ANNALES
POLONICI MATHEMATICI
XXXIV (1977)

On infinite systems of differential equations with deviated argument II

by BOGDAN RZEPECKI (Poznań)

Abstract. The aim of this paper is to indicate some effects caused by the deviated argument in infinite systems of differential equations. We give theorems on averaging and approximating by finite systems for the infinite system

$$\begin{aligned} x_k'(t) = f_k\big(t,\, & x_1(t), x_1(h_{11}(t)), x_1(h_{12}(t)), \ldots, \\ & x_2(t), x_2(h_{21}(t)), x_2(h_{22}(t)), \ldots, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\big) \end{aligned} \quad (k = 1, 2, \ldots).$$

In [10] there is a discussion of the existence and uniqueness of the solution for infinite system of differential equations with deviated argument. The purpose of this paper is to point out some other effects which the deviated argument has on infinite systems of differential equations.

We shall consider infinite system of differential equations

$$\text{(I)} \qquad \begin{aligned} x_k'(t) = f_k\big(t;\, & x_1(t), x_1(h_{11}(t)), x_1(h_{12}(t)), \ldots; \\ & x_2(t), x_2(h_{21}(t)), x_2(h_{22}(t)), \ldots; \\ & \ldots\ldots\ldots\ldots\ldots\ldots\big) \end{aligned} \quad (k = 1, 2, \ldots)$$

with the initial conditions

$$\text{(II)} \qquad x_k(0) = x_k^0 \quad (k = 1, 2, \ldots).$$

We shall study the averaging of (I) and its aproximation by finite systems.

0. Let s denote the space of all sequences of real numbers with the usual metrics. The sets $\mathbf{s}_1 = \mathbf{s} \times \mathbf{s} \times \ldots$, $\mathbf{s}_2 = [\dot{0}, a] \times \mathbf{s}_1$ are considered with the "product" metrics.

Let $\bar{a} = (\bar{t}, \bar{\xi}) \in \mathbf{s}_2$, where $\bar{\xi} = (\bar{u}_n) \in \mathbf{s}_1$, $\bar{u}_i = (\bar{u}_{in}) \in \mathbf{s}$. A function f: $\mathbf{s}_2 \to (-\infty, \infty)$ is continuous at the point $\bar{a}$ if and only if for every $\varepsilon > 0$ a number $\eta > 0$ and natural numbers $N_1, N_2, \ldots, N_M$ exist such that

$$|f(t;\ u_{11}, u_{12}, \ldots; \ldots) - f(\bar{t};\ \bar{u}_{11}, \bar{u}_{12}, \ldots; \ldots)| < \varepsilon,$$

whenever $|t-\bar{t}| < \eta$, $|u_{ki} - \bar{u}_{ki}| < \eta$ for $i = 1, 2, \ldots, N_k$, $k = 1, \ldots, M$.

Let $\|\cdot\|$ be a norm in $C[0, a]$. In the vector space $C_0 = C[0, a] \times \times C[0, a] \times \ldots$ define a sequence (p_n) of seminorms $p_n(g) = \|g_n\|$, where $g = (g_1, g_2, \ldots)$. Then (cf. [1]) the functional $|||\cdot|||_0$ defined by

$$|||g|||_0 = \sum_{1}^{\infty} 2^{-n} \frac{\|g_n\|}{1+\|g_n\|}$$

is a paranorm in C_0. It is known that the space C_0 equipped with this paranorm is a B_0-space. Let $C^\infty = \{(x_1, x_2, \ldots) \in C_0: \sup_{n \geqslant 1} \|x_n\| < \infty\}$. Then the vector space C^∞ equipped with the norm

$$|||g||| = \sup\{\|g_n\|: n = 1, 2, \ldots\} \quad (g = (g_1, g_2, \ldots))$$

is a Banach space.

1. In this part we shall give theorems for the aproximation of a solution of the initial-value problem (I)–(II) by solutions of some finite systems.

We shall consider the initial-value problem (I)–(II) and the following system of differential equations:

$$(\mathrm{I}_m) \qquad \begin{aligned} y_l'(t) = f_l\big(t;\ & y_1(t), y_1(h_{11}(t)), y_1(h_{12}(t)), \ldots; \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & y_m(t), y_m(h_{m1}(t)), y_m(h_{m2}(t)), \ldots; \\ & x_{m+1}^0, x_{m+1}^0, x_{m+1}^0, \ldots\ldots; \\ & x_{m+2}^0, x_{m+2}^0, x_{m+2}^0, \ldots\ldots; \\ & \ldots\ldots\ldots\ldots\ldots\ldots\big) \end{aligned} \qquad (l = 1, 2, \ldots, m)$$

with the initial conditions

$$(\mathrm{II}_m) \qquad y_l(0) = x_l^0 \quad (l = 1, 2, \ldots, m)$$

and

$$y_j(t) \equiv x_j^0 \quad \text{for } j = m+1, m+2, \ldots$$

We shall give conditions for the right-hand sides of (I) by means of which we can reduce the consideration of (I) to the finite system (I_m). This problem for infinite systems without deviated argument was considered by K. P. Persidski [8], [9] and expanded in [12], [2], [7].

In this part we assume that $f_k(t, \xi)$ $(k = 1, 2, \ldots)$ is a real function defined for $(t, \xi) \in \mathbf{s}_2$ and $h_{kj}(t)$ $(k, j = 1, 2, \ldots)$ is a function defined for $t \in [0, a]$ and takes its values in $[0, a]$.

We introduce

ASSUMPTION (A). *Suppose that*

1° *there exists a constant* $q_k \geqslant 0$ *such that*

$$|f_k(t, \xi)| \leqslant q_k \quad (k = 1, 2, \ldots)$$

for every $t \in [0, a]$, $\xi \in \mathbf{s}_1$,

2° $(q_k) \to 0$ *as* $k \to \infty$,

3° *there exists a bounded function* $a\colon [0, a] \to [0, \infty)$ *such that*

$$|f_k(t, \xi) - f_k(t, \bar{\xi})| \leqslant a(t) \sup\{|u_{ji} - \bar{u}_{ji}|\colon i, j = 1, 2, \ldots\}$$

for every $k = 1, 2, \ldots$, $t \in [0, a]$, $\xi = (u_n)$, $\bar{\xi} = (\bar{u}_n) \in \mathbf{s}_1$, *where* $u_j = (u_{jn})$, $\bar{u}_j = (\bar{u}_{jn}) \in \mathbf{s}$.

ASSUMPTION (B). *Let* m *denote a fixed natural number. Suppose that*

1° *the functions* h_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots$, *are continuous and fulfil the inequality* $h_{ij}(t) \leqslant t$ *for every* $t \in [0, a]$,

2° *the functions* f_i, $i = 1, 2, \ldots, m$ *are continuous.*

THEOREM 1. *Let assumption* (A) *be satisfied. Then for any* $\varepsilon > 0$ *there exists a natural number* m_0 *such that if for* $m \geqslant m_0$ *assumption* (B) *is satisfied and problems* (I)–(II), (I_m)–(II_m) *have the solutions* $x = (x_1, x_2, \ldots)$ *and* $y^{(m)} = (y_1, \ldots, y_m, x^0_{m+1}, x^0_{m+2}, \ldots)$, *respectively, then*

$$\sup_{0 \leqslant t \leqslant a} \max_{1 \leqslant i \leqslant m} |x_i(t) - y_i(t)| \leqslant C \cdot \varepsilon,$$

where $C = \big(a \cdot c_1 \exp(a \cdot c_1) + 1\big) \int\limits_0^a a(t)\,dt$, $c_1 = \sup\{a(t)\colon t \in [0, a]\}$ *and*

$$\sup_{0 \leqslant t \leqslant a} |x_{m+j}(t) - x^0_{m+j}| < \varepsilon \quad \textit{for } j = 1, 2, \ldots$$

Proof. Let m be a fixed number and let $k = 1, 2, \ldots, m$ and $t \in [0, a]$. Let us write

$$\begin{aligned} I_{1k}(t) = \Big| & f_k\big(t;\ x_1(t), x_1(h_{11}(t)), \ldots; \ldots;\ x_m(t), x_m(h_{m1}(t)), \ldots; \\ & \qquad x^0_{m+1}, x^0_{m+1}, \ldots;\ x^0_{m+2}, x^0_{m+2}, \ldots; \ldots\big) - \\ & - f_k\big(t;\ y_1(t), y_1(h_{11}(t)), \ldots; \ldots;\ y_m(t), y_m(h_{m1}(t)), \ldots; \\ & \qquad x^0_{m+1}, x^0_{m+1}, \ldots;\ x^0_{m+2}, x^0_{m+2}, \ldots; \ldots\big)\Big|, \end{aligned}$$

and

$$\begin{aligned} I_{2k}(t) = \Big| & f_k\big(t;\ x_1(t), x_1(h_{11}(t)), \ldots;\ x_2(t), x_2(h_{21}(t)), \ldots; \ldots\big) - \\ & - f_k\big(t;\ x_1(t), x_1(h_{11}(t)), \ldots; \ldots;\ x_m(t), x_m(h_{m1}(t)), \ldots; \\ & \qquad x^0_{m+1}, x^0_{m+1}, \ldots;\ x^0_{m+2}, x^0_{m+2}, \ldots; \ldots\big)\Big|. \end{aligned}$$

It is easy to prove that

$$|x_k(t) - y_k(t)| \leqslant \int_0^t \big(I_{1k}(s) + I_{2k}(s)\big)\,ds \tag{1}$$

and

$$I_{1k}(t) \leqslant a(t)\sup\big\{|x_i(t)-y_i(t)|,\ \big|x_i\big(h_{ij}(t)\big)-y_i\big(h_{ij}(t)\big)\big|:$$
$$i = 1, 2, \ldots, m,\ j = 1, 2, \ldots\big\},$$
$$I_{2k}(t) \leqslant a(t)\sup\big\{|x_{m+i}(t)-x^0_{m+i}|,\ \big|x_{m+i}\big(h_{m+i,j}(t)\big)-x^0_{m+i}\big|:\ i, j = 1, 2, \ldots\big\}.$$

It is obvious that

$$|x_j(t) - x^0_j| \leqslant \int_0^t \big|f_j\big(s;\ x_1(s), x_1\big(h_{11}(s)\big), \ldots; \ldots\big)\big|\,ds \quad \text{for } j = 1, 2, \ldots$$

and therefore $\limsup_{j\to\infty}\{|x_j(t)-x^0_j|:\ t\in[0, a]\} = 0$. Then, for every $\varepsilon > 0$ a number m_0 exists such that

$$\sup\{|x_{m+i}(t)-x^0_{m+i}|:\ t\in[0, a]\} < \varepsilon \qquad (i = 1, 2, \ldots) \tag{2}$$

for $m \geqslant m_0$.

Fix $\varepsilon > 0$. By (2)

$$\sup\big\{|x_{m+i}(t)-x^0_{m+i}|,\ \big|x_{m+i}\big(h_{m+i,j}(t)\big)-x^0_{m+i}\big|:\ i,\ j = 1, 2, \ldots\big\} < \varepsilon$$

for $m \geqslant m_0$ and $t \in [0, a]$. Consequently

$$I_{2k}(t) \leqslant \varepsilon\cdot a(t) \qquad (t\in[0, a]) \tag{3}$$

for $k = 1, 2, \ldots, m$, where $m \geqslant m_0$.

Let $t \in [0, a]$. By (1) it follows that

$$\big|x_k\big(h_{kj}(t)\big) - y_k\big(h_{kj}(t)\big)\big| \leqslant \int_0^{h_{kj}(t)} \big(I_{1k}(s) - I_{2k}(s)\big)\,ds$$
$$\leqslant \int_0^t \big(I_{1k}(s) - I_{2k}(s)\big)\,ds$$

for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots$ From this it follows that

$$I_{1k}(t) \leqslant a(t)\sup\Big\{\int_0^t \big(I_{1i}(s) + I_{2i}(s)\big)\,ds:\ i = 1, 2, \ldots, m\Big\}$$

for $k = 1, 2, \ldots, m$. Inequality (3) implies that

$$\sup\Big\{\int_0^t \big(I_{1i}(s) + I_{2i}(s)\big)\,ds:\ i = 1, 2, \ldots, m\Big\} \leqslant c_1 a\varepsilon +$$
$$+ \int_0^t \sup\{I_{1i}(s):\ i = 1, 2, \ldots, m\}\,ds$$

for $m \geqslant m_0$, and hence we have

$$\sup\{I_{1k}(t)\colon k = 1, 2, \ldots, m\} \leqslant c_1\Big(c_1 a\varepsilon + \int_0^t \sup\{I_{1k}(s)\colon k = 1, 2, \ldots, m\}\, ds\Big)$$

for $m \geqslant m_0$. Hence, applying Gronwall's lemma, we have

$$\sup\{I_{1k}(t)\colon k = 1, 2, \ldots, m\} \leqslant c_1^2 a\varepsilon \exp(c_1 a) \qquad \text{for } m \geqslant m_0, \quad t \in [0, a]. \tag{4}$$

By (1), (3) and (4), we get

$$|x_k(t) - y_k(t)| \leqslant c_1 a\big(c_1 a \cdot \exp(c_1 a) + 1\big)\cdot \varepsilon$$

for $t \in [0, a]$ and $k = 1, 2, \ldots, m$, where $m \geqslant m_0$. This ends the proof.

COROLLARY 1. *Let assumptions* (A) *and* (B) *for* $m = 1, 2, \ldots$ *be satisfied and let problems* (I)–(II), (I_m)–(II_m) *have the solutions* x *and* $y^{(m)}$, *respectively. Then* $|||x - y^{(m)}|||_0 \to 0$ *as* $m \to \infty$, *and, moreover, if* $x, y^{(m)} \in C^\infty$ $(m = 1, 2, \ldots)$, *then also* $|||x - y^{(m)}||| \to 0$ *as* $m \to \infty$.

Now we study problems (I_m)–(II_m) in the case where $x_j^0 = 0$ for $j = m+1, m+2, \ldots$

ASSUMPTION (C). *Suppose that*

1° *the function* h_{ij} $(i, j = 1, 2, \ldots)$ *is continuous and fulfils the inequality* $h_{ij}(t) \leqslant t$ *for every* $t \in [0, a]$,

2° *the function* f_k $(k = 1, 2, \ldots)$ *is continuous*,

3° *there exists an integrable function* $\beta\colon [0, a] \to [0, \infty)$ *such that*

$$|f_k(t, \theta)| \leqslant \beta(t) \qquad (k = 1, 2, \ldots)$$

for every $t \in [0, a]$, *where* θ *denotes the zero element of the space* $\mathbf{s}_1$,

4° *there exists a bounded function* $\alpha\colon [0, a] \to [0, \infty)$ *and the number sequence* (ε_m), $(\varepsilon_m) \to 0$ *as* $m \to \infty$, *such that*

$$|f_k(t, u_1, \ldots, u_{m-1}, u_m, u_{m+1}, \ldots) - f_k(t, u_1, \ldots, u_{m-1}, \bar{u}_m, \bar{u}_{m+1}, \ldots)|$$
$$\leqslant \alpha(t)\varepsilon_m \sup\{|u_{ji} - \bar{u}_{ji}|\colon j = m, m+1, \ldots, i = 1, 2, \ldots\}$$

for every $t \in [0, a]$, $k = 1, 2, \ldots$ *and* $u_1, u_2, \ldots \in \mathbf{s}$, $\bar{u}_m, \bar{u}_{m+1}, \ldots \in \mathbf{s}$, *where* $u_{m+j} = (u_{m+j,n})$, $\bar{u}_{m+j} = (\bar{u}_{m+j,n})$, $j = 0, 1, 2, \ldots$

THEOREM 2. *Let assumption* (C) *be satisfied and* $M = \sup\{|x_k^0|\colon k = 1, 2, \ldots\} < \infty$. *Assume, moreover, that problem* (I_m)–(II_m) $(m = 1, 2, \ldots)$, *where* $x_j^0 = 0$ *for* $j = m+1, m+2, \ldots$, *has the solution* $y^{(m)} = (y_{1m}, y_{2m}, \ldots, y_{mm}, 0, 0, \ldots)$.

Then there exists a solution $x = (x_1, x_2, \ldots)$ *of problem* (I)–(II) *such that*

$$\lim_{m\to\infty} y_{km}(t) = x_k(t) \quad \textit{uniformly on } [0, a]$$

for $k = 1, 2, \ldots$; *this solution consists of uniformly bounded and equicontinuous functions on the interval* $[0, a]$.

Proof. Let $x = (x_1, x_2, \ldots) \in C^\infty$ be a solution of (I)–(II). Put $g(t) = \sup\{|x_i(t)|, |x_i(h_{ij}(t))|: i, j = 1, 2, \ldots\}$ and $c_1 = \sup\{\alpha(t): t \in [0, a]\}$, $c_2 = \int_0^a \beta(t)\,dt$. We have

$$|x_i(t)| \leqslant |x_i^0| + \varepsilon_1 \int_0^t \alpha(s) g(s)\,ds + \int_0^t \beta(s)\,ds \leqslant M + c_1 \int_0^t g(s)\,ds + c_2$$

for $t \in [0, a]$, $\varepsilon_1 \leqslant 1$ and $i = 1, 2, \ldots$ This implies

$$g(t) \leqslant M + c_1 \int_0^t g(s)\,ds + c_2 \qquad (t \in [0, a]);$$

we infer from Gronwall's lemma that $g(t) \leqslant (M + c_2)\exp(c_1 a)$ for $t \in [0, a]$. From this it follows that

$$|x_i(t_1) - x_i(t_2)| \leqslant \big((M + c_2) c_1 \exp(c_1 a) + c_2\big) |t_1 - t_2|$$

for every $t_1, t_2 \in [0, a]$ and $i = 1, 2, \ldots$

Let (m_n) be a number sequence such that $\lim_{n\to\infty} m_n = +\infty$. Denote by $y_{m_n} = (y_{1m_n}, \ldots, y_{m_n m_n}, 0, 0, \ldots)$ the solution of problems (I_{m_n})–(II_{m_n}). It is obvious that

$$\|y_{km_n}\| \leqslant (M + c_2)\exp(c_1 a) \tag{5}$$

and

$$|y_{km_n}(t_1) - y_{km_n}(t_2)| \leqslant \big((M + c_2) c_1 \exp(c_1 a) + c_2\big) |t_1 - t_2| \tag{6}$$

for every $k, n = 1, 2, \ldots$ and $t_1, t_2 \in [0, a]$. So we see that the family $\{y_{m_n}: n = 1, 2, \ldots\}$ consists of functions uniformly bounded and equicontinuous; thus we can apply the Arzelà theorem.

Fix an index k. Repeating the argumentation of [8] (see also [4], [9]), we deduce that the sequence

$$y_{km_1}, y_{km_2}, \ldots, y_{km_n}, \ldots$$

contains a convergent subsequence (z_{kn}), where

$$z_{kn}(t) = x_k^0 + \int_0^t f_k\big(s;\ z_{1n}(s), z_{1n}(h_{11}(s)), z_{1n}(h_{12}(s)), \ldots;\ z_{2n}(s), z_{2n}(h_{21}(s)), \ldots; \ldots\big)\,ds.$$

Let

$$\lim_{n\to\infty} z_{kn}(t) = z_k(t) \qquad \text{uniformly on } [0, a].$$

Obviously functions z_n $(n = 1, 2, \ldots)$ satisfy inequalities (5) and (6). We now prove that the functions z_n, $n = 1, 2, \ldots$, satisfy problem (I)–(II).

Write

$$I(t) = f_k\big(t;\ z_{1n}(t), z_{1n}\big(h_{11}(t)\big), \ldots;\ z_{2n}(t), z_{2n}\big(h_{21}(t)\big), \ldots; \ldots\big) - \\ - f_k\big(t;\ z_1(t), z_1\big(h_{11}(t)\big), \ldots;\ z_2(t), z_2\big(h_{21}(t)\big), \ldots; \ldots\big),$$

$$I_{1l}(t) = f_k\big(t;\ z_{1n}(t), z_{1n}\big(h_{11}(t)\big), \ldots;\ z_{2n}(t), z_{2n}\big(h_{21}(t)\big), \ldots; \ldots\big) - \\ - f_k\big(t;\ z_1(t), z_1\big(h_{11}(t)\big), \ldots; \ldots;\ z_l(t), z_l\big(h_{l1}(t)\big), \ldots; \\ z_{l+1,n}(t), z_{l+1,n}\big(h_{l+1,1}(t)\big), \ldots;\ z_{l+2,n}(t), \ldots; \ldots\big),$$

$$I_{2l}(t) = f_k\big(t;\ z_1(t), z_1\big(h_{11}(t)\big), \ldots; \ldots;\ z_l(t), z_l\big(h_{l1}(t)\big), \ldots; \\ z_{l+1,n}(t), z_{l+1,n}\big(h_{l+1,1}(t)\big), \ldots;\ z_{l+2,n}(t), \ldots; \ldots\big) - \\ - f_k\big(t;\ z_1(t), z_1\big(h_{11}(t)\big), \ldots;\ z_2(t), z_2\big(h_{21}(t)\big), \ldots; \ldots\big)$$

and

$$h(t) = \sup\big\{|z_i(t) - z_{in}(t)|, \big|z_i\big(h_{ij}(t)\big) - z_{in}\big(h_{ij}(t)\big)\big|:\ i = l+1, l+2, \ldots, \\ j = 1, 2, \ldots\big\}.$$

Observe that

$$h(t) \leqslant 2(M + c_2)\exp(c_1 a) \qquad \text{for } t \in [0, a].$$

So we get

$$|I_{2l}(t)| \leqslant \varepsilon_{l+1} c_1 h(t) \leqslant 2\cdot\varepsilon_{l+1} c_1 (M + c_2)\exp(c_1 a) \tag{7}$$

for $t \in [0, a]$. Let $\eta > 0$. From (7) it follows that there exists a natural number q such that $|I_{2q}(t)| < 2^{-1}\eta$ for $t \in [0, a]$. We know that $\|z_{kn} - z_k\| \to 0$ as $n \to \infty$; thus for η there exists a natural number N such that

$$|I_{1q}(t)| \leqslant c_1 \sup\big\{|z_i(t) - z_{in}(t)|, \big|z_i\big(h_{ij}(t)\big) - z_{in}\big(h_{ij}(t)\big)\big|: \\ i = 1, 2, \ldots, q,\ j = 1, 2, \ldots\big\} < 2^{-1}\eta$$

for $n > N$ and $t \in [0, a]$. So we have

$$\Big|\int_0^t I(s)\,ds\Big| \leqslant \int_0^t \Big|I_{1q}(s) + I_{2q}(s)\Big|\,ds < a\cdot\eta$$

for every $t \in [0, a]$ and $n > N$. Thus

$$x_k^0 + \int_0^t f_k\big(s;\ z_1(s), z_1\big(h_{11}(s)\big), \ldots;\ z_2(s), z_2\big(h_{21}(s)\big), \ldots; \ldots\big)ds \\ = \lim_{n\to\infty}\Big(x_k^0 + \int_0^t f_k\big(s;\ z_{1n}(s), z_{1n}\big(h_{11}(s)\big), \ldots; \ldots\big)ds\Big) = z_k(t)$$

for $t \in [0, a]$ and $k = 1, 2, \ldots$ This ends the proof of the theorem.

COROLLARY 2. *Let the assumption of Theorem* 2 *be satisfied. Then* $|||x-y^{(m)}|||\to 0$ *as* $m\to\infty$.

2. In this section we study averaging in the infinite system

$$(+)\qquad x_k'(t) = \varepsilon F_k\big(t, x_1(t), x_2(t), \ldots\big) \quad (k = 1, 2, \ldots),$$

where $\varepsilon > 0$ is a small parameter. Let

$$F_{0k}(z) = \lim_{T\to\infty} T^{-1}\int_0^T F_k(t, z)\,dt \quad (k = 1, 2, \ldots)$$

and let an averaged system with $(+)$ have the form:

$$(++)\qquad X_k'(t) = \varepsilon\cdot F_{0k}\big(X_1(t), X_2(t), \ldots\big) \quad (k = 1, 2, \ldots).$$

The averaging method for a differential equation has been given by Bogolubov in [3]. The theorem of Bogolubov has been generalized in many directions and has a large and extensive bibliography (see e.g. [11], [4]). I. I. Gikhman [5] proved that the Bogolubov Principle can be obtained as a corollary of a certain theorem on continuous dependence of a solution of the equation $x' = f(t, x, \lambda)$ on parameter λ. In [13] and [2] the results of Gikhman were generalized to a countable system of differential equations. The theorems given below are obtained by a generalization of a method of Larinov and Filatov ([6], [4]).

ASSUMPTION (D). *Suppose that the functions* $F_k\colon [0, \infty)\times \mathbf{s}\to(-\infty, \infty)$, $k = 1, 2, \ldots$, *are such that*

1° *there exists a bounded function* $\gamma\colon [0, \infty)\to[0, \infty)$ *such that*

$$|F_k(t, z) - F_k(t, \bar z)| \leqslant \gamma(t)\cdot\sup\{|z_n - \bar z_n|\colon n = 1, 2, \ldots\}$$

for every $t\geqslant 0$, $z = (z_n)$, $\bar z = (\bar z_n)\in\mathbf{s}$ *and* $k = 1, 2, \ldots$,

2° *there exists a finite limit*

$$\lim_{T\to\infty} T^{-1}\int_0^T F_k(t, z)\,dt \quad (k = 1, 2, \ldots)$$

for every $z\in\mathbf{s}$.

THEOREM 3. *Let assumption* (D) *be satisfied and let systems* $(+)$, $(++)$ *have the solutions* $x = (x_1, x_2, \ldots)$ *and* $X = (X_1, X_2, \ldots)$, *respectively, such that* $x_k(0) = X_k(0)$ *for* $k = 1, 2, \ldots$ *Assume, moreover, that there exists a constant* $K > 0$ *such that*

$$\Big|\int_{t_1}^{t_2} F_{0k}\big(X_1(t), X_2(t), \ldots\big)\,dt\Big| \leqslant K|t_1 - t_2| \quad (k = 1, 2, \ldots)$$

for every $t_1, t_2\geqslant 0$.

Then for any $\eta > 0$ and $L > 0$ there exists an $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then

$$\sup\{|x_k(t) - X_k(t)|:\ k = 1, 2, \ldots\} < \eta$$

for every $t \in [0, \varepsilon^{-1}L]$.

Proof. Fix $L > 0$, $\varepsilon > 0$ and an index k. We write

$$I = [0, \varepsilon^{-1}L], \qquad C = \sup\{\gamma(t):\ t \geqslant 0\}$$

and

$$g_k(t, z) = F_k(t, z) - F_{0k}(t, z),$$

$$G_k(t) = \Big|\int_0^t g_k\big(s, X(s)\big)\,ds\Big|, \qquad H_k(t, z) = \Big|t^{-1}\int_0^t g_k(s, z)\,ds\Big|.$$

Let T be a point it which the function $G_k|_I$ has its maximum. Now we divide the interval $[0, T]$ into m equal parts $t_0 = 0 < t_1 < \ldots < t_m = T$. Let us fix $i = 0, 1, \ldots, m-1$ and $t \in I$. Since

$$|X_n(t) - X_n(t_i)| = \Big|\varepsilon\int_{t_i}^t F_{0n}\big(X(s)\big)\,ds\Big| \leqslant \varepsilon K|t - t_i|$$

for $n = 1, 2, \ldots$, we have

$$\begin{aligned}\big|g_k\big(t, X(t)\big) - g_k\big(t, X(t_i)\big)\big| &\leqslant \big|F_k\big(t, X(t)\big) - F_k\big(t, X(t_i)\big)\big| + \\ &\qquad + \big|F_{0k}\big(X(t)\big) - F_{0k}\big(X(t_i)\big)\big| \\ &\leqslant 2C\cdot\sup\{|X_n(t) - X_n(t_i)|:\ n = 1, 2, \ldots\} \leqslant 2CK\varepsilon\cdot|t - t_i|.\end{aligned}$$

Hence

$$\Big|\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\Big(g_k\big(s, X(s)\big) - g_k\big(s, X(t_i)\big)\Big)ds\Big| \leqslant m^{-1}KCL^2. \tag{8}$$

Since $\lim_{t\to\infty} H_k(t, z) = 0$ for every fixed $x \in \mathbf{s}$, we have

$$\begin{aligned}\varepsilon\Big|\int_0^{t_{i+1}} g_k\big(s, X(t_i)\big)\,ds\Big| &\leqslant \sup\Big\{\tau H_k\Big(\frac{\tau}{\varepsilon}, X(t_i)\Big):\ \tau \in [0, \varepsilon t_{i+1}]\Big\} \\ &\leqslant \sup\Big\{\tau H_k\Big(\frac{\tau}{\varepsilon}, X(t_i)\Big):\ \tau \in [0, (i+1)Lm^{-1}]\Big\}\end{aligned}$$

and

$$\varepsilon\Big|\int_0^{t_i} g_k\big(s, X(t_i)\big)\,ds\Big| \leqslant \sup\Big\{\tau H_k\Big(\frac{\tau}{\varepsilon}, X(t_i)\Big):\ \tau \in [0, m^{-1}iL]\Big\}.$$

Hence

$$(9)\quad \left|\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} g_k\big(s, X(t_i)\big)\,ds\right|$$
$$\leqslant \sum_{i=0}^{m-1}\sup\left\{\tau H_k\left(\frac{\tau}{\varepsilon}, X(t_i)\right):\ \tau\in[0, m^{-1}(i+1)L]\right\}+$$
$$+\sum_{i=1}^{m-1}\sup\left\{\tau H_k\left(\frac{\tau}{\varepsilon}, X(t_i)\right):\ \tau\in[0, m^{-1}iL]\right\}.$$

We have

$$|x_k(t)-X_k(t)|\leqslant C\cdot\varepsilon\cdot\int_0^t \sup\{|x_n(s)-X_n(s)|:\ n=1,2,\ldots\}\,ds+$$
$$+\varepsilon\cdot\max\{G_k(t):\ t\in I\}\leqslant C\cdot\varepsilon\cdot\int_0^t \sup\{|x_n(s)-X_n(s)|:\ n=1,2,\ldots\}\,ds+$$
$$+\left|\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\Big(g_k\big(s, X(s)\big)-g_k\big(s, X(t_i)\big)\Big)ds\right|+\left|\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} g_k\big(s, X(t_i)\big)\,ds\right|.$$

Hence, by (8) and (9), we get

$$(10)\quad |x_k(t)-X_k(t)|$$
$$\leqslant C\cdot\varepsilon\cdot\int_0^t \sup\{|x_k(s)-X_n(s)|:\ n=1,2,\ldots\}\,ds+m^{-1}KCL^2+$$
$$+\sum_{i=0}^{m-1}\sup\left\{\tau H_k\left(\frac{\tau}{\varepsilon}, X(t_i)\right):\ t\in[0, m^{-1}(i+1)L]\right\}+$$
$$+\sum_{i=1}^{m-1}\sup\left\{\tau H_k\left(\frac{\tau}{\varepsilon}, X(t_i)\right):\ t\in[0, m^{-1}iL]\right\}.$$

For $\mathfrak{S}>0$ there exists a natural number m_0 such that $2KCL^2<m_0\mathfrak{S}$. Since

$$\lim_{\varepsilon\to 0}\left(\sum_{i=0}^{m-1}\sup\left\{\tau H_k\left(\frac{\tau}{\varepsilon}, X(t_i)\right):\ \tau\in[0, m^{-1}(i+1)L]\right\}+\right.$$
$$\left.+\sum_{i=1}^{m-1}\sup\left\{\tau H_k\left(\frac{\tau}{\varepsilon}, X(t_i)\right):\ \tau\in[0, m^{-1}iL]\right\}\right)=0,$$

for $\mathfrak{S}$ there exists $\varepsilon_0 > 0$ such that

$$(11)\qquad m_0^{-1}KCL^2 + \sum_{i=0}^{m_0-1} \sup\left\{\tau H_k\left(\frac{\tau}{\varepsilon}, X(t_i)\right):\ \tau \in [0, m_0^{-1}(i+1)L]\right\} + \\ + \sum_{i=1}^{m_0-1} \sup\left\{\tau H_k\left(\frac{\tau}{\varepsilon}, X(t_i)\right):\ \tau \in [0, m_0^{-1}iL]\right\} < \mathfrak{S},$$

where $\varepsilon < \varepsilon_0$.

Let $\eta > 0$ and $0 < \mathfrak{S} < \exp(-CL)\eta$. From (10) and (11) we have

$$\sup\{|x_k(t) - X_k(t)|:\ k = 1, 2, \ldots\} \\ \leqslant C\cdot\varepsilon\cdot\int_0^t \sup\{|x_k(s) - X_k(s)|:\ k = 1, 2, \ldots\}ds + \mathfrak{S}$$

for $\varepsilon < \varepsilon_0$ and $t \in I$. Hence, applying Gronwall's lemma, we have

$$\sup\{|x_k(t) - X_k(t)|:\ k = 1, 2, \ldots\} \leqslant \mathfrak{S}\exp(\varepsilon Ct)$$

for $\varepsilon < \varepsilon_0$ and $t \in I$; therefore

$$\sup\{|x_k(t) - X_k(t)|:\ k = 1, 2, \ldots\} < \eta$$

for $\varepsilon < \varepsilon_0$ and $t \in [0, \varepsilon^{-1}L]$.

3. Let (I_ε) denote system (I) in which the right-hand side terms are multiplied by a small positive parameter ε. Let

$$f_{0k}(r_1, r_2, \ldots) = \lim_{T\to\infty} T^{-1}\int_0^T f_k(t;\ r_1, r_1, \ldots;\ r_2, r_2, \ldots;\ \ldots)\,dt$$

for $r_k \in (-\infty, \infty)$, $k = 1, 2, \ldots$ and let an averaged system with (I_ε) have the form

$$(\mathrm{III})\qquad X_k'(t) = \varepsilon f_{0k}\big(X_1(t), X_2(t), \ldots\big) \qquad (k = 1, 2, \ldots).$$

We introduce

ASSUMPTION (E). *Suppose that $h_{ij}(t)$ $(i, j = 1, 2, \ldots)$ is a function defined for $t \geqslant 0$ and takes its values in $[0, \infty)$ and the $f_k(t, \xi)$ $(k = 1, 2, \ldots)$ is a real function defined for $(t, \xi) \in [0, \infty) \times \mathbf{s}_1$ and such that*

1° $S = \sup\{|h_{ij}(t) - t|:\ t \geqslant 0,\ i, j = 1, 2, \ldots\} < \infty$,

2° *there exists a bounded function ψ: $[0, \infty) \to [0, \infty)$ such that*

$$|f_k(t, \xi) - f_k(t, \bar{\xi})| \leqslant \psi(t)\cdot\sup\{|u_{ij} - \bar{u}_{ij}|:\ i, j = 1, 2, \ldots\}$$

for every $t \geqslant 0$, $\xi = (u_n)$, $\bar{\xi} = (\bar{u}_n) \in \mathbf{s}_1$, where $u_i = (u_{in})$, $\bar{u}_i = (\bar{u}_{in}) \in \mathbf{s}$ and $k = 1, 2, \ldots,$

3° there exists a constant $M > 0$ such that

$$|f_k(t, \xi)| \leqslant M \qquad (k = 1, 2, \ldots)$$

for every $(t, \xi) \in [0, \infty) \times \mathbf{s}_1$.

ASSUMPTION (F). *Suppose that*

1° assumption (E) is satisfied,

2° there exists a finite limit

$$\lim_{T\to\infty} T^{-1} \int_0^T f_k(t;\, r_1, r_1, \ldots;\, r_2, r_2, \ldots;\, \ldots)\, dt$$

for every $r_k \in (-\infty, \infty)$ and $k = 1, 2, \ldots$

First, together with system (I_ε) we consider system (IV) obtained from (I_ε) when $h_{ij}(t) \equiv t$ $(i, j = 1, 2, \ldots)$:

$$\text{(IV)} \qquad y_k'(t) = \varepsilon f_k\big(t;\, y_1(t), y_1(t), \ldots;\, y_2(t), y_2(t), \ldots;\, \ldots\big) \qquad (k = 1, 2, \ldots).$$

The following lemma holds:

LEMMA. *Let assumption (E) be satisfied and systems (I_ε), (IV) have solutions $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, respectively, such that $x_k(0) = y_k(0)$ for $k = 1, 2, \ldots$*

Then for any $L > 0$ and for any $\eta > 0$ there exists an $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then

$$\sup\{|x_k(t) - y_k(t)|:\ k = 1, 2, \ldots\} < \eta$$

for every $t \in [0, \varepsilon^{-1}L]$.

Proof. To prove this, observe first that

$$\big|x_i\big(h_{ij}(t)\big) - x_i(t)\big| \leqslant \varepsilon \int_{h_{ij}(t)}^{t} \Big|f_i\big(s, x_1(s), x_1\big(h_{11}(s)\big), \ldots, \ldots\big)\Big|\, ds \leqslant \varepsilon MS$$

for $t \geqslant 0$ and $i, j = 1, 2, \ldots$ Setting $D = \sup\{\psi(t):\ t \geqslant 0\}$, we obtain

$$\begin{aligned}
&|x_k(t) - y_k(t)| \\
&\quad \leqslant \varepsilon \int_0^t \psi(s) \cdot \sup\Big\{|x_i(s) - y_i(s)|, \big|x_i\big(h_{ij}(s)\big) - y_i(s)\big|:\ i, j = 1, 2, \ldots\Big\}\, ds \\
&\quad \leqslant 2\varepsilon D \int_0^t \sup\{|x_i(s) - y_i(s)|:\ i = 1, 2, \ldots\}\, ds + \\
&\qquad\qquad + \varepsilon D \int_0^t \sup\Big\{\big|x_i\big(h_{ij}(s)\big) - x_i(s)\big|:\ i, j = 1, 2, \ldots\Big\}\, ds \\
&\quad \leqslant \varepsilon^2 DMSt + 2\varepsilon D \int_0^t \sup\{|x_i(s) - y_i(s)|:\ i = 1, 2, \ldots\}\, ds
\end{aligned}$$

for $t \geqslant 0$ and $k = 1, 2, \ldots$ So we have

$$\sup\{|x_k(t) - y_k(t)|:\ k = 1, 2, \ldots\} \leqslant \varepsilon DMSL + 2\varepsilon D \int_0^t \sup\{|x_k(s) - y_k(s)|:\ k = 1, 2, \ldots\} ds$$

for $t \in [0, \varepsilon^{-1}L]$, where $L > 0$ and $\varepsilon > 0$. Hence, applying Gronwall's lemma, we have

$$\sup\{|x_k(t) - y_k(t)|:\ k = 1, 2, \ldots\} \leqslant \varepsilon DMSL \exp(2\varepsilon Dt) \leqslant \varepsilon DMSL \exp(2DL)$$

for $t \in [0, \varepsilon^{-1}L]$, where $L > 0$ and $\varepsilon > 0$. Taking $\eta > 0$, $L > 0$ and setting

$$\varepsilon_0 = \frac{\eta}{DMSL \cdot \exp(2DL)},$$

we obtain

$$\sup\{|x_k(t) - y_k(t)|:\ k = 1, 2, \ldots\} < \eta \quad \text{for } t \in [0, \varepsilon^{-1}L], \text{ where } \varepsilon < \varepsilon_0.$$

This ends the proof.

By the lemma and Theorem 3 it follows

THEOREM 4. *Let assumption* (F) *be satisfied and systems* (I_ε), (III) *have solutions* $x = (x_1, x_2, \ldots)$ *and* $X = (X_1, X_2, \ldots)$, *respectively, such that* $x_k(0) = X_k(0)$ *for* $k = 1, 2, \ldots$

Then for any $L > 0$ *and for any* $\eta > 0$ *there exists an* $\varepsilon_0 > 0$ *such that if* $\varepsilon \in (0, \varepsilon_0)$, *then*

$$\sup\{|x_k(t) - X_k(t)|:\ k = 1, 2, \ldots\} < \eta$$

for every $t \in [0, \varepsilon^{-1}L]$.

Remark. Let ($\mathrm{I}_{m\varepsilon}$) denote system ($\mathrm{I}_m$) in which the right-hand side terms are multiplied by a small positive parameter ε. From Theorem 1 and Theorem 4 if follows that under the same initial conditions, the solution of ($\mathrm{I}_{m\varepsilon}$) obtained by the averaging method for a sufficiently large m is an approximate solution of (I_ε) on some sufficiently large but finite interval.

References

[1] A. Alexiewicz, *Analiza funkcjonalna,* Warszawa 1968.

[2] В. Т. Яцюк, *К исследованию счетных систем дифференциальных уравнений в пространстве* C^∞, Сборник *Асимптотические и качественные методы в теории нелинейных колебаний,* Киев 1971, p. 218–239.

[3] Н. Н. Боголюбов, *О некоторых статистических методах в математической физике,* Киев 1945.

[4] А. Н. Филатов, *Методы усреднения в дифференциальных и интегро-дифференциальных уравнениях,* Ташкент 1971.

[5] И. И. Гихман, *По поводу одной теоремы Н. Н. Боголюбова*, Украинский Математический Журнал 4 (1952), p. 215–219.

[6] Г. С. Ларинов, А. Н. Филатов, *О методе усреднения в нелинейной механике*, Известия Академии Наук УзССР, серия техн. наук 2 (1969).

[7] K. Moszyński et A. Pokrzywa, *Sur les systèmes infinis d'équations différentielles ordinaires dans certains espaces de Fréchet*, Diss. Math. 115 (1974), p. 1–37.

[8] К. П. Персидский, *Об устойчивости решений бесконечной системы уравнений*, Прикладная Математика и Механика 12 (1948), p. 597–612.

[9] К. П. Персидский, *Счетные системы дифференциальных уравнений и устойчивость их решений*, Известия Академии Наук Казахской ССР, вып. 7 (11) (1959), p. 52–71; вып. 8 (12) (1959), p. 45–64; вып. 9 (13) (1961), p. 11–34.

[10] B. Rzepecki, *On infinite systems of differential equations with deviated argument I*, Ann. Polon. Math. 31 (1975), p. 159–169.

[11] В. М. Волосов, *Усреднение в системах обыкновенных дифференциальных уравнений*, Успехи Мат. Наук 17 вып. 6 (108) (1962), p. 3–126.

[12] О. А. Жаутыков, *О счетной системе дифференциальных уравнений, содержащей пе-*
[13] *ременные параметры*, Мат. Сб. 49, 91 : 3 (1959), p. 317–320.
О. А. Жаутыков, *Принцип усреднения в нелинейной механике, применительно к счетным системам уравнений*, Украинский Математический Журнал 17 (1965), p. 39–46.

Reçu par la Rédaction le 22. 1. 1975