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The interpolation theorem for holomorphic generalized functions

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Abstract. We obtain the interpolation theorem in the setting of generalized functions introduced by the second author. This shows the richness of the class of the holomorphic generalized functions. The proof is more complicated than the classical proof.

A new concept of generalized functions was introduced in Colombeau [4], [5] and Aragona–Colombeau [2] (see Appendix). If Ω is any open subset of $\boldsymbol{R}^n$ we consider the algebra $G(\Omega)$ of generalized functions on Ω. If $G \in G(\Omega)$ and if $x \in \Omega$ then the pointvalue $G(x)$ is defined as a generalized complex number. These generalized complex numbers form an algebra containing $\boldsymbol{C}$ denoted by $\bar{\boldsymbol{C}}$. In the case Ω is an open subset of $\boldsymbol{C}^n$, the generalized holomorphic functions are studied in Colombeau [4], (Chapter 8) and [5] (Appendix 5), and Colombeau–Galé [6] and [7]. The purpose of this article is to prove the interpolation theorem for the holomorphic generalized functions. Its interest lies in that it shows a deep connection between our generalized functions and generalized numbers and also it shows the richness of the class of our holomorphic generalized functions. The proof is inspired from the classical proof but it is much more technical. The statement of the theorem is exactly similar to the classical case:

Theorem. *Let $(x_n)_{n \geqslant 1}$ be a sequence of different points of the complex plane without accumulation point and let $(c_n)_{n \geqslant 1}$ be an arbitrary sequence of generalized numbers (elements of $\bar{\boldsymbol{C}}$). Then there exists a generalized holomorphic function G on $\boldsymbol{C}$ such that $G(x_n) = c_n$ for each $n \geqslant 1$.*

If $r > 0$ we set $B_r := \{z \in \boldsymbol{C} \mid |z| < r\}$ and we denote by $\bar{B}_r$ the closure of B_r. If f is a complex valued function and A a subset of its domain on which f is bounded we set $\|f\|_A := \sup_{z \in A} |f(z)|$.

If Ω is an open subset of $\boldsymbol{C}$ we denote by $H(\Omega)$ the space of all usual holomorphic functions on Ω. The theorem works for the concept of generalized functions defined in [5], Part 1, in [8], Part 2, and in [3], Chapter 1. For simplification in notation we expose the proof in the case of the simplified concept defined in appendix and in [3], 1.8.

LEMMA. *Let $k > 0$, $N \in N^*$ and $\xi \in C$ with $|\xi| > k$ be given. Then for each $c > 0$ and $\varepsilon > 0$ there is $f_{c,\varepsilon} \in H(C)$ such that*

(a) $f_{c,\varepsilon}(\xi) = 1$;

(b) $\|f_{c,\varepsilon}\|_{\bar{B}_k} \leqslant c\varepsilon^N$;

(c) *For every compact subset K of C there exist $c' > 0$ and $N' \in N$, both independent on ε, such that*

$$\|f_{c,\varepsilon}\|_K \leqslant c' \varepsilon^{-N'}.$$

Proof. Seek for $f_{c,\varepsilon}$ of the form

$$f_{c,\varepsilon}(z) = (z/\xi)^{n(\varepsilon)}$$

with some $n(\varepsilon) \in N$ that we shall determine ($n(\varepsilon)$ will also depend on c but this does not matter). (a) is obvious; for (b) it suffices, setting $|\xi| = r$, to have

$$(k/r)^{n(\varepsilon)} \leqslant c\varepsilon^N,$$

i.e.,

$$n(\varepsilon) \geqslant -\log(c\varepsilon^N)/\log(r/k).$$

Choose for $n(\varepsilon)$ the smaller integer which is larger than this value. We check easily that (c) also holds with this choice of $n(\varepsilon)$.

Proof of the theorem. Let $(K_n)_{n \in N}$ be a sequence of closed balls of center 0 ($K_0 = \{0\}$) such that for all $n \geqslant 1$ there is $p \geqslant 1$ such that $x_p \in K_n \cap \complement K_{n-1}$. Thus for all $n \geqslant 1$ there is $\mu(n) \in N^*$ such that ($\mu(0) = 0$):

(1) $$\{x_{\mu(n-1)+1}, \ldots, x_{\mu(n)}\} \subset K_{n+1} \cap \complement K_n.$$

(2) $$\mu(n-1) < \mu(n) \quad \text{for all } n \in N^*.$$

(3) $$p > \mu(n) \Rightarrow x_p \notin K_{n+1}.$$

We fix an arbitrary representative $\hat{c}_n$ of c_n for every $n \in N^*$. We are going to construct a sequence $(g_n)_{n \geqslant 1}$ of functions from $]0, 1] \times C$ into C that have the following set (H_n) of properties:

(a) $g_n(\varepsilon, \cdot) \in H(C)$ for all $\varepsilon \in]0, 1]$ and $n \in N^*$,

(b) $\|g_n(\varepsilon, \cdot)\|_{K_n} \leqslant 2^{-n}$ for all $\varepsilon \in]0, 1]$ and $n \in N^*$,

(c) $g_n \in E_M[C]$ for all $n \in N^*$;

(d) For every $n \in N^*$, the inequality $j \leqslant \mu(n-1)$ implies $g_n(\varepsilon, x_j) = 0$ for all $\varepsilon \in]0, 1]$.

(e) For all $n \in N^*$, the inequalities $\mu(n-1) < j \leqslant \mu(n)$ imply that there is $\beta(n) > 0$ such that

$$g_n(\varepsilon, x_j) = \hat{c}_j(\varepsilon) - \sum_{k=1}^{n-1} g_k(\varepsilon, x_j)$$

whenever $0 < \varepsilon < \beta(n)$.

Now we are going to define inductively the sequence (g_n). Let us assume that there are $g_1, \ldots, g_{n-1}$ satisfying respectively $(H_1), \ldots, (H_{n-1})$. We are going to construct from them a function g_n satisfying (H_n).

For each $n \in N^*$ and i with $\mu(n-1) < i \leqslant \mu(n)$ we set

$$v_{i,n}(x) = \frac{\prod\limits_{\substack{1 \leqslant j \leqslant \mu(n) \\ j \neq i}} (x - x_j)}{\prod\limits_{\substack{1 \leqslant j \leqslant \mu(n) \\ j \neq i}} (x_i - x_j)} \qquad (x \in C).$$

Therefore we get

(4) For each $n \in N^*$ and i with $\mu(n-1) < i \leqslant \mu(n)$ we have $v_{i,n}(x_i) = 1$ and $v_{i,n}(x_j) = 0$ if $1 \leqslant j \leqslant \mu(n)$ and $j \neq i$.

For every $\varepsilon \in]0, 1]$ and i with $\mu(n-1) < i \leqslant \mu(n)$ we set

$$Q_{i,n}(\varepsilon) = \hat{c}_i(\varepsilon) - \sum_{k=1}^{n-1} g_k(\varepsilon, x_i),$$

$$T_{i,n}(\varepsilon) = 2^n [\mu(n) - \mu(n-1)] \cdot |Q_{i,n}(\varepsilon)| \cdot \|v_{i,n}\|_{K_n}.$$

In order to simplify the notations we shall introduce the two following finite sets of indices

$$I(n) = \{i \in N^* |\ \mu(n-1) < i \leqslant \mu(n)\} \qquad (n \in N^*)$$

$$J(n, \varepsilon) = \{i \in I(n) |\ Q_{i,n}(\varepsilon) \neq 0\} \qquad (n \in N^*,\ \varepsilon \in]0, 1]).$$

Before the construction of g_n we remark that since by the induction assumption $g_1, \ldots, g_{n-1}$ are moderate and $\hat{c}_i \in E_M$, it follows that the functions $Q_{i,n}$ are moderate hence for every $n \in N^*$ there are $N = N(n) \in N^*$, $c = c(n) > 0$ and $\eta = \eta(n) \in]0, 1[$ such that

(5) $\qquad |Q_{i,n}(\varepsilon)| \leqslant c\varepsilon^{-N}$ whenever $i \in I(n)$ and $0 < \varepsilon < \eta$ $(n \in N^*)$.

Since $n \in N^*$ and N are fixed and $0 < \varepsilon < \eta < 1$, there exists $N' \in N^*$, $N' > N$ such that

(6) $$2^n [\mu(n) - \mu(n-1)]\, c\varepsilon^{N'-N} \sup_{i \in I(n)} \|v_{i,n}\|_{K_n} \leqslant i^{-1}$$

for every $i \in I(n)$ and $\varepsilon \in]0, \eta[$. Now, by definition of $T_{i,n}(\varepsilon)$, (5) and (6), it follows that

(7) $\qquad i\varepsilon^{N'} \leqslant (T_{i,n}(\varepsilon))^{-1}$ for all $n \in N^*$, $0 < \varepsilon < \eta$ and $i \in J(n, \varepsilon)$.

By the preceding lemma (where the constants k, c, ξ and N of the lemma are substituted by the radius of K_n, i, x_i and N' respectively) there is, for every $n \in N^*$, a family $(f_{i,\varepsilon,n})_{\substack{i \in I(n) \\ 0 < \varepsilon \leqslant 1}}$ in $H(C)$ verifying the following conditions:

(8) $\qquad f_{i,\varepsilon,n}(x_i) = 1$ for all $n \in N^*$, $i \in I(n)$ and $0 < \varepsilon \leqslant 1$;

(9) For all $K \Subset C$ and $n \in N^*$ there are $c' = c'(n, K) > 0$ and $N'' = N''(n, K) \in N^*$ such that $\|f_{i,\varepsilon,n}\|_K \leqslant c' \varepsilon^{-N''}$ for every $i \in I(n)$ and $0 < \varepsilon \leqslant 1$,

(10) $\|f_{i,\varepsilon,n}\|_{K_n} \leqslant (T_{i,n}(\varepsilon))^{-1}$ for all $n \in N^*$, $0 < \varepsilon < \eta$ and $i \in J(n, \varepsilon)$

where (10) follows from (7) and condition (b) of the lemma.

For every $n \in N^*$ we define a function g_n: $]0, 1] \times C \to C$ by the formula

$$g_n(\varepsilon, x) = \begin{cases} 0 & \text{if } \eta(n) \leqslant \varepsilon \leqslant 1, \\ \sum_{i \in A} Q_{i,n}(\varepsilon) f_{i,\varepsilon,n}(x)\, v_{i,n}(x) & \text{if } 0 < \varepsilon < \eta(n), \end{cases}$$

where one may indifferently take $A = I(n)$ or $A = J(n, \varepsilon)$.

Now we have to check that g_n satisfies the properties (H_n). (a) is obvious. (b) follows at once from the definition of $T_{i,n}(\varepsilon)$ and from (7). For (c), since $g_n(\varepsilon, \cdot) \in H(C)$ it follows from Cauchy's inequalities that it suffices to check that:

for all $K \subset \subset C$ there exists $N \in N^*$, $c > 0$ and $\eta \in]0, 1]$ such that $|g_n(\varepsilon, x)| \leqslant c\varepsilon^{-N}$ whenever $x \in K$ and $0 < \varepsilon < \eta$.

For $n = 1$ this inequality follows from (5) (which is trivial in this case since $Q_{i,1} = \hat{c}_i$), (9) and from the formula of g_1. For $n > 1$, we have already proved that the induction assumption implies (5) and hence the proof of the moderation of g_1 works for g_n. (d) follows immediately from the definition of g_n and (4). (e) follows from (4), (8) and from the definitions of $Q_{i,n}$ and g_n for $\beta(n) = \eta(n)$.

Now let us consider the function g: $]0, 1] \times C \to C$ defined by

$$g(\varepsilon, x) = \sum_{n=1}^{\infty} g_n(\varepsilon, x).$$

From condition (b) in (H_n) this series is uniformly convergent on every compact subset of C. Therefore, from (a) it defines an entire function on C for every fixed $\varepsilon \in]0, 1]$. From (b) and (c) in (H_n) we get $g \in E_M[C]$. By consequence the class G of g is a holomorphic generalized function. We shall show finally that $G(x_j) = c_j$ for all $j \in N^*$ and for this it is enough to prove that, for fixed $j \in N^*$ we have

(11) There exists $\beta > 0$ such that $g(\varepsilon, x_j) = \hat{c}_j(\varepsilon)$ if $0 < \varepsilon < \beta$.

For fixed $j \in N^*$ there is a unique $p \in N^*$ such that $j \in I(p)$ and (4) implies $v_{i,q}(x_j) = 0$ whenever $q > p$ and $i \in I(q)$. Hence $g_q(\varepsilon, x_j) = 0$ for all $q > p$. The definition of g and condition (e) in (H_n) implies

$$g(\varepsilon, x_j) = \sum_{k=1}^{p} g_k(\varepsilon, x_j) = \hat{c}_j(\varepsilon) \quad \text{if } 0 < \varepsilon < \beta := \beta(p)$$

which proves (11). □

Appendix. In this appendix we give a simplified definition of these new generalized functions as it can be found in [3], 1.8. This definition is a simplification of the one in [3], [5], [8], but everything works with this last one (easy modifications in proofs).

In what follows Ω denotes an open subset of $\boldsymbol{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}^n$ and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$ is a derivation operator in $\boldsymbol{R}^n$. We denote by $E_M[\Omega]$ the set of functions $f\colon]0, 1] \times \Omega \to \boldsymbol{C}$ verifying the two following conditions:

(a) $f(\varepsilon, \cdot) \in C^\infty(\Omega)$ for all $\varepsilon \in]0, 1]$.

(b) For every derivation operator ∂^α in $\boldsymbol{R}^n$ and every compact subset K of Ω there are $N \in \boldsymbol{N}^*$ and two real numbers $c > 0$ and $\eta \in]0, 1]$ such that

$$\text{(A.1)} \qquad \sup_{x \in K} |\partial^\alpha f(\varepsilon, x)| \leqslant c\varepsilon^{-N} \quad \text{for all } \varepsilon \in]0, \eta[,$$

where

$$\text{(A.2)} \qquad \partial^\alpha f(\varepsilon, x) := \partial^\alpha [f(\varepsilon, \cdot)](x) \quad \text{for all } x \in \Omega \text{ and } \varepsilon \in]0, 1].$$

Given $f \in E_M[\Omega]$ and a derivation operator ∂^α in $\boldsymbol{R}^n$, (A.2) shows that we have a function

$$\partial^\alpha f\colon (\varepsilon, x) \in]0, 1] \times \Omega \to \boldsymbol{C}$$

which clearly belongs to $E_M[\Omega]$, hence ∂^α defines a map, still denoted by ∂^α, which maps $f \in E_M[\Omega]$ into $\partial^\alpha f \in E_M[\Omega]$. It is easy to see that $E_M[\Omega]$ endowed with the pointwise operations of addition, multiplication and multiplication by a complex number, is a complex algebra.

We are going to define the algebra $G(\Omega)$ as a quotient of $E_M[\Omega]$ by the ideal $N[\Omega]$ of $E_M[\Omega]$ defined as the set of all $f \in E_M[\Omega]$ such that, for every $K \Subset \Omega$ and every derivation operator ∂^α in $\boldsymbol{R}^n$ there exists $N \in \boldsymbol{N}^*$ such that for every $q \geqslant N$ we can find $c > 0$ and $\eta \in]0, 1]$ such that

$$\sup_{x \in K} |\partial^\alpha f(\varepsilon, x)| \leqslant c\varepsilon^q \quad \text{for all } \varepsilon \in]0, \eta[.$$

Then we define our generalized functions as the elements of the quotient algebra

$$G(\Omega) := E_M[\Omega]/N[\Omega].$$

If ∂^α is a derivation operator in $\boldsymbol{R}^n$ clearly we have $\partial^\alpha f \in N[\Omega]$ whenever $f \in N[\Omega]$, therefore the map $\partial^\alpha\colon E_M[\Omega] \to E_M[\Omega]$ induces a map, still denoted by ∂^α, of $G(\Omega)$ into $G(\Omega)$. Given $G \in G(\Omega)$, the element $\partial^\alpha G$ of $G(\Omega)$ is called the *derivative of order* α *of* G.

If $\Omega \subset \boldsymbol{C}^n$, the *holomorphic generalized functions on* Ω are the elements F in $G(\Omega)$ such that

$$\partial F/\partial \bar{z}_j = 0 \quad \text{for all } j = 1, 2, \ldots, n.$$

The complex algebra $\bar{\boldsymbol{C}}$ of *generalized complex numbers* is defined by the formula

$$\bar{\boldsymbol{C}} = E_M/J,$$

where E_M is the complex algebra, for the pointwise operations of addition, multiplication and scalar multiplication, of all functions $\boldsymbol{R}$: $]0, 1] \to \boldsymbol{C}$ for which there exists $N \in \boldsymbol{N}^*$ and real numbers $c > 0$ and $\eta \in]0, 1]$ such that $|R(\varepsilon)| \leqslant c\varepsilon^{-N}$ whenever $0 < \varepsilon < \eta$, and J is the ideal of E_M of all functions $R \in E_M$ for which there exists $N \in \boldsymbol{N}^*$ such that for every $q \geqslant N$ we can find real numbers $c > 0$ and $\eta \in]0, 1]$ verifying $|R(\varepsilon)| \leqslant c\varepsilon^q$ whenever $0 < \varepsilon < \eta$.

Let Ω be an open subset of $\boldsymbol{R}^n$. If $F \in G(\Omega)$ and $x \in \Omega$ it is clear that the function R: $\varepsilon \in]0, 1] \to f(\varepsilon, x) \in \boldsymbol{C}$, where f is an arbitrary representative of F, belongs to E_M. The point value $F(x) \in \bar{\boldsymbol{C}}$ is defined as the class of the function R.

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