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# *THE PERIMETER OF MINKOWSKI UNIT DISCS*

BY

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Let $K$ be a convex body in the plane and let $z \in \operatorname{int} K$; a norm $\| \ \|_{K,z}$ (not symmetric unless $z$ is the center of symmetry of $K$) is defined by

$$\|x\|_{K,z} = \inf\{\lambda > 0 \mid x - z \in \lambda(-z+K)\}.$$

Using the (non-symmetric in general) distance function derived from $\| \ \|_{K,z}$ it is possible to define the arc length of oriented arcs (see, e. g., Gołąb [1]). For an oriented closed curve $C$ let the length of $C$ in this metric be denoted by $L_{K,z}$ $(C)$. Let $\operatorname{bd} K$ denote the boundary of $K$ in either of its two possible orientations.

Gołąb [1] and Hammer [4] conjectured that

(i) $\sup_{K} \min_{z \in \operatorname{int} K} L_{K,z}(\operatorname{bd} K) \leqslant 9$

and

(ii) $L_{K,z}$ $(\operatorname{bd} K) \geqslant 6$ for all $K$ and $z \in \operatorname{int} K$.

The first conjecture was recently established in [3]. The second, however, while well known and easily proved in case $K$ is centrally symmetric with center $z$, seems not to have been settled so far. It is the purpose of the present paper* to give an elementary solution of (ii) by establishing the following

THEOREM. *For every planar convex body $K$ and every $z \in \operatorname{int} K$ the inequality $L_{K,z}$ $(\operatorname{bd} K) \geqslant 6$ holds. Moreover, $L_{K,z}$ $(\operatorname{bd} K) = 6$ if and only if $K$ is an affine-regular hexagon and $z$ its center.*

Proof. We shall prove explicitly only the first assertion of the theorem, since the second follows by inspection of the possible cases of equality in different steps of the proof. The proof is elementary throughout, but its length makes it desirable to split it into a number of stages.

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1. It is well known (see, e. g., Gołąb [1]) that if $C_1$ and $C_2$ are convex curves having the same orientation and such that $C_2$ encloses $C_1$, then $L_{K,z}(C_1) \leqslant L_{K,z}(C_2)$. It is also well known (see [2] for a list of references) that for each convex curve there exists an affine-regular hexagon inscribed into it (i. e., having all its vertices on the curve). Therefore, denoting by $H$ the boundary of any affine-regular hexagon inscribed into bd $K$, we have $L_{K,z}$ (bd $K$) $\geqslant L_{K,z}$ $(H)$. We shall prove the theorem by showing that the last number is at least 6. Note that since $H$ is centrally symmetric, its length is independent of its orientation.

2. Let $H^*$ be a polygon with at most six sides, determined by supporting lines of $K$ passing through the vertices of $H$. Then obviously $L_{K,z}$ $(H) \geqslant L_{H^*,z}$ $(H)$. We shall show that $L_{H^*,z}$ $(H) \geqslant 6$.

3. If $z$ is not inside $H$, then (in the notation of Fig. 1) the edges $H_1$ and $H_3$ have length at least 1 each, while the sum of the lengths of $H_2$ and $H_5$ is at least 4. Thus the theorem is established in this case; in the sequel we shall assume that $z$ is inside $H$.

4. We shall have occasion to use the following easy exercise in analytic geometry: In an (oblique) coordinate system let be given a point $(x_0, y_0)$ such that $x_0 \geqslant y_0 > 0$; if the intercepts of a straight line through $(x_0, y_0)$ are $a > x_0$ and $b > y_0$, then the function $(1/a)+(1/b)$ is

(i) independent of $a$, provided $x_0 = y_0$;

(ii) monotonically decreasing for decreasing $a$, provided $x_0 > y_0$.

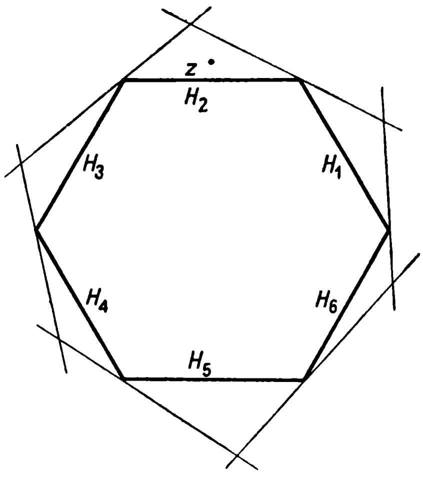


Fig. 1

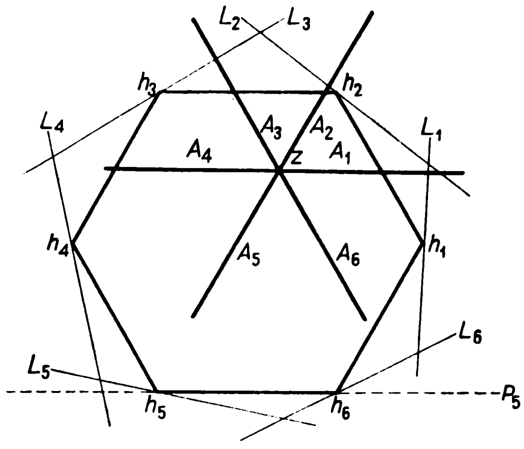


Fig. 2

This fact shall be used repeatedly in the sequel, in assertions of the type: "If a certain line is appropriately rotated about a certain point, the sum of the lengths of certain segments will not increase".

5. There exists a polygon $P$ with at most five sides, containing all the vertices of $H$, and such that $L_{H^*,z}$ $(H) \geqslant L_{P,z}$ $(H)$. Indeed (see Fig. 2

for the notation), if $H^*$ is changed by rotating its two sides $L_5$ and $L_6$ about $h_5$ resp. $h_6$ until they both coincide with the line $P_5$ determined by $h_5$ and $h_6$, the sum $\sigma$ of the lengths of the 6 heavily drawn segments $A_i$ (i. e. the length of $H$) can only decrease.

Thus in order to complete the proof of the theorem it is enough to establish it for the special case of (possibly degenerated) pentagons $P$ with the properties: (i) One side of the pentagon contains that side of the hexagon $H$ which is opposite to the "sextant" of $H$ which contains $z$; (ii) each side of $P$ contains a vertex of $H$. In the remaining part of the proof we shall have to consider three cases, according to the position of $z$ (see Fig. 3, in which $c_2$ (resp. $c_3$) is the midpoint of $c$ and $h_2$ (resp. $h_3$)).

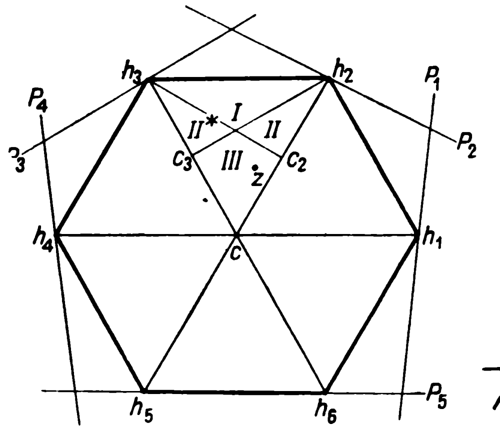

Fig. 3

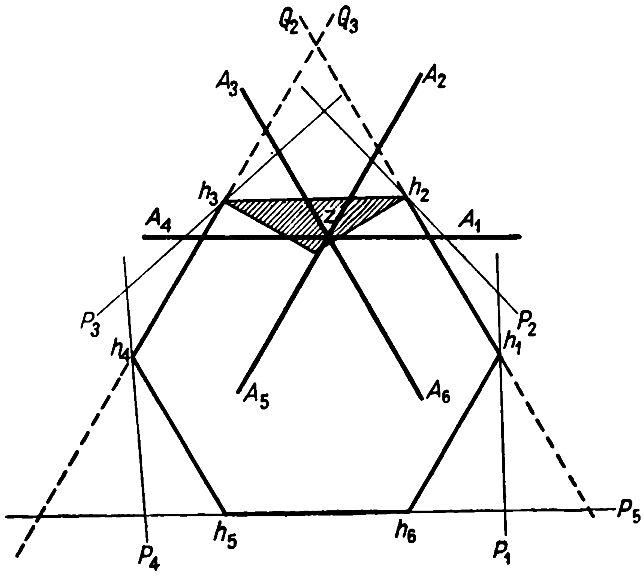

Fig. 4

6. First case. $z$ belongs to the triangle denoted by $I$ in Fig. 3. Then (see Fig. 4) $\sigma$ does not increase if the lines $P_2$ and $P_3$ are rotated to the new positions $Q_2$ and $Q_3$, and if the lines $P_1$ and $P_4$ are omitted. This yields instead of $P$ a triangle $T$ (bounded by $Q_2$, $Q_3$, and $P_5$); an elementary computation shows that $\|A_1\|_{T,z} + \|A_4\|_{T,z} \geqslant 3$ and $\|A_2\|_{T,z} + \|A_3\|_{T,z} \geqslant 3$. This completes the proof in the first case.

7. Second case. $z$ belongs to the triangle denoted by II in Fig. 3. (Clearly a similar reasoning applies if $z$ belongs to II*). Then $\sigma$ does not increase if $P_1$ and $P_4$ are omitted, $P_3$ rotated to $Q_3$ (see Fig. 5), and $P_2$ is rotated about $h_2$ until in position $Q_2$ it meets the intersection of $Q_3$ with $A_3$. Thus $P$ is replaced by a triangle $T$ bounded by $P_5, Q_2, Q_3$. Then clearly $\|A_1\|_{T,z} \geqslant 1$, $\|A_2\|_{T,z} \geqslant 2$, $\|A_3\|_{T,z} \geqslant 1$, $\|A_4\|_{T,z} \geqslant 1$, $\|A_5\|_{T,z} \geqslant \frac{1}{2}$ $\|A_6\|_{T,z} \geqslant \frac{1}{2}$, and the proof of the second case is completed.

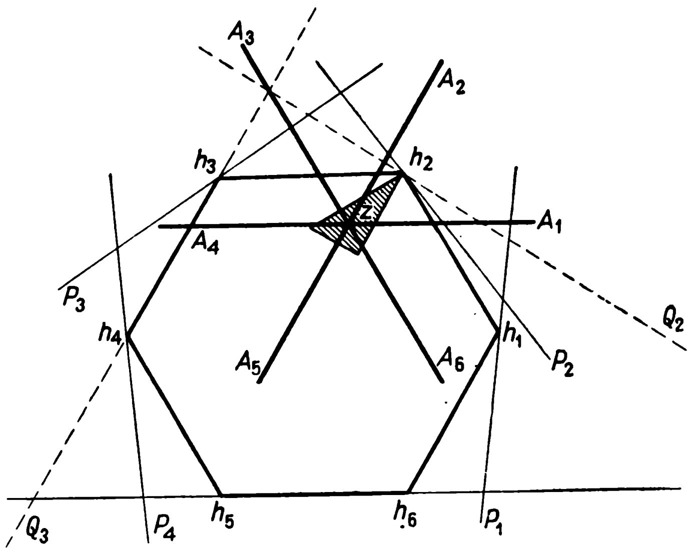


Fig. 5

8. Third case. $z$ belongs to the triangle denoted by III in Fig. 3. Then, without increasing $\sigma$, it is possible to rotate $P_2$ about $h_2$ until its intersection with $P_1$ is on $A_1$ or its continuation. Similarly, $P_3$ may be rotated about $h_3$ until it intersects $P_4$ on $A_4$ or its extension. Next, again without increasing $\sigma$, $P_1$ and $P_4$ may be rotated about $h_1$ (resp. $h_4$) in such a way that they intersect $P_5$ on the continuation of $A_6$ resp. $A_5$. Finally, $P_2$ and $P_3$ may be again rotated (about $h_2$ resp. $h_3$) until they intersect

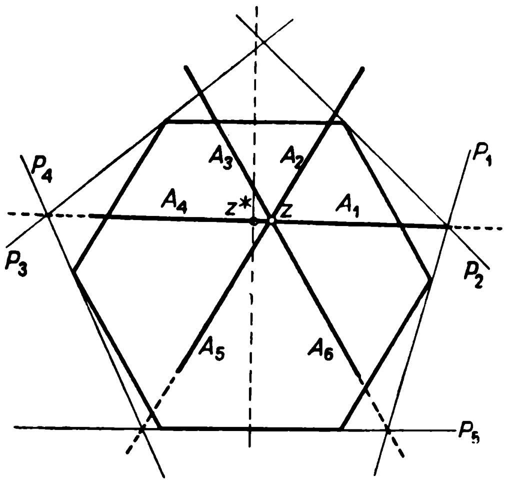


Fig. 6

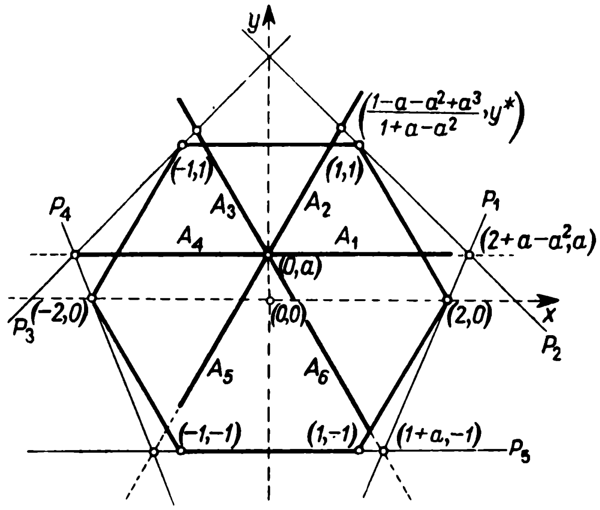


Fig. 7

the new positions of $P_1$ resp. $P_4$ on $A_1$ resp. $A_4$ or their extensions. Thus we see that in the present case $\sigma$ is minimized by a configuration of the type represented in Fig. 6, for which $P_1$, $P_5$ and $A_6$ or its extension are concurrent, and the same applies to $P_1$, $P_2$ and $A_1$, to $P_3$, $P_4$ and $A_4$, and to $P_4$, $P_5$ and $A_5$.

Now we consider, within this class of pentagons, the pentagon $P^*$ obtained from $P$ by moving $z$ to $z^*$, where $z^*$ is the intersection of the vertical line passing through the center of $H$ and the horizontal line through $z$. Clearly, this transition leaves $\|A_5\|+\|A_6\|$ unchanged, and it does not increase $\|A_1\|+\|A_4\|$; it is easily checked that if $z \neq z^*$ the last sum decreases. Also, it is not hard to compute that $\|A_2\|+\|A_3\|$ does not increase. Thus it is sufficient to consider laterally symmetric configurations. Now we introduce an orthogonal coordinate system $(x, y)$ in which $z^* = (0, a)$ and the vertices $h_i$ of $H$ have coordinates as indicated in Fig. 7. (Note that $0 \leqslant a \leqslant \frac{2}{3}$.) Then the coordinates of points which determine $\|A_i\|_{P^*,z^*}$ are easily computed (they are indicated in Fig. 7), and we have

$$\sigma = \sum_{i=1}^{6} \|A_i\|_{P^*,z^*} = 2\left\{\frac{1}{1+a} + \frac{2}{2+a-a^2} + \frac{1+a-a^2}{1-a-a^2+a^3}\right\}$$
$$= 2\,\frac{6-8a+3a^2}{2-3a-a^2+3a^3-a^4}.$$

In view of the positivity of the denominator for the values of $a$ considered, we have $\sigma \geqslant 6$ if and only if

$$6-8a+3a^2 \geqslant 3(2-3a-a^2+3a^3-a^4),$$

which simplifies to

$$3a^4-9a^3+6a^2+a \geqslant 0.$$

Writing the left hand side in the form $a\{3a(2-a)(1-a)+1\}$ we see that it is indeed non-negative for $0 \leqslant a \leqslant 2/3$, and equals zero if and only if $a = 0$.

This completes the proof of the theorem.

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