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Hardy norm, Bergman norm, and univalency

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Abstract. Sufficient conditions for f meromorphic in $|z| < 1$ to be univalent are proposed in terms of the Hardy norm and the Bergman norm of the Schwarzian derivative of f (see (1.5) and (1.6)). Various applications of the fundamental inequalities in Theorem 1 will be proposed.

1. Introduction. Let f be a function holomorphic in $D = \{|z| < 1\}$, and let

$$\|f\|_p = \lim_{r \to 1}\left[\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right]^{1/p}, \quad 0 \leqslant r < 1,\ 0 < p < \infty,$$

$$\|f\|_\infty = \sup_{z \in D}|f(z)|,$$

be the Hardy norm of f of order $0 < p \leqslant \infty$. Let

$$\|f\|_{p,B} = \left[\frac{1}{\pi}\iint_D |f(z)|^p\,dx\,dy\right]^{1/p}, \quad z = x+iy,\ 0 < p < \infty,$$

$$\|f\|_{\infty,B} = \|f\|_\infty,$$

be the Bergman norm of f of order $0 < p \leqslant \infty$. Thus, the Hardy class H^p (the Bergman space B^p, respectively) is the family of f with $\|f\|_p < \infty$ ($\|f\|_{p,B} < \infty$, resp.), where $0 < p \leqslant \infty$.

The non-Euclidean distance in D is defined by

$$\sigma(w, z) = \tanh^{-1}(|w-z|/|1-\bar{z}w|),$$

so that

$$H(z, \gamma) = \{w \in D;\ \sigma(w, z) < \gamma\} \quad (0 < \gamma \leqslant \infty)$$

and

$$\Gamma(z, \gamma) = \{w \in D;\ \sigma(w, z) = \gamma\} \quad (0 < \gamma < \infty)$$

are the non-Euclidean disk and the non-Euclidean circle, respectively, of center $z \in D$ and radius γ. For convenience, we set $\Gamma(z, \infty) = C$ for each $z \in D$, where $C = \{|w| = 1\}$ is the unit circle.

An objective of the present paper is to prove

THEOREM 1. *Let f be a function holomorphic in D. Then, for each $0 < \gamma \leqslant \infty$, for each $0 < p \leqslant \infty$, and at each $z \in D$, the following inequalities hold*

$$(1.1)\qquad (1-|z|^2)^{1/p}|f(z)| \leqslant (\tanh\gamma)^{-1/p}\left[\frac{1}{2\pi}\int_{\Gamma(z,\gamma)}|f(\zeta)|^p|d\zeta|\right]^{1/p} \leqslant \|f\|_p,$$

$$(1.2)\qquad (1-|z|^2)^{2/p}|f(z)| \leqslant (\tanh\gamma)^{-2/p}\left[\frac{1}{\pi}\iint_{H(z,\gamma)}|f(\zeta)|^p\,d\xi d\eta\right]^{1/p} \leqslant \|f\|_{p,B},$$

$$\zeta = \xi + i\eta.$$

Furthermore, both inequalities are sharp for each trio γ, p, and z.

Here, in the case $p = \infty$, the second term in (1.1) ((1.2), resp.) is interpreted as $\|f\|_\infty = \|f\|_{\infty,B}$. In the case $\gamma = \infty$, the second term in (1.1) ((1.2), resp.) is interpreted as $\|f\|_p$ ($\|f\|_{p,B}$, resp.).

The estimate (1.1) is a precision of the known one:

$$(1-|z|^2)^{1/p}|f(z)| \leqslant \|f\|_p, \quad z \in D, f \in H^p,\ 0 < p \leqslant \infty;$$

see [4], p. 144.

Let f be a function meromorphic in D such that the Schwarzian derivative

$$S_f = (f''/f')' - \tfrac{1}{2}(f''/f')^2$$

is holomorphic in D. Two sufficient conditions for f to be univalent in D in terms of $\|S_f\|_p$ are obtained by Z. Nehari and D. London:

(N) $\qquad \|S_f\|_\infty \leqslant \pi^2/2 \quad$ ([8], Theorem II, p. 549);

(L_1) $\qquad \|S_f\|_1 \leqslant 4 \quad$ ([7], Theorem 6, p. 990).

London [7], Theorem 1, p. 981, also obtained a sufficient condition in the Bergman norm as follows:

(L_2) $\qquad \|S_f\|_{1,B} \leqslant 2.$

As an application of Theorem 1, we shall propose sufficient conditions for f to be univalent in D, which contain the above (N), (L_1), and (L_2) as the special cases.

To describe our result we consider

$$c(p) = \begin{cases} 2(3-1/p) & \text{for } \tfrac{1}{2} \leqslant p < 1, \\ 2^{3/p-1}\pi^{2-2/p} & \text{for } 1 \leqslant p \leqslant \infty. \end{cases}$$

The function $c(p)$ is strictly increasing in $[\tfrac{1}{2}, \infty]$, together with $c(1) = 4$, $c(\infty) = \pi^2/2 = 4.9348\ldots$

THEOREM 2. *Let f be a function meromorphic in D. Suppose that S_f is holomorphic in D. Then, f is univalent in D if one of the following two inequalities is valid*:

$$\sup_{z\in D}\left[\frac{1}{2\pi}\int_{\Gamma(z,\gamma)}|S_f(\zeta)|^p|d\zeta|\right]^{1/p}\leqslant c(p)(\tanh\gamma)^{1/p} \tag{1.3}$$

for a certain pair $0<\gamma\leqslant\infty$, $\frac{1}{2}\leqslant p\leqslant\infty$,

$$\sup_{z\in D}\left[\frac{1}{\pi}\iint_{H(z,\gamma)}|S_f(\zeta)|^p\,d\xi d\eta\right]^{1/p}\leqslant c(p/2)(\tanh\gamma)^{2/p} \tag{1.4}$$

for a certain pair $0<\gamma\leqslant\infty$, $1\leqslant p\leqslant\infty$.

We do not repeat the obvious remarks in the case of $\gamma=\infty$ or $p=\infty$. It follows from Theorem 1 and Theorem 2 that, f is univalent in D if one of the following conditions is satisfied:

$$\|S_f\|_p\leqslant c(p)\quad \text{for a } \tfrac{1}{2}\leqslant p\leqslant\infty; \tag{1.5}$$

$$\|S_f\|_{p,B}\leqslant c(p/2)\quad \text{for a } 1\leqslant p\leqslant\infty. \tag{1.6}$$

Now, (N) is the case $p=\infty$ in (1.5), (L_1) is the case $p=1$ in (1.5), and (L_2) is the case $p=1$ in (1.6). The sharpness of $c(\infty)$ is known [8], p. 550. Since $\|S_f\|_\infty\geqslant\|S_f\|_p$ for each $0<p\leqslant\infty$, it follows that, for each $\frac{1}{2}\leqslant p<\infty$, the constant $c(p)$ in (1.5) may never be replaced by any constant strictly larger than $c(\infty)$. By the same reasoning, the constant $c(p/2)$ in (1.6) may never be replaced by any constant strictly larger than $c(\infty)$. The sharpness of $c(p)$ in (1.5) and $c(p/2)$ in (1.6) for $p\neq\infty$ is still open.

We notice that Theorem 2 extends our former results [16], Theorem 2 and Theorem 3.

2. Proofs of Theorem 1 and Theorem 2. We may assume that $0<p<\infty$. For the proof of (1.1) ((1.2), resp.) we may further assume that $f\in H^p$ ($f\in B^p$, resp.). To prove (1.1), we fix $z\in D$ and set

$$h_\gamma(w)=(\beta w+z)/(1+\beta\bar{z}w),\qquad \beta=\tanh\gamma,\ 0<\gamma\leqslant\infty.$$

Then h_γ is holomorphic on the closed disk $\bar{D}=\{|w|\leqslant 1\}$, and h_γ maps C one-to-one onto $\Gamma(z,\gamma)$. Since $h_\gamma(w)=h_\infty(\beta w)$, $w\in D$, it follows that the function

$$F_\gamma(w)=f\big(h_\gamma(w)\big)h_\gamma'(w)^{1/p}$$

is subordinate [4], p. 10, to the function

$$F_\gamma^*(w)=\beta^{1/p}f\big(h_\infty(w)\big)h_\infty'(w)^{1/p},\qquad w\in D.$$

In effect, $F_\gamma(w)=F_\gamma^*(\beta w)$, $w\in D$. It then follows from Littlewood's subordination theorem [4], Theorem 1.7, p. 10, together with the subharmonicity of $|F_\gamma|^p$, that

$$(2.1)\quad \beta(1-|z|^2)|f(z)|^p = |F_\gamma(0)|^p$$
$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} |F_\gamma(re^{i\theta})|^p\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(h_\gamma(re^{i\theta}))|^p\,|h'_\gamma(re^{i\theta})|\,d\theta$$
$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} |F^*_\gamma(re^{i\theta})|^p\,d\theta, \qquad 0<r<1.$$

We next show that $F^*_\gamma \in H^p$. Since $f \in H^p$, and since $h'_\infty \in H^\infty$, it follows that $f \in N^+$, and $h'_\infty \in N^+$ (see [4], p. 26; $N^+ = S(D)$ in the sense of [13]), whence $F^*_\gamma \in N^+$. For, because $\log^+|f\circ h_\infty|$ and $\log^+|h'_\infty|$ have quasi-bounded harmonic majorants, the same is true of $\log^+|F^*_\gamma|$, so that $F^*_\gamma \in N^+$ (see [13], Theorem 1). On the other hand, the boundary value $F^*_\gamma(e^{i\theta})$ of F^*_γ exists for almost every $\theta \in [0, 2\pi]$, and

$$(2.2)\quad \frac{1}{2\pi}\int_0^{2\pi} |F^*_\gamma(e^{i\theta})|^p\,d\theta = \frac{\beta}{2\pi}\int_0^{2\pi} |f(h_\infty(e^{i\theta}))|^p\,|h'_\infty(e^{i\theta})|\,d\theta = \beta\,\|f\|_p^p < \infty.$$

It then follows from [4], Theorem 2.11, p. 28, that $F^*_\gamma \in H^p$.

Now, letting $r \to 1$ in (2.1), and considering (2.2), together with

$$\frac{1}{2\pi}\int_0^{2\pi} |f(h_\gamma(e^{i\theta}))|^p\,|h'_\gamma(e^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_{\Gamma(z,\gamma)} |f(\zeta)|^p\,|d\zeta|,$$

one observes that

$$\beta(1-|z|^2)|f(z)|^p \leqslant \frac{1}{2\pi}\int_{\Gamma(z,\gamma)} |f(\zeta)|^p\,|d\zeta| \leqslant \beta\,\|f\|_p^p,$$

whence follows (1.1).

For the proof of (1.2) we consider the function

$$\Phi_\gamma(w) = f(h_\gamma(w))\,h'_\gamma(w)^{2/p}, \qquad w \in D,$$

being subordinate to the function

$$\Phi^*_\gamma(w) = \beta^{2/p} f(h_\infty(w))\,h'_\infty(w)^{2/p}, \qquad w \in D,$$

in D. It then follows from the subordination theorem that

$$\frac{1}{\pi}\int_0^{2\pi} |\Phi_\gamma(re^{i\theta})|^p\,d\theta \leqslant \frac{1}{\pi}\int_0^{2\pi} |\Phi^*_\gamma(re^{i\theta})|^p\,d\theta,$$

whence

$$(2.3)\quad \|\Phi_\gamma\|_{p,B}^p \leqslant \|\Phi^*_\gamma\|_{p,B}^p = \beta^2\,\|f\|_{p,B}^p.$$

On the other hand, since $|\Phi_\gamma|^p$ is subharmonic, it follows that

(2.4) $$\beta^2(1-|z|^2)^2|f(z)|^p = |\Phi_\gamma(0)|^p \leqslant \|\Phi_\gamma\|_{p,B}^p$$
$$= \frac{1}{\pi}\iint_D |f(h_\gamma(w))|^p |h'_\gamma(w)|^2\,dxdy \qquad (w = x+iy)$$
$$= \frac{1}{\pi}\iint_{H(z,\gamma)} |f(\zeta)|^p\,d\xi d\eta.$$

Combining (2.3) and (2.4) one obtains

$$\beta^2(1-|z|^2)^2|f(z)|^p \leqslant \frac{1}{\pi}\iint_{H(z,\gamma)} |f(\zeta)|^p\,d\xi d\eta \leqslant \beta^2\|f\|_{p,B}^p,$$

whence follows (1.2).

For the proof of the sharpness of (1.1) we consider

$$\varphi_1(w) = h'_\infty(-w)^{1/p}, \qquad w \in \bar{D}\ (0 < p < \infty).$$

Then

$$(1-|z|^2)^{1/p}|\varphi_1(z)| = 1 = \|\varphi_1\|_p.$$

In the case $p = \infty$, we consider $\varphi_1 = 1$.

For the proof of the sharpness of (1.2) we consider

$$\varphi_2(w) = h'_\infty(-w)^{2/p}, \qquad w \in D\ (0 < p < \infty).$$

Then

$$(1-|z|^2)^{2/p}|\varphi_2(z)| = 1 = \|\varphi_2\|_{p,B}.$$

In the case $p = \infty$, we consider $\varphi_2 = 1$.

Now, let f be holomorphic in D, and let

$$\|f\|_{\lambda,\infty} = \sup_{z\in D}(1-|z|^2)^\lambda|f(z)|, \qquad 0 < \lambda < \infty,$$
$$\|f\|_{0,\infty} = \|f\|_\infty,$$

be the weighted H^∞ norm of f of order $0 \leqslant \lambda < \infty$. It then follows from Theorem 1 that

(2.5) $$\|f\|_{1/p,\infty} \leqslant (\tanh\gamma)^{-1/p}\sup_{z\in D}\left[\frac{1}{2\pi}\int_{\Gamma(z,\gamma)} |f(\zeta)|^p|d\zeta|\right]^{1/p},$$

(2.6) $$\|f\|_{2/p,\infty} \leqslant (\tanh\gamma)^{-2/p}\sup_{z\in D}\left[\frac{1}{\pi}\iint_{H(z,\gamma)} |f(\zeta)|^p\,d\xi d\eta\right]^{1/p}.$$

Let f be meromorphic in D such that S_f is holomorphic in D. P. R.

Beesack ([2], (2.6), p. 217, and the italicized sentence in [2], p. 218, line 9) proved that f is univalent in D provided that

$$(\mathrm{B}_1) \qquad \|S_f\|_{\lambda,\infty} \leqslant c(1/\lambda)$$

for a $0 \leqslant \lambda \leqslant 2$. Condition (B_1) contains Nehari's [8], Theorem I, as $\lambda = 2$, Nehari's (N) as $\lambda = 0$, and the result of V. V. Pokornyi [9] (see [7], Theorem 5, p. 988) as $\lambda = 1$.

Theorem 2 is now a consequence of (2.5), (2.6), both applied to S_f, and (B_1).

3. Holomorphic case. Let f be a function non-constant and holomorphic in D. Then it is well known that f is univalent in D if one of the following is satisfied:

$$(\mathrm{P}) \qquad \|f''/f'\|_{0,\infty} = \|f''/f'\|_\infty \leqslant 2\sqrt{2} \quad ([11], \text{ p. } 179);$$

$$(\mathrm{B}_2) \qquad \|f''/f'\|_{1,\infty} \leqslant 1 \quad ([1], \text{ Corollary 4.1, p. } 36).$$

Condition (B_2) is an improvement of the result due to P. L. Duren, H. S. Shapiro, and A. L. Shields [3], Theorem 2, that f is univalent in D if the constant 1 on the right-hand side of (B_2) is replaced by $2(\sqrt{5}-2) = 0.47\ldots$ It follows from (B_2) that if

$$(3.1) \qquad \|f''/f'\|_{\lambda,\infty} \leqslant 1$$

for a $0 < \lambda \leqslant 1$, then f is univalent in D, because of the inequality $\|f''/f'\|_{\lambda,\infty} \geqslant \|f''/f'\|_{1,\infty}$. *Is there a sufficient condition like* (B_1), S_f *being replaced by* f''/f'? Unfortunately we have a reasonable answer only for small λ.

For $0 < \lambda \leqslant 1$ we consider the function

$$G(\lambda) = 2(2-\lambda)[\lambda^{-\lambda}(\lambda+1)^{\lambda+1} - \lambda^{\lambda}(\lambda+1)^{-\lambda+1} + 1]^{-1}.$$

The function G is strictly decreasing in $(0, 1]$, and satisfies

$$\lim_{\lambda \to 0} G(\lambda) = 4, \qquad G(1) = \tfrac{1}{2},$$

so that there exist λ_1 and λ_2 such that

$$0 < \lambda_2 < \lambda_1 < 1, \qquad G(\lambda_2) = 2, \qquad G(\lambda_1) = 1.$$

A computation by a programmable calculator (for example, TI Programmable 57) teaches us that

$$\lambda_2 = 0.1813\ldots, \qquad \lambda_1 = 0.5578\ldots$$

We now define $K(\lambda)$ by

$$K(\lambda) = \begin{cases} 2, & \text{if } 0 < \lambda \leqslant \lambda_2, \\ G(\lambda), & \text{if } \lambda_2 < \lambda \leqslant \lambda_1. \end{cases}$$

THEOREM 3. *Let f be a function non-constant and holomorphic in D. Then f is univalent in D if*

$$\|f''/f'\|_{\lambda,\infty} \leqslant K(\lambda) \tag{3.2}$$

for a $0 < \lambda \leqslant \lambda_1$.

Since $K(\lambda) > 1$ for $0 < \lambda < \lambda_1$, condition (3.2) is significant in view of (3.1). Theorem 3 improves our former result [17], Theorem.

As an obvious application of Theorem 1 to (3.2), one can now easily prove sufficient conditions for holomorphic f to be univalent in D in terms of $\|f''/f'\|_p$ or $\|f''/f'\|_{p,B}$. More precisely, one obtains

THEOREM 4. *Let f be a function non-constant and holomorphic in D. Then one of the following two inequalities asserts the univalency of f in D*:

$$\sup_{z\in D} \int_{\Gamma(z,\gamma)} |f''(\zeta)/f'(\zeta)|^p \,|d\zeta| \leqslant 2\pi K(1/p)^p \tanh\gamma \tag{3.3}$$

for a certain pair $0 < \gamma \leqslant \infty$, $\lambda_1^{-1} \leqslant p < \infty$,

$$\sup_{z\in D} \iint_{H(z,\gamma)} |f''(\zeta)/f'(\zeta)|^p \,d\xi\, d\eta \leqslant \pi K(2/p)^p (\tanh\gamma)^2 \tag{3.4}$$

for a certain pair $0 < \gamma \leqslant \infty$, $2\lambda_1^{-1} \leqslant p < \infty$.

In effect, one can easily prove that f''/f' is holomorphic in D if (3.3) or (3.4) is satisfied; actually, $\lambda_1^{-1} > 1$.

For the proof of Theorem 3 we shall make use of

LEMMA. *Let g be holomorphic in D, let $0 < \lambda \leqslant 1$, and suppose that*

$$\|g\|_{\lambda,\infty} \leqslant M(\lambda) \leqslant 2.$$

Then

$$\sup_{z\in D} (1-|z|^2)^{\lambda+1} \big(|g'(z)| + \tfrac{1}{2}|g(z)|^2\big) \leqslant 2(2-\lambda)\, M(\lambda)\, G(\lambda)^{-1}. \tag{3.5}$$

Proof. M. S. Robertson [12], Theorem A, proved that, if h is holomorphic in D, and if $\|h\|_{\lambda,\infty} \leqslant 1$, then, at each $z \in D$,

$$(1-|z|^2)^{\lambda+1} |h'(z)| \leqslant \lambda^{-\lambda}(\lambda+1)^{\lambda+1} [1-\lambda^{2\lambda}(\lambda+1)^{-2\lambda}(1-|z|^2)^{2\lambda}|h(z)|^2].$$

Then, at each $z \in D$,

$$\begin{aligned} (1-|z|^2)^{\lambda+1}\big(|h'(z)| + |h(z)|^2\big) &\leqslant (1-|z|^2)^{\lambda+1}|h'(z)| + (1-|z|^2)^{2\lambda}|h(z)|^2 \\ &\leqslant \lambda^{-\lambda}(\lambda+1)^{\lambda+1} + [1-\lambda^{\lambda}(\lambda+1)^{-\lambda+1}](1-|z|^2)^{2\lambda}|h(z)|^2 \\ &\leqslant \lambda^{-\lambda}(\lambda+1)^{\lambda+1} + 1 - \lambda^{\lambda}(\lambda+1)^{-\lambda+1} = 2(2-\lambda)/G(\lambda). \end{aligned} \tag{3.6}$$

We now apply (3.6) to $h = g/M(\lambda)$, where we may assume, without any loss of generality, that $M(\lambda) \neq 0$. Inequality (3.4) is then a consequence of $M(\lambda)^{-1} \geqslant \frac{1}{2}$.

Proof of Theorem 3. We apply the lemma to $g = f''/f'$, together with $M(\lambda) = K(\lambda)$, $0 < \lambda \leqslant \lambda_1$. It then follows from (3.5), together with $K(\lambda) \leqslant G(\lambda)$, that

$$\|S_f\|_{\lambda+1,\infty} \leqslant 2(2-\lambda).$$

Since $1 < \lambda+1 \leqslant \lambda_1+1 < 2$ for $0 < \lambda \leqslant \lambda_1$, it follows that

$$c(1/(\lambda+1)) = 2(2-\lambda),$$

whence

$$\|S_f\|_{\mu,\infty} \leqslant c(1/\mu), \qquad 1 < \mu \leqslant \lambda_1+1 < 2.$$

Therefore f is univalent in D by the Beesack criterion (B_1).

Remark. The cited result of J. Becker [1], Corollary 4.1, p. 36, asserts much more. That is, if f is holomorphic in D, $f'(0) \neq 0$, and if

$$\|g\|_{1,\infty} \leqslant 1, \qquad g(z) = zf''(z)/f'(z),$$

then f is univalent in D. Again, we obtain the sufficient conditions for f to be univalent in D in terms of $\|g\|_1$ or $\|g\|_{2,B}$. The details are easy excercises.

4. Further applications of Theorem 1. In the present section we shall give three applications of Theorem 1.

4.1. *Gavrilov's extremal theorem.* Fix $z \in D$ and a complex number A once and for all in the present subsection. Let $\mathscr{G}(z, A)$ be the family of all holomorphic functions f in D such that $f(z) = A$. V. I. Gavrilov [5], Theorem 3, p. 843, proved that

$$\min_{f \in \mathscr{G}(z,A)} \|f\|_{2,B}^2 = (1-|z|^2)^2 |A|^2; \tag{4.1}$$

the minimum is attained by the function

$$f_2(w) = A(1-|z|^2)\, h'_{\infty}(-w), \qquad w \in D.$$

It now follows from (1,2) in Theorem 1 that, for each $0 < p < \infty$,

$$\min_{f \in \mathscr{G}(z,A)} \|f\|_{p,B}^p = (1-|z|^2)^2 |A|^p; \tag{4.2}$$

the minimum is attained by the function

$$f_p(w) = A(1-|z|^2)^{2/p} h'_{\infty}(-w)^{2/p}, \qquad w \in D.$$

Gavrilov's result (4.1) is the special case $p = 2$ in (4.2). Furthermore, it follows from (1.1) that, for each $0 < p < \infty$,

$$\min_{f \in \mathscr{G}(z,A)} \|f\|_p^p = (1-|z|^2)|A|^p;$$

the minimum is attained by the function

$$A(1-|z|^2)^{1/p}\, h'_z(-w)^{1/p}$$

of $w \in D$.

4.2. *α-Bloch function.* A function f holomorphic in D is called *α-Bloch* $(0 \leqslant \alpha < \infty)$ if $\|f'\|_{\alpha,\infty} < \infty$. A *1-Bloch function* is simply a Bloch function [10]. A holomorphic function f in D is continuous on $\bar{D}$ and $f(e^{i\theta}) \in \Lambda_\alpha$ $(0 < \alpha \leqslant 1)$ if and only if f is a $(1-\alpha)$-Bloch function [4], Theorem 5.1, p. 74.

THEOREM 5. *Let f be a function holomorphic in D, and let $0 < \alpha < \infty$. Then the following are equivalent:*

(5A) *f is α-Bloch.*

(5B) *There exists $0 < \gamma < \infty$ such that*

$$\sup_{z \in D} \iint_{H(z,\gamma)} |f'(\zeta)|^{2/\alpha}\, d\xi d\eta < \infty. \tag{4.3}$$

(5C) *There exists $0 < \gamma < \infty$ such that*

$$\sup_{z \in D} \int_{\Gamma(z,\gamma)} |f'(\zeta)|^{1/\alpha}\, |d\zeta| < \infty. \tag{4.4}$$

Proof. Both (5B) ⇒ (5A) and (5C) ⇒ (5A) are consequences of Theorem 1 applied to f'. For the proofs of (5A) ⇒ (5B) and (5A) ⇒ (5C) we assume that

$$(1-|\zeta|^2)^\alpha |f'(\zeta)| \leqslant M, \qquad \zeta \in D,$$

where $M > 0$ is a constant. One then observes that, for each $0 < \gamma < \infty$,

$$\begin{aligned}\iint_{H(z,\gamma)} |f'(\zeta)|^{2/\alpha}\, d\xi d\eta &\leqslant M^{2/\alpha} \iint_{H(z,\gamma)} (1-|\zeta|^2)^{-2}\, d\xi d\eta \\ &= M^{2/\alpha} \iint_{|w|<\beta} (1-|w|^2)^{-2} dx dy = M^{2/\alpha}\pi\beta^2/(1-\beta^2), \\ &\qquad \beta = \tanh\gamma,\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma(z,\gamma)} |f'(\zeta)|^{1/\alpha}\, |d\zeta| &\leqslant M^{1/\alpha} \int_{\Gamma(z,\gamma)} (1-|\zeta|^2)^{-1}\, |d\zeta| \\ &= M^{1/\alpha} \int_{|w|=\beta} (1-|w|^2)^{-1}\, |dw| = M^{1/\alpha}\, 2\pi\beta/(1-\beta^2).\end{aligned}$$

Remark. In the special case $\alpha = 1$, the integral in (4.3) ((4.4), resp.) is the area (length, resp.) of the Riemannian image of $H(z, \gamma)$ ($\Gamma(z, \gamma)$, resp.) by f. The cited criterion for $\alpha = 1$ and for $H(z, \gamma)$ is also obtained as a consequence of [15], Theorem 2. It might be of interest that the analogous criterion in terms of $f^{\#} = |f'|/(1+|f|^2)$ is false for f meromorphic in D to be normal, that is,

$$\sup_{z \in D}(1-|z|^2)f^{\#}(z) < \infty .$$

In effect, P. A. Lappan [6] constructed a function f_L holomorphic and non-normal in D, such that f_L is univalent in each $H(z, \gamma)$, $z \in D$, where $0 < \gamma < \infty$ is a constant independent of z. Then

$$\sup_{z \in D} \iint_{H(z,\gamma)} f_L^{\#}(\zeta)^2 \, d\xi d\eta \leqslant \pi,$$

where π is the area of the Riemann sphere. We remark that, if f is normal in D, then, for each $0 < \gamma < \infty$,

$$\sup_{z \in D} \int_{H(z,\gamma)} f^{\#}(\zeta)^2 \, d\xi d\eta < \infty .$$

4.3. LUS *function.* Now, Lappan's function f_L is LUS. Namely, a function f holomorphic in D is called *locally uniformly schlicht* (LUS, for short) in D if there exists $0 < \gamma \leqslant \infty$ such that f is univalent in each $H(z, \gamma)$, $z \in D$ (see [14]). We propose criteria in

THEOREM 6. *Let f be a function non-constant and holomorphic in D. Then the following are equivalent*:

(6A) f *is* LUS *in* D.

(6B) *There exists* $0 < \gamma < \infty$ *such that*

$$\sup_{z \in D} \iint_{H(z,\gamma)} |f''(\zeta)/f'(\zeta)|^2 \, d\xi d\eta < \infty . \tag{4.5}$$

(6C) *There exists* $0 < \gamma < \infty$ *such that*

$$\sup_{z \in D} \int_{\Gamma(z,\gamma)} |f''(\zeta)/f'(\zeta)| \, |d\zeta| < \infty . \tag{4.6}$$

Proof. Assume (6A). Then f' never vanishes in D, and further it follows from [14], Theorem 2, that each branch of $\log f'$ is Bloch in D. Therefore (6A) $\Rightarrow$ (6B) and (6A) $\Rightarrow$ (6C) both are consequences of Theorem 5, applied to $\log f'$ and $\alpha = 1$. Conversely, assume (6B) ((6C), resp.). It then follows from (4.5) ((4.6), resp.), together with a local consideration, that f''/f' has no pole in D. It now follows from Theorem 5, applied to $\log f'$ and $\alpha = 1$, that $\log f'$ is Bloch, whence f is LUS in D again by [14], Theorem 2.

Remark. A meromorphic analogue of Theorem 6, in terms of S_f instead of f''/f', is announced in [16], Theorem 4.

References

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