theorem 2 when $x = 0, y = -1, p = 1, q = 0$. But $\sum n u_n \epsilon R_{-1,0}$ if and only if $u_n = o(1/n)$ by Theorem 1. But $\frac{1}{n\log n} = o(1/n)$ and so the series $\sum \frac{n}{n\log n} \epsilon R_{-1,0}$. However, $\sum \frac{1}{n\log n} \notin R_{0,-1}$ since the series does not converge and so is certainly not summable $(C, -1)$.

Theorem 2 is a consequence of the following two lemmas:

LEMMA 2. *A necessary and sufficient condition for* $\sum u_n$ *to be in class* $R_{x,y}$ *is that* $\sum n u_n \epsilon R_{x-1,y+1}$ *provided* $x \neq 0$.

LEMMA 3. *In order for* $\sum v_n$ *to be in class* $R_{x,y}$ *it is necessary and sufficient that there exists a solution* u_n^* *such that* $v_n = \Delta^q u_n^*$ *and* $\sum u_n^* \epsilon R_{x-q,y}$, x, y *arbitrary.*

Proof of Lemma 2. Suppose $\sum u_n \epsilon R_{x,y}$. The following formula is well-known (see [2]). For γ arbitrary

$$\sum_{\nu=0}^{n+1} \nu u_\nu A_{n+1-\nu}^{\gamma} = -(\gamma+1) s_n^{\gamma+1} + (n+1) s_{n+1}^{\gamma}, \tag{6}$$

where $s_n^{(0)} = \sum_{\nu=0}^{n} u_\nu$. Assuming without loss of generality that $u^{(0)} = u^{(1)} = \ldots = u^{(x)} = 0$, it follows from Theorem 1 and (6) that

$$\sum_{\nu=0}^{n+1} \nu u_\nu A_{n+1-\nu}^{x+y+r} = o(n^{y+1+r}) + (n+1) o(n^{y+r}) = o(n^{y+r+1}).$$

Since the expression on the left is the $(x+y+r)$-th Cesàro sum of the series $\sum nu_n$ it follows that $\sum nu_n \in R_{x-1,y+1}$. In this direction the lemma holds without any restriction on x.

For the converse result we shall prove that $\sum u_n \in R_{x,y}$ implies $\sum \frac{u_n}{n+1} \in R_{x+1,y-1}$ provided $x \neq -1$. The following formulae are valid:

$$s_n^\gamma(v) = \frac{A_{n+1}^\gamma}{\gamma+2} \sum_{\nu=0}^{n-2} \frac{s_\nu^\gamma(u)}{A_\nu^{\gamma+2}} + \frac{s_n^\gamma(u)}{n+1} \tag{7}$$

if $\gamma \neq -1, -2, -3, \ldots$,

$$s_n^{-m}(v) = \sum_{k=0}^{m} s_{k+n-m}^{-m}(u) \sum_{\nu=0}^{m-k} A_{\nu+m-k}^{-m} A_\nu^{m-2} \cdot \frac{1}{\nu+k+n-m+1} \tag{8}$$

for $m = 1, 2, 3, \ldots$, where $s_n^\gamma(v)$ is the Cesàro sum of order γ of $\sum \frac{u_n}{n+1}$ and $s_n^\gamma(u)$ is the Cesàro sum of order γ of $\sum u_n$. To prove (8) observe that

$$\begin{aligned} s_n^{-m}(v) &= \sum_{\nu=0}^{n} A_{n-\nu}^{-m} \frac{u_\nu}{\nu+1} = \sum_{\nu=0}^{n} \sum_{k=0}^{\nu} A_{\nu-k}^{m-2} s_k^{-m}(u) \frac{A_{n-\nu}^{-m}}{\nu+1} \\ &= \sum_{k=0}^{n} s_k^{-m}(u) \sum_{\nu=k}^{n} A_{\nu-k}^{m-2} \frac{A_{n-\nu}^{-m}}{\nu+1} \\ &= \sum_{k=n-m}^{n} s_k^{-m}(u) \sum_{\nu=k}^{n} A_{n-\nu}^{-m} A_{\nu-k}^{m-2} \cdot \frac{1}{\nu+1} \\ &= \sum_{k=0}^{m} s_{n-m+k}^{-m}(u) \sum_{\nu=n-m+k}^{n} A_{n-\nu}^{-m} A_{\nu-k+m-n}^{m-2} \cdot \frac{1}{\nu+1} \\ &= \sum_{k=0}^{m} s_{k+n-m}^{-m}(u) \sum_{\nu=0}^{m-k} A_{\nu+m-k}^{-m} A_\nu^{m-2} \cdot \frac{1}{\nu+k+n-m+1}. \end{aligned}$$

The derivation of (7) is similar to the derivation of the formula given by Andersen [1], p. 61. However, since there are differences in the two derivations, the proof of (7) will be given here.

$$\begin{aligned} s_n^\gamma(v) &= \sum_{\nu=0}^{n} A_{n-\nu}^\gamma \cdot \frac{u_\nu}{\nu+1} = \sum_{k=0}^{n} s_k^\gamma(u) \sum_{\nu=k}^{n} \frac{A_{\nu-k}^{-\gamma-2} A_{n-\nu}^\gamma}{\nu+1} \\ &= \sum_{\nu=0}^{n} s_\nu^\gamma(u) \sum_{k=\nu}^{n} A_{k-\nu}^{-\gamma-2} A_{n-k}^\gamma \cdot \frac{1}{k+1} = \sum_{\nu=0}^{n} B_{n\nu} s_\nu^\gamma(u), \end{aligned}$$

where

$$B_{n,\nu} = \frac{A_0^{-\gamma-2}A_{n-\nu}^{\gamma}}{\nu+1} + \frac{A_1^{-\gamma-2}A_{n-\nu-1}^{\gamma}}{\gamma+2} + \ldots + \frac{A_{n-\nu}^{-\gamma-2}A_0^{\gamma}}{n+1}.$$

However, from the Cauchy product rule we have for $|x|<1$,

$$\Big(\sum_0^\infty A_n x^n\Big)\Big[\frac{x^{\nu+1}}{\nu+1}A_0^{-\gamma-2} + \frac{x^{\nu+2}}{\nu+2}A_1^{-\gamma-2} + \ldots\Big] = \sum_{n=\nu}^\infty B_{n,\nu}x^{n+1}.$$

But the left-hand side is just

$$\frac{1}{(1-x)^{\nu+1}}\int_0^x \xi^\nu(1-\xi)^{\gamma+1}d\xi = \frac{1}{(1-x)^{\gamma+1}}\Big[-\frac{x^\nu(1-x)^{\gamma+2}}{\gamma+2} -$$

$$-\frac{\nu x^{\nu-1}(1-x)^{\gamma+3}}{(\gamma+2)(\gamma+3)} - \ldots - \frac{\nu(\nu-1)\ldots 2}{(\gamma+2)(\gamma+3)\ldots(\gamma+\nu)}\cdot\frac{x(1-x)^{\gamma+1+\nu}}{\gamma+1+\nu} -$$

$$-\frac{\nu(\nu-1)\ldots 2\cdot 1}{(\gamma+2)(\gamma+3)\ldots(\gamma+1+\nu)}\cdot\frac{1}{\gamma+2+\nu}\big((1-x)^{\gamma+2+\nu}-1\big)\Big].$$

But

$$\frac{1}{(1-x)^{\gamma+1}}\cdot\frac{\nu(\nu-1)\ldots 2\cdot 1}{(\gamma+2)(\gamma+3)\ldots(\gamma+2+\nu)} = \frac{1}{(\gamma+2)A_\nu^{\gamma+2}}\sum_0^\infty A_n^\gamma x^n.$$

Hence for $n+1 \geqslant \nu+2$ or $\nu \leqslant n-1$,

$$B_{n,\nu} = \frac{1}{\gamma+2}\cdot\frac{A_{n+1}^\gamma}{A_\nu^{\gamma+2}} \quad \text{for } \nu = n,\ B_{n,n} = \frac{1}{n+1}.$$

Thus, we have

$$s_n^\gamma(v) = \frac{A_{n+1}^\gamma}{\gamma+2}\sum_{\nu=0}^{n-1}\frac{s_\nu^\gamma(u)}{A_\nu^{\gamma+2}} + \frac{s_n^\gamma}{n+1}.$$

We assume again that $u^{(0)} = u^{(1)} = \ldots = u^{(x)} = 0$ so that $s_n^{x+y+r}(u) = o(n^{y+r})$ for all non-negative integers r. If $x+y+r$ is not a negative integer, then from (7) it follows that

$$s_n^{x+y+r}(v) = O(n^{x+y+r})\sum_{\nu=0}^{n-2}o(\nu^{-x-2}) + \frac{o(n^{y+r})}{n+1}.$$

If $x < -1$, then

$$s_n^{x+y+r}(v) = O(n^{x+y+r})o(n^{-x-1}) + o(n^{y+r-1}) = o(n^{y+r-1}).$$

Thus $s_n^{x+y+r} = s_n^{x+1+y-1+r} = o(n^{y-1+r})$ and $\sum\frac{u_n}{n+1}\epsilon R_{x+1,y-1}$ by Theorem 1*.

If $x > -1$, then $\sum \frac{s_\nu^{x+y+r}}{A_\nu^{x+y+r+2}}$ converges to c_0, say, and

$$s_n^{x+y+r} = \frac{A_{n+1}^{x+y+r}}{x+y+r+2}\Big[c_0 - \sum_{\nu=n-1}^{\infty} o(\nu^{-x-2})\Big] + o(n^{y-1+r})$$

$$= \frac{c_0}{x+y+r+2} \cdot A_n^{x+y+r} + o(n^{y-1+r}).$$

Dividing both sides of the equation by A_n^{x+y+r} we see that $\frac{s_n^{x+y+r}(v)}{A_n^{x+y+r}}$ $\to \frac{c_0}{x+y+r+2}$ as $n \to \infty$ and hence $\frac{c_0}{x+y+r+2} = v^{(0)}$. We have therefore $s_n^{x+y+r}(v) = v^{(0)} A_n^{x+y+r} + o(n^{y-1+r})$. Thus $\sum \frac{u_n}{n+1} \in R_{x+1,y-1}$. However, if $x = -1$ the estimates fails since $\sum_{\nu=0}^{n-2} o(\nu^{-x-2}) = \sum_{\nu=0}^{n-2} o(\nu^{-1}) = o(\log n)$.

If $x+y+r$ is a negative integer, then it immediately follows from (8) that

$$s_n^{x+y+r} = o(n^{y+r-1}).$$

Now suppose $\sum n u_n \in R_{x-1,y+1}$. Then $\sum (n+1) u_{n+1} \in R_{x-1,y+1}$ and by what was just proved $\sum \frac{(n+1)u_{n+1}}{n+1} = \sum u_{n+1} \in R_{x,y}$ provided $x-1 \neq -1$. Hence $\sum u_n \in R_{x,y}$ provided $x \neq 0$. This completes the proof of Lemma 2.

Proof of Lemma 3. It suffices to prove the lemma for $q = 1$, since by repeated application of this special case we obtain the proof of the general case. Thus, suppose $\sum u_n \in R_{x-1,y}$. Let $v_n = \Delta u_n$ and r a non-negative integer. If $x+y+r-1$ is not a negative integer and $x-1 \geqslant 0$, then

$$s_n^{x+y+r}(v) = -\sum_{\nu=0}^{n+1} A_{n+1-\nu}^{x+y+r-1} u_\nu + u_0 A_{n+1}^{x+y+r}.$$

Thus

$$s_n^{x+y+r}(v) = -s_{n+1}^{x+y+r-1}(u) + u_0 A_n^{x+y+r} + u_0 A_{n+1}^{x+y+r-1}. \tag{9}$$

By hypothesis

$$s_n^{x+y+r-1}(u) = c_0 A_n^{x+y+r-1} + c_1 A_n^{x+y+r-2} + \ldots + c_k A_n^{y+r} + o(n^{y+r})$$

for all non-negative integers r, $x = k+a-1$.

Hence

$$s_n^{x+y+r}(v) = d_0 A_n^{x+y+r} + d_1 A_n^{x+y+r-1} + \ldots + d_k A_n^{y+r} + o(n^{y+r})$$

and $\sum v_n \in R_{x,y}$.

Conversely, assume $\sum v_n \epsilon R_{x,y}$ and assume $x+y+r$ is not a negative integer and $x \geqslant 0$. From (9) it follows that

$$s_{n+1}^{x-1+y+r}(u) = A_n^{x+y+r}u_0 + A_{n+1}^{x+y+r}u_0 - [c_0 A_n^{x+y+r} + \ldots + c_k A_n^{y+r} + o(n^{y+r})]$$
$$= (u_0 - c_0)A_n^{x+y+r} + \ldots + c_k A_n^{y+r} + o(n^{y+r}).$$

Choosing $u_0^* = c_0 - u_0$ then $\Delta u_n^* = v_n$ and

$$s_{n+1}^{x-1+y+r}(u) = 0\cdot A_n^{x+y+r} - (u_0^* - v^{(1)})A_n^{x+y+r-1} + \ldots + (-1)^k v^{(x)} A_n^{y+r} + o(n^{y+r})$$

and hence $\sum u_n^* \epsilon R_{x-1,y}$.

If $x+y+r$ is a negative integer, then from (9) we see that $s_n^{x+y+r}(v) = -s_{n-1}^{x+y+r-1}(u)$. Since $s_n^{x+y+r}(v) = o(n^{y+r})$ it follows that $\sum u_n \epsilon R_{x-1,y}$.

If $x<0$, then choosing $u_n^* = u_n - u_0$ it follows from (9) that

$$s_n^{x+y+r}(v) = -s_{n+1}^{x+y+r-1}(u^*) + A_n^{x+y+r}u_0^* + A_{n+1}^{x+y+r-1}u_0^*.$$

Since $u_0^* = 0$,

$$s_n^{x+y+r}(v) = -s_{n+1}^{x+y+r-1}(u^*) = o(n^{y+r}).$$

This completes the proof of the lemma.

Returning to the theorem, we have $\sum n^p \Delta^q u_n \epsilon R_{x+q-p,y+p}$ if and only if $\sum n^{p-1}\Delta^q u_n \epsilon R_{x+q-p+1,y+p-1}$ by Lemma 2, provided $x+q-p+1 \neq 0$. Applying Lemma 2 $p-1$ more times we see that $n^p \Delta^q u_n \epsilon R_{x+p-q,y+p}$ if and only if $\sum \Delta^q u_n \epsilon R_{x+q,y}$ provided $x+q-j \neq -1$ for $j = 1, 2, \ldots, p$. From Lemma 3 this is true if and only if $\sum u_n^* \epsilon R_{x,y}$.

Before proving Theorem 2* we need the following theorem which gives us the inclusion property on the order of convergence and on the order of summability.

THEOREM 3. (i) *If* $\sum u_n \epsilon R_{x,y}$, *then* $\sum u_n \epsilon R_{x,y'}$ *for* $y' > y$.

(ii) *If* $\sum u_n \epsilon R_{x,y}$, *then* $\sum u_n \epsilon R_{x',y}$ *for* $x' < x$.

Proof. For $x \geqslant 0$, $y > -1$, (i) is clearly true since $\sum u_n^{(x)}$ is summable (C, y) implies that $\sum u_n^{(x)}$ is summable (C, y'). For $x \geqslant 0$, $y \leqslant -1$, if k is a positive integer such that $y+k > -1$, then

$$\sum u_n \epsilon R_{x,y} \Rightarrow \sum n^k \Delta^k u_n \epsilon R_{x,y+k} \Rightarrow \sum n^k \Delta^k u_n \epsilon R_{x,y'+k} \Rightarrow \sum u_n \epsilon R_{x,y'}$$

by Theorem 2. For $x < 0$, if k is a positive integer such that $x+k \geqslant 0$, then

$$\sum u_n \epsilon R_{x,y} \Rightarrow \sum \Delta^k u_n \epsilon R_{x-k,y} \Rightarrow \sum \Delta^k u_n \epsilon R_{x+k,y'} \Rightarrow \sum u_n \epsilon R_{x,y}$$

by Lemma 3. This completes the proof of (i).

In order to prove (ii) we first let $y \geqslant 0$ and $x' \geqslant 0$. Let $x = a+k-1$, $0 < a \leqslant 1$. We have

$$s_n^{x'+y} = \sum_{\nu=0}^{n} A_{n-\nu}^{x'-x-1} s_\nu^{x+y}.$$

By hypothesis and Theorem 1

$$s_n^{x'+y} = \sum_{\nu=0}^{n} A_{n-\nu}^{x'-x-1}[c_0 A_\nu^{x+y} + c_1 A_\nu^{x+y-1} + \ldots + c_k A_\nu^y + o(\nu^y)]$$

$$= c_0 A_n^{x'+y} + c_1 A_n^{x'+y-1} + \ldots + c_k A_n^{x'-x+y} + o(n^y)$$

since $x' - x - 1 < -1$ and $\sum A_{n-\nu}^{x'-x-1} o(\nu^y) = o(n^y)$.

Also, if $x' < y + m$ for some integer m, $0 \leqslant m \leqslant k$, then $A_n^{x'-m+y} = o(n^y)$. Thus we have

$$s_n^{x'+y} = c_0 A_n^{x'+y} + c_1 A_n^{x'+y-1} + \ldots + c_k A_n^y + o(n^y).$$

Hence $\sum u_n \epsilon R_{x',y}$.

Suppose x and y are arbitrary. Let p be a positive integer such that $y+p \geqslant 0$ and q a positive integer such that $x'-p+q \geqslant 0$. Then

$$\sum u_n \epsilon R_{x,y} \Rightarrow \sum n^p \Delta^q u_n \epsilon R_{x-p+q,\, y+p}$$
$$= \sum n^p \Delta^q u_n \epsilon R_{x'-p+q,\, y+p} \Rightarrow \sum u_n \epsilon R_{x',y}$$

by Theorem 2.

We can now easily prove Theorem 2* using Theorem 3 and noting that

$$A_n^p = c_0 + c_1 n + c_2 n^2 + \ldots + c_p n^p \quad \text{for some constants } c_j,\ j = 0, \ldots, n.$$

From this fact and from Theorems 2 and 2* we can also prove

THEOREM 4. *A necessary and sufficient condition for* $\sum A_n^\gamma u_n$ *to be in class* $R_{x,y}$ *is that* $\sum n^\gamma u_n \epsilon R_{x,y}$ *for* x, y *arbitrary*, $\gamma \neq -1, -2, -3, \ldots$

So far we have considered only the transformations

$$v_n = A_n^p \Delta^q u_n \quad \text{and} \quad v_n = n^p \Delta^q u_n$$

for p, q non-negative integers. If p is not necessarily an integer, say $p = \gamma$, it still follows in many cases that

$$\sum A_n^\gamma \Delta^q u_n \epsilon R_{x+q-\gamma,\, y+\gamma} \Leftrightarrow \sum u_n \epsilon R_{x,y}$$

although further restrictions on x are necessary.

In comparing $\sum u_n$ and $\sum A_n^\gamma \Delta^q u_n$, $\gamma > 0$, q a non-negative integer, if we let $a_n = \Delta^q u_n$, then the problem reduces to finding the relation between $\sum a_n$ and $\sum A_n^\gamma a_n$ and then applying Lemma 3 which connects $\sum a_n$ and $\sum u_n$. The following theorems give us the desired information:

THEOREM 5. *If* $\sum a_n \epsilon R_{x,y}$, *then* $\sum A_n^\gamma a_n \epsilon R_{x-\gamma,\, y+\gamma}$ *for* $\gamma > 0$, *non-integral, provided* $x - \gamma \neq 0, 1, 2, \ldots$

THEOREM 6. *If* $\sum A_n^\gamma a_n \epsilon R_{x-\gamma,\, y+\gamma}$, *then* $\sum a_n \epsilon R_{x,y}$ *for* $\gamma > 0$, *non-integral, provided* $x \neq 0, 1, 2, \ldots$

Combining Theorems 5 and 6 we have

THEOREM 6*. *If $\gamma > 0$, non-integral, x and $x-\gamma \neq 0, 1, 2, \ldots$, then $\sum a_n \epsilon R_{x,y}$ if and only if $\sum A_n^\gamma a_n \epsilon R_{x-\gamma, y+\gamma}$.*

If γ is non-integral, positive, and $x = \gamma$, then Theorem 6 extends the results of the classical theorem mentioned in the introduction. Thus, if $\sum A_n^\gamma a_n$ is summable $(C, y+\gamma)$, then $\sum a_n$ not only is summable (C, y) but is in $R_{\gamma, y}$.

Borwein [3] has defined Cesàro summability of order $\leqslant -1$ of a series $\sum a_n$ in the following way:

A series $\sum a_n$ is summable $(C, -\gamma)$, $\gamma \geqslant 1$ if

$$(10) \qquad \sum A_{n-\nu}^{-\gamma-1} A_\nu^\gamma s_\nu^{(0)} \to L \quad \text{as } n \to \infty, \text{ where } s_n^{(0)} = \sum_{\nu=0}^{n} a_\nu.$$

From Theorem 5 and Theorem 6 we readily show that this definition coincides with the one used in this paper. We may suppose (10) converges to zero as $n \to \infty$. Then

$$\sum_{\nu=0}^{n} A_{n-\nu}^{-\gamma-1} A_\nu^\gamma s_\nu^{(0)} \to 0 \quad \text{as } n \to \infty \Leftrightarrow \sum A_n^\gamma s_n^{(0)} \epsilon R_{-\gamma-1, 0}$$
$$\Leftrightarrow \sum s_n^{(0)} \epsilon R_{-1, -\gamma} \Leftrightarrow \sum a_n \epsilon R_{0, -\gamma}.$$

The first equivalence follows from Theorem 1, the second from Theorem 5 and Theorem 6 and the third, from Lemma 3.

Both theorems are immediate consequences of the following lemma:

LEMMA 4. *If $\sum a_n \epsilon R_{x,y}$ and if $\{f_n\}$ is a sequence such that*

$$\Delta^j f_n = O(n^{\gamma-j}), \quad j = 0, 1, 2, \ldots,$$

then

$$\sum a_n f_n \epsilon R_{x-\gamma, y+\gamma}$$

provided $x-\gamma \neq 0, 1, 2, \ldots$

The proof is similar to the proofs of Lemmas A and B of Marcinkiewicz and Zygmund [11]. However, in that paper the authors deal only with trigonometric series and the proofs correspond to special cases of the above lemma. Since we are concerned here with a more general situation, one in which the conclusion of the lemma fails for certain values of the parameters, we give the complete proof.

Proof of Lemma 4. The lemma will first be proved for $x+y$ an integer. Let r be any non-negative integer such that $x+y+r \geqslant 0$ and let $t = x+y+r$, $\eta_\nu = A_{n-\nu}^t f_\nu$. We also suppose for the time being that

$\gamma > x$ and without loss of generality $s_n^t = \sum_{\nu=0}^{n} A_{n-\nu}^t a_\nu = o(n^{y+r})$. Now

$$\sum_{\nu=0}^{n} A_{n-\nu}^t a_\nu f_\nu = \sum_{\nu=0}^{n} a_\nu \eta_\nu = (-1)^{t+1} \sum_{\nu=0}^{n-t-1} s_\nu^t \Delta^{t+1} \eta_\nu + \sum_{k=0}^{t} s_{n-k}^k \Delta^k \eta_{n-k} = P_n + Q_n.$$

We have

$$\Delta^k \eta_{n-k} = O(n^\gamma) \quad \text{for } k = 0, 1, \ldots, t$$

and since the relation $s_n^k = o(n^{y+r})$ implies $s_{n-k}^k = o(n^{y+r})$ for $n \to \infty$ it follows that

$$Q_n = o(n^{y+r}) O(n^\gamma) = o(n^{y+\gamma+r}).$$

We use here the fact that $\sum a_n \epsilon R_{x,y}$ implies $\sum a_n \epsilon R_{k-y-r,y}$ since $k-y-r \leqslant x$. Hence $s_n^{k-r+r^*} = o(n^{y+r^*})$ for all non-negative integers r^* and in particular, for $r^* = r$, $s_n^k = o(n^{y+r})$. In order to estimate the expression P_n we use the formula

$$\Delta^t \xi_n \eta_n = \sum_{k=0}^{t} c_{t,k} \Delta^k \xi_n \Delta^{t-k} \eta_{n+k}, \quad t = 0, 1, 2, \ldots$$

Hence, taking into account that $\Delta^k f_\nu = O(\nu^{\gamma-k})$ we obtain

$$\Delta^{t+1} \eta_\nu = \sum_{k=0}^{t+1} c_{t+1,k} A_{n-\nu}^{t-k+1} O(\nu^{\gamma+k-t-1}).$$

It follows that P_n is equal to the sum of the expressions

$$c_{t+1,k} \sum_{\nu=0}^{n-t-1} s_\nu^t A_{n-\nu}^{t-k} O(\nu^{\gamma+k-t-1}) \quad \text{for } k = 0, 1, \ldots, t.$$

This expression in absolute value does not exceed

$$c_{t+1,j} \sum_{\nu=0}^{n-t-1} o(\nu^{y+r}) O(\nu^{\gamma+k-t-1}) A_{n-\nu}^{t-k} = c_{t+1,j} \sum_{\nu=0}^{n-t-1} o(\nu^{\gamma-x+k-1}) A_{n-\nu}^{t-k} = o(n^{\gamma+y+r})$$

since $t-k > -1$ and for $\gamma > x$, $\gamma - x + k - 1 > -1$.

If $\gamma \leqslant x$, then for k a large enough positive integer $\sum n^k a_n \epsilon R_{x-k,y+k}$ and $\gamma > x-k$. By what was proved above it follows that $\sum n^k a_n f_n \epsilon R_{x-k-\gamma,y+k+\gamma}$ and by Theorem 2 $\sum a_n f_n \epsilon R_{x-\gamma,y+\gamma}$ provided $x-\gamma \neq 0, 1, \ldots, k-1$.

Thus if $x-\gamma$ is a positive integer, then the smallest value for k we can choose so that $\gamma > x-k$ is $x-\gamma+1$ and the latter condition would fail to hold. Thus the theorem is true for this case provided $x-\gamma \neq 0, 1, 2, \ldots$

Suppose that $x+y+r = -t-1$ is a negative integer. We may write

$$\sum_{\nu=0}^{n} A_{n-\nu}^{-t-1} a_\nu f_\nu = (-1)^t \Delta^t [a_{n-t+1} f_{n-t+1}]$$

$$= (-1)^t \sum_{k=0}^{t} c_{t,k} \Delta^k a_{n-t} \Delta^{t-k} f_{n-t+k}$$

$$= (-1)^t \sum_{k=0}^{t} c_{t,k} (-1)^k s_{n-t+j}^{-k-1} \Delta^{t-k} f_{n-t+k}$$

$$= \sum_{k=0}^{t-1} o(n^{y+r+t-k}) O(n^{\gamma-t+k}) = o(n^{\gamma+y+r}).$$

This proves the theorem for $x+y$ integral.

In the case of fractional $x+y$, we assume for the time being that $x+y > -1$, $\gamma > x$ and $y > -1$. We write

$$t = [x+y]+1, \quad \eta_\nu = A_{n-\nu}^{x+y}(f_\nu - f_n).$$

Now

$$\sum_{\nu=0}^{n} A_{n-\nu}^{x+y} a_\nu f_\nu = f_n \sum_{\nu=0}^{n} A_{n-\nu}^{x+y} a_\nu + \sum_{\nu=0}^{n} a_\nu \eta_\nu = o(n^{y+\gamma}) + \sum_{\nu=0}^{n} a_\nu \eta_\nu.$$

To the last sum we apply Abel's transformation $t+1$ times,

$$\sum_{\nu=0}^{n} a_\nu \eta_\nu = \sum_{\nu=0}^{n-t-1} s_\nu^t \Delta^{t+1} \eta_\nu + \sum_{k=1}^{t} s_{n-k}^k \Delta^k \eta_{n-k} = P_n + Q_n, \quad \text{say}.$$

Now

$$\Delta^k \eta_\nu = \sum_{j=0}^{k-1} c_{k,j} A_{n-\nu}^{x+y-j} \Delta^{k-j} f_{\nu+j} + A_{n-\nu}^{x+y-k}(f_{\nu+k} - f_n).$$

For $\nu = n-k$ it is easy to see from the above equation that

$$\Delta^k \eta_{n-k} = O(n^{\gamma-k+j}) = O(n^{\gamma-1}) \quad \text{for } j = 0, 1, \ldots, k-1,$$

$s_n^t = o(n^{t-x})$ by Theorem 3 and so $s_{n-k}^k = o(n^{t-x})$, $k = 0, 1, \ldots, t$. Hence

$$Q_n = \sum_{k=1}^{t} o(n^{t-x}) O(n^{\gamma-1}) = o(n^{t-x-1+\gamma}) = o(n^{\gamma+y}).$$

It remains to extimate the sum P_n. On account of the equation

$$\Delta^k \eta_\nu = \sum_{j=0}^{k-1} c_{k,j} A_{n-\nu}^{x+y-j} \Delta^{k-j} f_{\nu+j} + A_{n-\nu}^{x+y-k}(f_{\nu+k} - f_n)$$

with $k = t+1$, P_n is a sum of $t+1$ expressions

$$c_{t+1,j}\sum_{\nu=0}^{n-t-1} s_\nu^t A_{n-\nu}^{x+y-j}\Delta^{t+1-j}f_{\nu+j}, \quad j = 0, 1, \ldots, t,$$

and of the expression

$$\sum_{\nu=0}^{n-t-1} s_\nu^t A_{n-\nu}^{x+y-t-1}(f_{\nu+t+1}-f_n).$$

The first sum is

$$\sum_{\nu=0}^{n-t-1} o(\nu^{t-x})A_{n-\nu}^{x+y-j}O(\nu^{\gamma-t-1+j}) = \sum_{\nu=0}^{n-t-1} o(\nu^{\gamma-x-1+j})A_{n-\nu}^{x+y-j} = o(n^{\gamma+y})$$

provided $\gamma > x$. Now

$$f_\nu - f_n = \sum_{k=\nu}^{n-1}\Delta f_k = \sum_{k=\nu}^{n-1} O(k^{\gamma-1}) \quad \text{for } \gamma \geqslant 1$$

and

$$f_\nu - f_n = \sum_{k-\nu}^{n-1} O(\nu^{\gamma-1}) \quad \text{for } \gamma < 1.$$

Thus in any case we can write

$$f_\nu - f_n = (n-\nu)O(n^{\gamma-1}) + O(\nu^{\gamma-1})$$

and the latter sum does not exceed in absolute value

$$\sum_{\nu=0}^{n-t-1} o(\nu^{t-x})A_{n-\nu}^{x+y-t-1}(n-\nu)O(n^{\gamma-1}) + O(\nu^{\gamma-1}) = o(n^{\gamma+y}).$$

If $\gamma \leqslant x$ or $y \leqslant -1$, then for k a large enough positive integer $\sum n^k a_n \epsilon R_{x-k,y+k}$ and $\gamma > x-k$, $y+k > -1$. By what was proved above $\sum n^k a_n \epsilon R_{x-k-\gamma,y+k+\gamma}$ and by Theorem 2, $\sum a_n f_n \epsilon R_{x-\gamma,y+\gamma}$ provided $x-\gamma \neq 0, 1, 2, \ldots$ This proves the theorem for $x+y > -1$.

Assume now that $x+y \leqslant -1$ and in addition assume also that $y > 0$, $\gamma+y > 0$. We write

$$\sum_{\nu=0}^{n} A_{n-\nu}^{x+y} a_\nu f_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{x+y}(a_\nu f_\nu - a_\nu f_n) + f_n \sum_{\nu=0}^{n} A_{n-\nu}^{x+y} a_\nu.$$

The last term is equal to $f_n s_n^{x+y} = O(n^\gamma)o(n^y) = o(n^{\gamma+y})$. Consider

$$\sum_{\nu=0}^{n} A_{n-\nu}^{x+y} a_\nu(f_\nu - f_n) = \sum_{\nu=0}^{n} A_{n-\nu}^{x+y}\eta_\nu, \quad \text{where } \eta_\nu = a_\nu(f_\nu - f_n).$$

Letting $-t-1 < x+y \leqslant -t$, where t is a non-negative integer we have

$$(11) \qquad \sum_{\nu=0}^{n} A_{n-\nu}^{x+y} \eta_\nu = (-1)^t \sum_{\nu=t}^{n} A_{n-\nu}^{x+y+t} \Delta^t \eta_{\nu-t} + \sum_{j=0}^{t-1} A_{n-t+1+j}^{x+y+t-j} \Delta^{t-1-j} \eta_0 .$$

But

$$\Delta^t \eta_{\nu-t} = \sum_{k=0}^{t} c_{t,k} \Delta^{t-k} a_{\nu-t+k} \Delta^k (f_{\nu-t} - f_n) \text{ and } \Delta^{t-k}(a_{\nu-t+k}) = (-1)^{-k} s_\nu^{k-t-1}.$$

From (11) we obtain

$$(12) \qquad \sum_{\nu=0}^{n} A_{n-\nu}^{x+y} \eta_\nu = (-1)^t \sum_{\nu=t}^{n} A_{n-\nu}^{x+y+t} \sum_{k=0}^{t} c_{t,k} (-1)^{t-k} s_\nu^{k-t-1} \Delta^k (f_{\nu-t} - f_n) + \\ + \sum_{j=0}^{t-1} A_{n-t+j+1}^{x+y+t-j} \Delta^{t-j-1} \eta_0 = P_n + Q_n .$$

Since $\Delta^{t-1-j}\eta_0 = o(1)$ if $\gamma \leqslant 0$ and $O(n^\gamma)$ if $\gamma > 0$ for $j = 0, 1, 2, \ldots$ $\ldots, t-1$ we can write in any case

$$\Delta^{t-1-j}\eta_0 = O(1) \qquad \text{and} \qquad Q_n = O(n^{x+y+t}) = o(n^{\gamma+y})$$

assuming, as we do, that $y+\gamma > 0$, $y > 0$ and $x+y \leqslant -t$.

Now

$$P_n = (-1)^t \sum_{\nu=t}^{n} A_{n-\nu}^{x+y+t} \sum_{k=1}^{t} c_{t,k} [\Delta^k (f_{\nu-t} - f_n)] (-1)^{t-k} s_\nu^{k-t-1} + \\ + \sum_{\nu=t}^{n} A_{n-\nu}^{x+y+t} c_{t,0} (f_{\nu-t} - f_n) (-1)^t s_\nu^{-t-1} = P'_n + P''_n .$$

$$P'_n = \sum_{\nu=t}^{n} O\big((n-\nu)^{x+y+t}\big) O(\nu^{\gamma-k}) o(\nu^{-x-t+k-1}) \\ = \sum_{\nu=t}^{n} O(n-\nu)^{x+y+t} O(\nu^{\gamma-t-x-1}) = o(n^{y+\gamma})$$

since $\gamma+y > 0$ and $x+y \leqslant -t$ implies $\gamma-t-x-1 > 1$. Substituting $s_\nu^{-t} - s_{\nu-1}^{-t}$ for s_ν^{-t-1} and expanding, we see that P''_n can be written

$$(-1)^n c_{t,0} \Big\{ \sum_{\nu=t}^{n-1} A_{n-\nu}^{x+y+t} s_\nu^{-t} (f_{\nu-t} - f_n) + \sum_{\nu=t}^{n-1} A_{n-\nu}^{x+y+t-1} s_\nu^{-t} (f_{\nu+1-t} - f_n) + \\ + (f_{n-t} - f_n) s_n^{-t} - A_{n-t}^{x+y+t} (1 - f_n) s_{t-1}^{-t} \Big\}.$$

The last two terms are respectively $O(n^{\gamma-1})o(n^{-x-t}) = o(n^{y+\gamma})$ and since $1-f_n = [O(1)+O(n^{\gamma})]$, the last term is $O(n^{x+y+t})[O(1)+O(n^{\gamma})] = o(n^{y+\gamma})$ since $y+\gamma > 0$. The first term is

$$\sum_{\nu=t}^{n-1} O(n-\nu)^{x+y+t}O(\nu^{\gamma-1})o(\nu^{-x-t}) = \sum_{\nu=t}^{n-1} O(n-\nu)^{x+y+t}O(\nu^{\gamma-1-x-t}) = o(n^{y+\gamma}).$$

Finally, since $f_{\nu+1-t}-f_n = (n-\nu)O(\nu^{\gamma-1})+O(n^{\gamma-1})$, the second term is

$$\sum_{\nu=t}^{n-1} O(n-\nu)^{x+y+t-1}o(\nu^{-x-t})(n-\nu)[O(\nu^{\gamma-1})+O(n^{\gamma-1})]$$

$$= \sum_{\nu=t}^{n-1} O(n-\nu)^{x+y+t}o(\nu^{-x-\gamma-t-1}) + O(n^{\gamma-1})\sum_{\nu=t}^{n-1} O(n-\nu)^{x+y+t}O(\nu^{-x-t})$$

$$= o(n^{\gamma+y})+O(n^{\gamma-1})o(n^{y+1}) = o(n^{\gamma+y}).$$

In the general case, we let k be a positive integer such that $y+k > 0$ and $\gamma+y+k > 0$. Then $\sum n^k a_n \in R_{x-k,\,y+k}$ and by what has just been proved $\sum n^k a_n f_n \in R_{x-k-\gamma,\,y+k+\gamma}$ and by Theorem 2. $\sum a_n f_n \in R_{x-\gamma,\,y+\gamma}$ provided $x-\gamma \neq 0, 1, 2, \ldots$ This completes the proof of the lemma.

If $\sum a_n \in R_{x,0}$, x non-integral and greater than 0, then it is not true that $\sum A_n^x a_n \in R_{0,x}$. It is known that a Fourier series $S[f]$ is in $R_{a,0}$ uniformly if and only if $f(x)$ is in the Lipschitz class λ_a, see [13], and it is well known that $f(x)$ being in λ_a does not imply that $\sum n^a(a_n \cos nx + b_n \sin nx)$ is summable (C, a) for all $x \in (0, 2\pi)$. However, we do have the following result:

THEOREM 7. *Let $x \geqslant 0$. A necessary and sufficient condition for $\sum A_n^x a_n$ to be in class $R_{0,x}$ (i. e. to be summable (C, x)) is that $\sum A_n^x a_n$ is summable by Abel's method and $\sum a_n \in R_{x,0}$.*

Proof. If $\sum A_n^x a_n \in R_{0,x}$, then it is summable (A) and by Theorem 6, $\sum a_n \in R_{x,0}$.

Conversely, suppose $\sum A_n^x a_n$ is summable (A) and $\sum a_n \in R_{x,0}$.

LEMMA 5. *Suppose $\sum a_n \in R_{x,0}$ and $a^{(0)} = a^{(1)} = \ldots = a^{(x)} = 0$. Then $\sum A_n^x a_n$ is equisummable (C, x) with $\sum s_n^x \Delta^{x+1}(A_n^x)$, where $x = k+a-1$, $0 < a \leqslant 1$, $x \geqslant 0$ and where $\Delta^{x+1}(A_n^x)$ is the fractional difference of A_n^x of order $x+1$.*

Proof. The following formula is valid: $s-1 < x < s$,

$$\sum_{\nu=0}^{n} A_\nu^x a_\nu = \sum_{\nu=0}^{n} s_\nu^x \Delta^{x+1}(A_\nu^x) + \sum_{j=1}^{s} s_{n+1-j}^{j-1}\Delta^{j-1}(A_{n+2-j}^x) + o(1). \tag{13}$$

To see this, observe that

$$\sum_{\nu=0}^{n} A_\nu^x a_\nu = \sum_{\nu=0}^{n} \left(\Delta_{n-\nu}^{x+1}(A_\nu^x)\right) s_\nu^x = \sum_{\nu=0}^{n} s_\nu^x \Delta^{x+1}(A_\nu^x) - \sum_{\nu=0}^{n} s_\nu^x \sum_{p=n-\nu+1}^{\infty} A_p^{-x-2} A_{\nu+p}^x .$$

Now

$$\sum_{\nu=0}^{n} s_\nu^x \sum_{p=n-\nu+1}^{\infty} A_p^{-x-2} A_{\nu+p}^x = \sum_{\nu=0}^{n} s_\nu^x \sum_{p=n+1}^{\infty} A_p^{-x-2} A_{\nu+p}^x + \sum_{\nu=0}^{n} s_\nu^x \sum_{p=n-\nu+1}^{n} A_p^{-x-2} A_{\nu+p}^x .$$

The first term on the right is

$$\sum_{\nu=0}^{n} s_\nu^x \sum_{p=n+1}^{\infty} O(p^{-x-2}) O\left((\nu+p)^x\right) = \sum_{\nu=0}^{n} s_\nu^x \sum_{p=n+1}^{\infty} o(p^{-x-2}) p^x (1+\nu/p)^x .$$

Since $(1+\nu/p) \leqslant 2$ the sum is

$$\sum_{\nu=0}^{n} s_\nu^x \sum_{p=n+1}^{\infty} p^{-x-2+x} o(1) = \sum_{\nu=0}^{n} s_\nu^x O(n^{-1}) = \sum_{\nu=0}^{n} o(1) O(n^{-1}) = o(1).$$

Consider $\sum_{\nu=0}^{n} s_\nu^x \sum_{p=n-\nu+1}^{n} A_p^{-x-2} A_{\nu+p}^x$. Summing the inner sum by parts s times, where $s-1 < x < s$ yields

$$\sum_{p=n-\nu+1}^{n} A_p^{-x-2} A_{\nu+p}^x = \sum_{p=n-\nu+1}^{n-s} A_p^{-x-2+s} \Delta^s (A_{\nu+p}^x) +$$
$$+ \sum_{j=1}^{s} [(A_{n+1-j}^{-x-2+j} \Delta^{j-1} A_{\nu+n+1-j}^x) - A_{n+1-\nu-j}^{-x-2+j} \Delta^{j-1} A_{n+2-j}^x].$$

Now

$$\left| \sum_{p=n-\nu+1}^{n-s} A_p^{-x-2+s} \Delta^s (A_{\nu+p}^x) \right| = O(n^{x-s}) \sum_{p=n-\nu+1}^{n-s} |A_p^{-x-2+s}|$$

and

$$\left| \sum_{\nu=0}^{n} s_\nu^x \sum_{p=n-\nu+1}^{n} A_p^{-x-2} A_{\nu+p}^x \right| = O(n^{x-s}) \sum_{\nu=0}^{n} |s_\nu^x| \sum_{p=n-\nu+1}^{n-s} |A_p^{-x-2+s}| .$$

But A_p^{-x-2+s} is of constant sign for $p \geqslant 1$. Hence we can omit the absolute value signs. Thus

$$\left| \sum_{p=n-\nu+1}^{n-s} A_p^{-x-2+s} \Delta^s (A_{\nu+p}^x) \right| = O(n^{x-s}) \sum_{\nu=0}^{n} o(1) \left(A_{n-s}^{-x-1+s} - A_{n-\nu}^{-x-1+s}\right)$$
$$= O(n^{x-s}) o(n^{-x+s}) = o(1).$$

Also

$$\sum_{\nu=0}^{n} s_\nu^x A_{n+1-\nu-j}^{-x-2+j} \Delta^{j-1} A_{n+2-j}^x = \Delta^{j-1} A_{n+2-j}^x \sum_{\nu=0}^{n+1-j} s_\nu^x A_{n+1-j-\nu}^{-x-2+j}$$
$$= \Delta^{j-1} (A_{n+2-j}^x) s_{n+1-j}^{j-1} \quad \text{for } j = 1, 2, \ldots, s.$$

It is clear that

$$\sum_{\nu=0}^{n} s_\nu^x \sum_{j=1}^{s} A_{n+1-j}^{-x-2+j} \Delta^{j-1} A_{\nu+n+1-j}^{x} = o(1).$$

This completes the proof of (13).

If $\sum a_n \in R_{x,0}$, then $\sum s_n^{(j)} \in R_{x-j-1,0}$ by Lemma 3 for $j = 0, 1, \ldots, k-1$. By Theorem 6

$$\sum A_n^{x-j} s_n^{(j)} \in R_{-1, x-j} \qquad \text{for } j = 0, 1, \ldots, k-1.$$

Thus the $(C, x-j)$ means of the sequence $\{A_n^{x-j} s_n^{(j)}\}$ tend to 0. Taking the $(C, x-1)$ means of each side of (13) show that $\sum A_n^x a_n$ is equisummable (C, x) with $\sum s_n^x \Delta^{x+1}(A_n^x)$ thus completing the proof of the lemma.

Return to the theorem. We will show that $\sum s_n^x \Delta^{x+1}(A_n^x)$ has terms whose order of magnitude is $o(1/n)$. For that purpose, we state a theorem by B. Kuttner (see [9], Theorem B):

If $s > -1$, $r+s > -1$ and $r+s$ is not an integer, then $\Delta^{r+s} a_n = \Delta^r(\Delta^s a_n)$ is valid whenever the expressions on both sides of the equation exist.

Now suppose x is fractional. Then $\Delta^{x+1}(A_n^x) = \Delta^{x-k} \Delta^{k+1}(A_n^x)$ since we can let $r = x-k$, $s = k+1$ in Kuttner's theorem. Then $s > -1$, $r+s = x+1 > -1$ and $r+s$ is not an integer if x is fractional. But $\Delta^{k+1} A_n^x = (-1)^{k+1} A_{n+k+1}^{x-k-1} = (-1)^{k+1} A_{n+k+1}^{a-2}$ and $x-k = a-1$. Thus

$$\Delta^{x+1}(A_x^n) = (-1)^{k+1} \Delta^{a-1}(A_{n+k+1}^{a-2})$$

$$= \sum_{p=0}^{\infty} A_p^{-a} O(n+p)^{a-2} = \sum_{p=0}^{n} + \sum_{p=n+1}^{\infty} = P+Q,$$

$$|P| = \Big|\sum_{p=0}^{n} A_p^{-a} O(n+p)^{a-2}\Big| = \Big|\sum_{p=0}^{n} A_p^{-a} n^{a-2}(1+p/n)^{a-2} O(1)\Big|$$

$$= n^{a-2} \sum_{p=0}^{n} p^{-a} 2^{a-2} O(1) = n^{a-2} O(n^{-a+1}) = O(n^{-1}),$$

$$|Q| = \sum_{p=n+1}^{\infty} A_p^{-a}(2p)^{a-2} O(1) = 2^{a-2} \sum_{p=n+1}^{\infty} p^{-2} = O(n^{-1}).$$

Thus, $\Delta^{x+1}(A_n^x) = O(1/n)$ and since $s_n^x = o(1)$, the terms of the series have an order of magnitude $o(n^{-1})$. If x is integral the result trivially follows.

Hence if the series is summable (A), then it converges. From (13) if $\sum A_n^x a_n$ is summable (A), $\sum s_n^x \Delta^{x+1}(A_n^x)$ converges and from the lemma $\sum A_n^x a_n$ is summable (C, x). This completes the proof of the theorem.

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