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Linear independence in linear rings with abstract differentiation

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Abstract. A commutative ring K is considered with an operation D, such that $D(x+y) = Dx+Dy$ and $D(xy) = x\cdot Dy+y\cdot Dx$ for any $x, y \in K$. The equation

$$fx \equiv a_n D^n x + \ldots + a_1 Dx + a_0 x = 0,$$

in the unknown $x \in K$, with coefficients a_i $(i = 0, 1, \ldots, n)$ from a sub-ring $K_1 \subseteq K$, has linearly independent solutions

$$x_{1,1}, x_{1,2}, \ldots, x_{1,n}.$$

In particular, K_1 may be a sub-ring of constants if $Da_i = 0$ for $i = 0, 1, \ldots, n$. Let the equation $f^k x = 0$, obtained by superposition, have kn solutions:

$$x_{\varkappa,p} \quad (\varkappa = 1, 2, \ldots, k;\ p = 1, \ldots, n).$$

Are the solutions $x_{\varkappa,p}$ linearly independent? In the simplest case of constant coefficients and when there exists a linear operation t such that $D(tx) = tDx+x$, Theorem 4 yields a positive answer.

In the general case, the solutions $x_{\varkappa,p}$ $(\varkappa > 1)$ are not expressed by $x_{1,p}$ $(p = 1, \ldots, n)$. Wronskians have been investigated and a relation between the wronskian of the $x_{\varkappa,p}$ and the wronskian of the $x_{1,p}$ has been found. Some other determinants, called *eliminants*, have turned out to be important factors. In the particular case of constant coefficients, the eliminant is the discriminant of the polynomial f, i.e., the resultant of the polynomial f and its derivative.

If the superposition fgx is considered and a relation between wronskians is required, again a determinant is found to be a factor. In the particular case of constant coefficients, the determinant is the resultant of the polynomials f and g.

1. A new theory of operational calculus has been created by J. Mikusiński, who wrote a book [7] containing the theory and its applications. The foundations of the theory ([1]–[4]) are as follows:

Continuous complex functions of a real variable t, defined for $t \geqslant 0$, are added in the usual way and multiplied by the convolution

$$f(t)*g(t) = \int_0^t f(t-\tau)\cdot g(\tau)\cdot d\tau.$$

The commutative ring has no unit element and by Titchmarsh's theorem ([10], [11]) the ring has no divisors of zero. Then it is possible to extend the ring to a field A, the elements of which are called *operators*. The field of operators contains a sub-field which is isomorphic to the field of complex numbers; so the field has the characteristic zero. Operational function $a(\lambda)$ is a function which assigns operators to numbers λ. In particular, an ordinary continuous function $a(\lambda, t)$ of two variables, defined for $t \geqslant 0$ and for some values λ, is an operational function $a(\lambda)$. A function is said to be *differentiable at a point* λ_0 if it can be represented in a neighbourhood of the point as the product $a(\lambda) = q * f(\lambda, t)$, where q is an operator and $f(\lambda, t)$ is an ordinary function such that the quotient

$$\frac{f(\lambda, t) - f(\lambda_0, t)}{\lambda - \lambda_0}$$

tends uniformly to the limit in every finite interval $0 \leqslant t \leqslant t_0$ as λ tends to λ_0. A derivative of the operational function $a(\lambda)$ at the point λ_0 is the product $q * \frac{\partial}{\partial \lambda} f(\lambda_0, t)$, which may be denoted by $Df(\lambda_0)$. J. Mikusiński applied his theory to partial differential equations with constant coefficients. Hence the partial differential equation

$$(1) \qquad \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} \frac{\partial^{\mu+\nu} x(\lambda, t)}{\partial^{\mu} \lambda \partial^{\nu} t} = \varphi(\lambda, t) \qquad (a_{\mu\nu} = \text{const})$$

can be written in the following operational form:

$$(2) \qquad \sum_{\mu=0}^{m} b_{\mu} * D^{\mu} x(\lambda) = f(\lambda),$$

where the function $x(\lambda)$ is the required operational function and the coefficients b_{μ} are given operators. There is, therefore, a need to investigate some equations such as $a_m D^m x + \ldots + a_0 x = 0$, where the coefficients a_i $(i = 0, 1, \ldots, m)$ belong to a field or a ring and the operation D has some properties of the ordinary derivative.

2. Some necessary definitions and remarks will be introduced here. It is assumed that an operation D is defined in a commutative ring K, and so an element $Dx \in K$ is assigned to any $x \in K$. By applying the operation D i-times successively, $D^i x$ is obtained. The operation D satisfies the conditions [1]

$$(3) \qquad D(x+y) = Dx + Dy,$$

$$(4) \qquad D(xy) = x \cdot Dy + y \cdot Dx$$

[1] Conditions (3) and (4) were assumed in J. F. Ritt's and E. R. Kolchin's papers. In 1950 a book *Differential algebra*, written by J. F. Ritt, was published (Amer. Math. Soc. Publ. 33, New York 1950).

for any x, y belonging to the ring K. It is evident from (3) that $D(0) = 0$, and it follows from (4) that if the ring K has a unit element ε, then $D\varepsilon = 0$. The elements satisfying the equation $Dx = 0$ will be called constants; they constitute a sub-ring $K_0 \subseteq K$. The equation

$$\sum_{i=0}^{m} a_i D^i x = 0 \tag{5}$$

will be considered. Element x is the required element of the ring K, and the coefficients a_i $(i = 0, 1, \ldots, m)$ are given elements from a ring $K_1 \subseteq K$. In particular, we may have $K_1 = K_0$ (the sub-ring of constants). Equation (5) is of the order m if $a_m \neq 0$. Elements $x_1, \ldots, x_m$ $(m \geqslant 1)$ of the ring K are linearly independent over K_0 if the equality $c_1 x_1 + \ldots + c_m x_m = 0$ $(c_i \in K_0,\ x_i \in K)$ occurs only if $c_1 = \ldots = c_m = 0$. The determinant

$$W(x_1, \ldots, x_m) = \begin{vmatrix} x_1, \ldots, & x_m \\ \ldots & \ldots \\ D^{m-1}x_1, \ldots, & D^{m-1}x_m \end{vmatrix}$$

is the wronskian of the elements $x_1, \ldots, x_m$.

The following condition will be used later:

(A) An equation $c_1 Dx + c_0 x = 0$ $(c_1 \neq 0,\ c_0, c_1 \in K_1,\ x \in K,\ K_1 \subseteq K)$ cannot have a solution, being a divisor of zero.

This assumption will be useful when wronskians are considered. Liouville's well-known theorem points out that the wronskian W of the solutions of the equation

$$a_n D^n x + a_{n-1} D^{n-1} x + \ldots + a_0 x = 0 \tag{6}$$

satisfies the equation $a_n DW + a_{n-1} W = 0$. Thus the wronskian cannot be a divisor of zero if condition (A) is assumed. Any product of wronskians of the solutions of equation (6) will also be different from zero. If x_1 satisfies the equation $a_1 Dx + a_0 x = 0$ and x_2 satisfies the equation $b_1 Dx + b_0 x = 0$, the product $x_1 x_2$ is a solution of the equation $a_1 b_1 Dx + (a_0 b_1 + a_1 b_0) x = 0$; thus the product cannot be a divisor of zero according to (A), and similarly for any finite number of factors.

Another important conclusion derived from (A), namely that a constant cannot be a divisor of zero, will now be proved. If $cx = 0$ $(c \in K_0,\ x \in K)$, then either $c = 0$ or $x = 0$. In fact, $D(cx) = D(0) = 0$, $D(cx) = c \cdot Dx + 0 = 0$. The element $x \neq 0$ would be a solution of the equation $c \cdot Dx + cx = 0$ in spite of assumption (A) if $c \neq 0$, $x \neq 0$, $cx = 0$.

Condition (4) and equation (5) clearly indicate an application of the results for differential equations. It is not intended, however, to use such terms as for instance "value of the function at a point" because the problem of linear independence is an algebraic one. Functions will be presented as elements of a ring or a field.

3. Let the elements $y_1, \ldots, y_k$ satisfy the equation

$$fx \equiv \sum_{i=0}^{k} a_i D^i x = 0 \quad (x \in K, a_i \in K_1 \subseteq K) \tag{7}$$

and let the elements $z_1, \ldots, z_n$ be the solutions of the equation

$$gx \equiv \sum_{j=0}^{n} b_j D^j x = 0 \quad (x \in K, b_j \in K_1 \subseteq K). \tag{8}$$

The question arises whether the $(k+n)$ elements $y_1, \ldots, y_k, z_1, \ldots$ $\ldots, z_n$ are linearly independent. When the coefficients a_i $(i = 0, 1, \ldots, k)$ and b_j $(j = 0, 1, \ldots, n)$ are constant, the $(k+n)$ elements will be the solutions of the equation $fgx = 0$ which is obtained from f and g by superposition. In the particular case of constant coefficients, the problem is solved by Theorem 4, which is valid in a linear space K over the field C_0 of constants. The linear independence of $y_1, \ldots, y_k, z_1, \ldots, z_n$ follows directly from the linear independence of those $y_1, \ldots, y_k$ and $z_1, \ldots, z_n$ (see the proof of Theorem 4). Another idea is to use wronskians, and for constant coefficients a relation between resultants, discriminants and wronskians has been established (see Theorem 3^a). The coefficients are not necessarily constant in Theorems 1, 2 and 3, but condition (10) has to be introduced to ensure symmetry for the superposition. It is obvious that condition (10) restricts the expressions fx and gx very much.

4. The elements $x_1, x_2, \ldots, x_m$ are linearly dependent if constants $c_1, \ldots, c_m$ exist such that $c_1 x_1 + \ldots + c_m x_m = 0$ and not all c_i $(i = 1, 2, \ldots$ $\ldots, m)$ are zeros. Let $c_m \neq 0$. The wronskian $W(x_1, \ldots, x_m)$ is the determinant of the system of equations

$$c_1 D^\mu x_1 + \ldots + c_m D^\mu x_m = 0 \quad (\mu = 0, 1, \ldots, m-1).$$

The determinant, expanded from the last column, can be written in the form

$$W(x_1, \ldots, x_m) = A_0 x_m + \ldots + A_{m-1} D^{m-1} x_m,$$

where $A_0, \ldots, A_{m-1}$ are the cofactors. Adding the equations

$$A_0 \cdot (c_1 x_1 + \ldots + c_m x_m) = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_{m-1} \cdot (c_1 D^{m-1} x_1 + \ldots + c_m D^{m-1} x_m) = 0,$$

we obtain $c_m W(x_1, \ldots, x_m) = 0$ because

$$A_0 x_k + \ldots + A_{m-1} D^{m-1} x_k = W(x_1, \ldots, x_{m-1}, x_k) = 0 \quad \text{for } k < m.$$

Therefore, the elements $x_1, \ldots, x_m$ are linearly independent over K_0 if $W(x_1, \ldots, x_m) \neq 0$, and besides a constant cannot be a divisor of zero. The inequality $W(x_1, \ldots, x_m) \neq 0$ is sufficient for the linear independence of the elements $x_1, \ldots, x_m$ when condition (A) is assumed.

The inequality $W(x_1, \ldots, x_m) \neq 0$ may serve as a definition of "strict linear independence" (see [12]) because the condition is a little stronger than linear independence. However, linear independence and the condition $W \neq 0$ can be equivalent in some special cases when additional information on the ring K is given.

5. If the equation $hx \equiv c_m D^m x + \ldots + c_0 x = 0$ is satisfied by the elements $x_1, \ldots, x_m$, then

$$c_m \cdot W(x_1, \ldots, x_m, x) = W(x_1, \ldots, x_m) \cdot hx. \tag{9}$$

In fact,

$$\begin{aligned} & c_m \cdot W(x_1, \ldots, x_m, x) \\ &= c_m \cdot \begin{vmatrix} x_1, \ldots, & x_m, & x \\ \ldots & \ldots & \ldots \\ D^{m-1}x_1, \ldots, & D^{m-1}x_m, & D^{m-1}x \\ D^m x_1, \ldots, & D^m x_m, & D^m x \end{vmatrix} = \begin{vmatrix} x_1, \ldots, & x_m, & x \\ \ldots & \ldots & \ldots \\ D^{m-1}x_1, \ldots, & D^{m-1}x_m, & D^{m-1}x \\ hx_1, \ldots, & hx_m, & hx \end{vmatrix} \\ &= W(x_1, \ldots, x_m) \cdot hx. \end{aligned}$$

Let fgx denote the superposition of the expressions fx and gx. We assume that the solutions $y_1, \ldots, y_k$ of equation (7) and the solutions $z_1, \ldots, z_n$ of equation (8) must satisfy the condition

$$fgx = gfx \quad \text{for } x = y_1, \ldots, y_k \text{ and for } x = z_1, \ldots, z_n. \tag{10}$$

This means that $fx = 0$ for $x = gy_1, \ldots, gy_k$ and $gx = 0$ for $x = fz_1, \ldots, fz_n$. Condition (10) is always fulfilled if the coefficients of the expressions fx and gx are constant or if $f \equiv g$, but it may happen that $fgx = gfx$ identically (for any x) in some other cases. For example, in the case of the expressions $fx = Dx + bx$ and $gx = Dx + (b+c)x$, the condition $fgx = gfx$ is fulfilled for any x and any b if $c = \text{const}$.

By putting $hx \equiv fgx$ (the leading coefficient will be $c_m = a_k b_n$) and using condition (10), the identity

$$a_k b_n \cdot W(y_1, \ldots, y_k, z_1, \ldots, z_n, x) = W(y_1, \ldots, y_k, z_1, \ldots, z_n) \cdot fgx$$

can be obtained from (9).

6. A consequence of equations (7) and (8) is a system of $(k+n)$ equations

$$\begin{aligned} D^\mu fx &\equiv D^\mu(a_k D^k x + \ldots + a_0 x) = 0, \quad \mu = 0, 1, \ldots, n-1; \\ D^\nu gx &\equiv D^\nu(b_n D^n x + \ldots + b_0 x) = 0, \quad \nu = 0, 1, \ldots, k-1 \end{aligned} \tag{11}$$

in the variables $x, Dx, \ldots, D^{k+n-1}x$. A necessary condition for the existence of a common solution of equations (7) and (8), which is not a divisor of zero and is different from zero, is that the determinant of system (11) should be equal to zero. The determinant will be called the *eliminant*. Introducing the notation

$$(12) \qquad D^{\mu}fx \equiv \sum_{p=0}^{k+\mu} a_{\mu,p}\cdot D^{p}x, \quad D^{\nu}gx \equiv \sum_{q=0}^{n+\nu} b_{\nu,q}\cdot D^{q}x$$

$$(\mu = 0, 1, \ldots, n-1; \nu = 0, 1, \ldots, k-1)$$

we can write eliminant in the forms

$$E(f, g) = \begin{vmatrix} a_{n-1,k+n-1}, \ldots, a_{n-1,0} \\ \ldots\ldots\ldots\ldots \\ 0 \quad a_{0,k}, \ldots, a_{0,0} \\ b_{k-1,k+n-1}, \ldots, b_{k-1,0} \\ \ldots\ldots\ldots\ldots \\ 0 \quad b_{0,n}, \ldots, b_{0,0} \end{vmatrix} = \begin{vmatrix} b_{0,0}, \ldots, b_{0,n} \quad 0 \\ \ldots\ldots\ldots\ldots \\ b_{k-1,0}, \ldots, b_{k-1,k+n-1} \\ a_{0,0}, \ldots, a_{0,k} \quad 0 \\ \ldots\ldots\ldots\ldots \\ a_{n-1,0}, \ldots, a_{n-1,k+n-1} \end{vmatrix}.$$

In the particular case where a_i, b_j $(i = 0, 1, \ldots, k; j = 0, 1, \ldots, n)$ are constant the eliminant is well known in algebra, namely it is the resultant of the polynomials $f(w) = a_k w^k + \ldots + a_0$ and $g(w) = b_n w^n + \ldots \ldots + b_0$. The discriminant of the polynomial $f(w)$ can be obtained, in particular, if

$$g(w) = \sum_{r=1}^{k} r\cdot a_r\cdot w^{r-1}.$$

7. Lemma 1 is needed for the proof of Theorem 1 and Lemma 2 will be used to prove Theorem 2.

LEMMA 1. *If* $fx \equiv \sum_{i=0}^{k} a_i D^i x = 0$ *for* $x = x_j$ $(j = 1, \ldots, k)$, $gx \equiv \equiv \sum_{r=0}^{n} b_r D^r x$, *then* $a_k^n\cdot W(gx_1, \ldots, gx_k) = E(f, g)\cdot W(x_1, \ldots, x_k)$.

Proof. Notation (12) is adopted and it is noted that the leading coefficients are equal to a_k for arbitrary $\mu = 0, 1, \ldots$, and so $a_{n-1,k+n-1} = \ldots = a_{1,k+1} = a_{0,k} = a_k$. Multiplying the rows of the eliminant by the columns of the extended wronskian $W(x_1, \ldots, x_k)$, we obtain the result $E(f, g)\cdot W(x_1, \ldots, x_k) = a_k^n\cdot W(gx_1, \ldots, gx_k)$:

$$\begin{vmatrix} b_{0,0}, \ldots, b_{0,n} & 0 \\ \cdots & \cdots \\ b_{k-1,0}, \ldots, b_{k-1,k+n-1} & \\ a_{0,0}, \ldots, a_{0,k} & 0 \\ \cdots & \cdots \\ a_{n-1,0}, \ldots, a_{n-1,k+n-1} & \end{vmatrix} \cdot \begin{vmatrix} x_1, \ldots, x_k, & 0, \ldots, 0 \\ \cdots & \cdots \\ D^{k-1}x_1, \ldots, D^{k-1}x_k, & 0, \ldots, 0 \\ D^k x_1, \ldots, D^k x_k, & 1 \quad\quad 0 \\ \cdots & \ddots \\ D^{k+n-1}x_1, \ldots, D^{k+n-1}x_k, & \quad\quad 1 \end{vmatrix}$$

$$= \begin{vmatrix} gx_1, \ldots, gx_k, & \\ \cdots & \\ D^{k-1}gx_1, \ldots, D^{k-1}gx_k, & \\ fx_1, \ldots, fx_k, & a_{0,k} \quad\quad 0 \\ \cdots & \ddots \\ D^{n-1}fx_1, \ldots, D^{n-1}fx_k, & \quad\quad a_{n-1,k+n-1} \end{vmatrix}$$

$$= \begin{vmatrix} gx_1, \ldots, gx_k, & \\ \cdots & \\ D^{k-1}gx_1, \ldots, D^{k-1}gx_k, & \\ 0, \ldots, 0, & a_k \quad\quad 0 \\ \cdots & \ddots \\ 0, \ldots, 0, & \quad\quad a_k \end{vmatrix} = a_k^n \cdot W(gx_1, \ldots, gx_k).$$

Now Lemma 2 should be derived from Lemma 1. Given an expression fx of order n, the superposition is defined in a recurrent way: $f^{\varkappa}x = ff^{\varkappa-1}x$ $(\varkappa = 1, 2, \ldots)$, $f^0x = x$. Let $x_{\varkappa-1,p}$ $(\varkappa = 2, \ldots, k+1;\ p = 1, 2, \ldots, n)$ be the solutions of the equation $f^{\varkappa-1}x = 0$. It is obvious that the elements $x_{\varkappa-1,p}$ satisfy the equation $f^{\varkappa}x = 0$ as well. The equation $fx = 0$ is satisfied by $y_{\varkappa-1,p} = f^{\varkappa-2}x_{\varkappa-1,p}$ $(\varkappa = 2, \ldots, k+1;\ p = 1, 2, \ldots, n)$ because

$$f^{\varkappa-1}x_{\varkappa-1,p} = ff^{\varkappa-2}x_{\varkappa-1,p} = fy_{\varkappa-1,p} = 0.$$

Putting $gx \equiv h_{\varkappa}x$ $(\varkappa = 2, 3, \ldots)$, we obtain the following lemma from Lemma 1:

LEMMA 2. *Let* $fx \equiv \sum_{j=0}^{k} a_j D^j x$ *and* $h_{\varkappa}x \equiv \sum_{r=0}^{n-1} d_{\varkappa,r} D^r x$. *If* $y_{\varkappa,p} = h_{\varkappa} y_{\varkappa-1,p}$ *and* $fy_{\varkappa-1,p} = 0$ *for* $\varkappa = 2, 3, \ldots, k$; $p = 1, 2, \ldots, n$, *then* $a_n^{n-1} \cdot W(y_{\varkappa,1}, \ldots$ $\ldots, y_{\varkappa,n}) = E(f, h_{\varkappa}) \cdot W(y_{\varkappa-1,1}, \ldots, y_{\varkappa-1,n})$.

8. Two theorems on wronskians will be proved.

THEOREM 1. *Let* $fx \equiv \sum_{i=0}^{k} a_i D^i x$, $gx \equiv \sum_{j=0}^{n} b_j D^j x$ $(a_i, b_j \in K_1 \subseteq K)$. *If* $fy_i = 0$ *for* $i = 1, 2, \ldots, k$ *and* $gz_j = 0$ *for* $j = 1, 2, \ldots, n$, *then*

$$a_k^n \cdot b_n^k \cdot W(y_1, \ldots, y_k, z_1, \ldots, z_n) = E(g, f) \cdot W(y_1, \ldots, y_k) \cdot W(z_1, \ldots, z_n).$$

Proof. The last row of the determinant $W(y_1, \ldots, y_k, z_1, \ldots, z_n)$ is multiplied by b_n. After that, other rows, multiplied suitably, are added to the last row in order to obtain $D^{k-1} gz_j$ $(j = 1, 2, \ldots, n)$. By this way n elements of the last row have been made equal to zero. The next step will be made similarly. The last but one row is multiplied by b_n and then the previous rows, multiplied suitably, are added to the last but one row in order to obtain $D^{k-2} gz_j$ $(j = 1, 2, \ldots, n)$. A rectangle having k rows and n columns will contain only zeros after k such steps:

$$a_k^n \cdot b_n^k \cdot W(z_1, \ldots, z_n, y_1, \ldots, y_k)$$

$$= a_k^n \cdot b_n^k \cdot \begin{vmatrix} z_1, \ldots, & z_n, & y_1, \ldots, & y_k \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1} z_1, \ldots, & D^{n-1} z_n, & D^{n-1} y_1, \ldots, & D^{n-1} y_k \\ D^n z_1, \ldots, & D^n z_n, & D^n y_1, \ldots, & D^n y_k \\ \cdots & \cdots & \cdots & \cdots \\ D^{k+n-1} z_1, \ldots, & D^{k+n-1} z_n, & D^{k+n-1} y_1, \ldots, & D^{k+n-1} y_k \end{vmatrix}$$

$$= a_k^n \cdot \begin{vmatrix} z_1, \ldots, & z_n, & y_1, \ldots, & y_k \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1} z_1, \ldots, & D^{n-1} z_n, & D^{n-1} y_1, \ldots, & D^{n-1} y_k \\ gz_1, \ldots, & gz_n, & gy_1, \ldots, & gy_k \\ \cdots & \cdots & \cdots & \cdots \\ D^{k-1} gz_1, \ldots, & D^{k-1} gz_n, & D^{k-1} gy_1, \ldots, & D^{k-1} g y_k \end{vmatrix}$$

$$= a_k^n \cdot \begin{vmatrix} z_1, \ldots, & z_n, & y_1, \ldots, & y_k \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1} z_1, \ldots, & D^{n-1} z_n, & D^{n-1} y_1, \ldots, & D^{n-1} y_k \\ 0, \ldots, & 0, & gy_1, \ldots, & gy_k \\ \cdots & \cdots & \cdots & \cdots \\ 0, \ldots, & 0, & D^{k-1} gy_1, \ldots, & D^{k-1} gy_k \end{vmatrix}$$

$$= a_k^n \cdot \begin{vmatrix} gy_1, \ldots, & gy_k \\ \cdots & \cdots \\ D^{k-1}gy_1, \ldots, & D^{k-1}gy_k \end{vmatrix} \cdot \begin{vmatrix} z_1, \ldots, & z_n \\ \cdots & \cdots \\ D^{n-1}z_1, \ldots, & D^{n-1}z_n \end{vmatrix}.$$

Lemma 1 will now be used:

$$a_k^n \cdot \begin{vmatrix} gy_1, \ldots, & gy_k \\ \cdots & \cdots \\ D^{k-1}gy_1, \ldots, & D^{k-1}gy_k \end{vmatrix} = E(f, g) \cdot W(y_1, \ldots, y_k).$$

Thus,

$$a_k^n \cdot b_n^k \cdot W(z_1, \ldots, z_n, y_1, \ldots, y_k) = E(f, g) \cdot W(y_1, \ldots, y_k) \cdot W(z_1, \ldots, z_n).$$

In addition, it might be noted that

$$\begin{aligned} a_k^n \cdot b_n^k \cdot W(y_1, \ldots, y_k, z_1, \ldots, z_n) &= a_k^n \cdot b_n^k \cdot (-1)^{kn} W(z_1, \ldots, z_n, y_1, \ldots, y_k) \\ &= (-1)^{kn} E(f, g) \cdot W(y_1, \ldots, y_k) \cdot W(z_1, \ldots, z_n) \\ &= E(g, f) \cdot W(y_1, \ldots, y_k) \cdot W(z_1, \ldots, z_n). \end{aligned}$$

THEOREM 2. *Let* $fx \equiv \sum_{j=0}^{n} a_j D^j x$ $(a_j \in K_1 \subseteq K)$. *If*

(i) $f^\varkappa x_{\varkappa,p} = 0$ *for* $\varkappa = 1, 2, \ldots, k$; $p = 1, 2, \ldots, n$,

(ii) *linear combinations* $h_\varkappa x \equiv \sum_{j=0}^{n-1} d_{\varkappa,j} D^j x$ $(d_{\varkappa,j} \in K_1)$ $(\varkappa = 2, \ldots, k)$ *exist such that the* n *coefficients* $d_{\varkappa,0}, d_{\varkappa,1}, \ldots, d_{\varkappa,n-1}$ *satisfy the* n *linear algebraic equations* $h_\varkappa f^{\varkappa-2} x_{\varkappa-1,p} = f^{\varkappa-1} x_{\varkappa,p}$ $(p = 1, 2, \ldots, n)$ *for each combination* $h_\varkappa x$ $(\varkappa = 2, \ldots, k)$,

then the following formula holds:

$$(13) \qquad a_n^{(n-\frac{1}{2})k(k-1)} \cdot W(x_{1,1}, \ldots, x_{1,n}; \ldots; x_{k,1}, \ldots, x_{k,n}) = \prod_{\varkappa=2}^{k} E^{k-\varkappa+1}(f, h_\varkappa) \cdot W^k(x_{1,1}, \ldots, x_{1,n}).$$

Proof. The leading coefficient of the expression $f^\varkappa x$, denoted by $a_{\varkappa,\varkappa n}$, is equal to $a_n^\varkappa$. The formula

$$(14) \qquad \prod_{\varkappa=1}^{k-1} a_{\varkappa,\varkappa n}^n W(x_{1,1}, \ldots, x_{1,n}; \ldots; x_{k,1}, \ldots, x_{k,n}) = \prod_{\varkappa=1}^{k} W(f^{\varkappa-1} x_{\varkappa,1}, \ldots, f^{\varkappa-1} x_{\varkappa,n})$$

will be proved by mathematical induction. Equality (14) is assumed and the determinant

$$W(x_{1,1},\ldots,x_{1,n};\ldots;x_{k+1,1},\ldots,x_{k+1,n}) \tag{15}$$

will be considered. We begin with the last row as has been done in the proof of Theorem 1. By suitable multiplication and elementary operations, n^2k elements of the n last rows of determinant (15) are equated to zero:

$$\prod_{\varkappa=1}^{k} a^n_{\varkappa,\varkappa n}\cdot W(x_{1,1},\ldots,x_{1,n};\ldots;x_{k+1,1},\ldots,x_{k+1,n}) = \prod_{\varkappa=1}^{k-1} a^n_{\varkappa,\varkappa n}\cdot a^n_{k,kn}\cdot$$

$$\cdot\begin{vmatrix} x_{1,1}, \ldots, & x_{1,n}, \ldots, & x_{k+1,1}, \ldots, & x_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1}x_{1,1}, \ldots, & D^{n-1}x_{1,n}, \ldots, & D^{n-1}x_{k+1,1}, \ldots, & D^{n-1}x_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ D^{kn-n}x_{1,1}, \ldots, & D^{kn-n}x_{1,n}, \ldots, & D^{kn-n}x_{k+1,1}, \ldots, & D^{kn-n}x_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ D^{kn-1}x_{1,1}, \ldots, & D^{kn-1}x_{1,n}, \ldots, & D^{kn-1}x_{k+1,1}, \ldots, & D^{kn-1}x_{k+1,n} \\ D^{kn}x_{1,1}, \ldots, & D^{kn}x_{1,n}, \ldots, & D^{kn}x_{k+1,1}, \ldots, & D^{kn}x_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ D^{kn+n-1}x_{1,1}, \ldots, & D^{kn+n-1}x_{1,n}, \ldots, & D^{kn+n-1}x_{k+1,1}, \ldots, & D^{kn+n-1}x_{k+1,n} \end{vmatrix}$$

$$= \prod_{\varkappa=1}^{k-1} a^n_{\varkappa,\varkappa n}\cdot$$

$$\cdot\begin{vmatrix} x_{1,1}, \ldots, & x_{1,n}, \ldots, & x_{k+1,1}, \ldots, & x_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1}x_{1,1} \ldots, & D^{n-1}x_{1,n}, \ldots, & D^{n-1}x_{k+1,1}, \ldots, & D^{n-1}x_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ D^{kn-n}x_{1,1}, \ldots, & D^{kn-n}x_{1,n}, \ldots, & D^{kn-n}x_{k+1,1}, \ldots, & D^{kn-n}x_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ D^{kn-1}x_{1,1}, \ldots, & D^{kn-1}x_{1,n}, \ldots, & D^{kn-1}x_{k+1,1}, \ldots, & D^{kn-1}x_{k+1,n} \\ f^k x_{1,1}, \ldots, & f^k x_{1,n}, \ldots, & f^k x_{k+1,1}, \ldots, & f^k x_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1}f^k x_{1,1}, \ldots, & D^{n-1}f^k x_{1,n}, \ldots, & D^{n-1}f^k x_{k+1,1}, \ldots, & D^{n-1}f^k x_{k+1,n} \end{vmatrix}$$

$$= \prod_{\varkappa=1}^{k-1} a_{\varkappa,\varkappa n}^{n} \cdot$$

$$\cdot \begin{vmatrix} x_{1,1}, & \ldots, & x_{1,n}, & \ldots, & x_{k+1,1}, & \ldots, & x_{k+1,n} \\ \ldots & & & & & & \\ D^{n-1}x_{1,1}, & \ldots, & D^{n-1}x_{1,n}, & \ldots, & D^{n-1}x_{k+1,1}, & \ldots, & D^{n-1}x_{k+1,n} \\ \ldots & & & & & & \\ \ldots & & & & & & \\ D^{kn-n}x_{1,1}, & \ldots, & D^{kn-n}x_{1,n}, & \ldots, & D^{kn-n}x_{k+1,1}, & \ldots, & D^{kn-n}x_{k+1,n}, \\ \ldots & & & & & & \\ D^{kn-1}x_{1,1}, & \ldots, & D^{kn-1}x_{1,n}, & \ldots, & D^{kn-1}x_{k+1,1}, & \ldots, & D^{kn-1}x_{k+1,n} \\ 0, & \ldots & , & 0, & f^k x_{k+1,1}, & \ldots, & f^k x_{k+1,n} \\ \ldots & & & & & & \\ 0, & \ldots & , & 0, & D^{n-1}f^k x_{k+1,1}, & \ldots, & D^{n-1}f^k x_{k+1,n} \end{vmatrix}$$

$$= \prod_{\varkappa=1}^{k-1} a_{\varkappa,\varkappa n}^{n} \cdot W(x_{1,1}, \ldots, x_{1,n}; \ldots; x_{k,1}, \ldots, x_{k,n}) \cdot W(f^k x_{k+1,1}, \ldots, f^k x_{k+1,n})$$

$$= \prod_{\varkappa=1}^{k} W(f^{\varkappa-1}x_{\varkappa,1}, \ldots, f^{\varkappa-1}x_{\varkappa,n}) \cdot W(f^k x_{k+1,1}, \ldots, f^k x_{k+1,n})$$

$$= \prod_{\varkappa=1}^{k+1} W(f^{\varkappa-1}x_{\varkappa,1}, \ldots, f^{\varkappa-1}x_{\varkappa,n}).$$

Equality (14) needs checking. For $k = 2$, it is possible to transform the determinant $W(x_{1,1}, \ldots, x_{1,n}; x_{2,1}, \ldots, x_{2,n})$ according to the procedure which has been described earlier.

After identity (14) has been proved, Lemma 2 will be used to show that

$$a_n^{n-1} \cdot W(f^{\varkappa-1}x_{\varkappa,1}, \ldots, f^{\varkappa-1}x_{\varkappa,n}) = E(f, h_\varkappa) \cdot W(f^{\varkappa-2}x_{\varkappa-1,1}, \ldots, f^{\varkappa-2}x_{\varkappa-1,n})$$
$$\text{for } \varkappa = 2, 3, \ldots, k.$$

The elements $f^{\varkappa-2}x_{\varkappa-1,p}$ $(p = 1, \ldots, n)$ satisfy the equation $fx = 0$ on the strength of assumption (i). Putting $y_{\varkappa,p} = f^{\varkappa-1}x_{\varkappa,p}$ $(\varkappa = 1, 2, \ldots \ldots, k; p = 1, \ldots, n)$ and using assumption (ii), we can obtain the last identity by Lemma 2.

It may be proved by mathematical induction that

$$a_n^{(n-1)(\varkappa-1)} \cdot W(f^{\varkappa-1}x_{\varkappa,1}, \ldots, f^{\varkappa-1}x_{\varkappa,n}) = \prod_{\nu=2}^{\varkappa} E(f, h_\nu) \cdot W(x_{1,1}, \ldots, x_{1,n})$$

for $\varkappa = 2, \ldots, k$, and so

$$a_n^{\frac{(n-1)k(k-1)}{2}} \cdot \prod_{\varkappa=2}^{k} W(f^{\varkappa-1}x_{\varkappa,1}, \ldots, f^{\varkappa-1}x_{\varkappa,n})$$

$$= \prod_{\varkappa=2}^{k} E^{k-\varkappa+1}(f, h_\varkappa) \cdot W^{k-1}(x_{1,1}, \ldots, x_{1,n})$$

and result (13) will be established if the product $\prod_{\varkappa=1}^{k-1} a_{\varkappa,\varkappa n}^{n}$ is written as $a_n^{\frac{1}{2}nk(k-1)}$,

$$a_n^{\frac{1}{2}(n-1)k(k-1)} \cdot a_n^{\frac{1}{2}nk(k-1)} = a_n^{(n-\frac{1}{2})k(k-1)}.$$

9. A superposition of linear expressions will now be considered

$$f_j x \equiv a_{n_j,j} \cdot D^{n_j} x + \ldots + a_{0,j} x, \qquad F_j x \equiv f_j^{a_j} x.$$

Let the elements $x_{\varkappa,p,j}$ ($\varkappa = 1, \ldots, a_j$; $p = 1, \ldots, n_j$) be the solutions of the equation $F_j x = 0$ of the order $n_j a_j$. Substituting $j = 1, 2, \ldots, m$, we may consider the superposition $Fx \equiv F_1 F_2 \ldots F_m x$ and it is possible to apply first Theorem 2 m-times and then Theorem 1 $(m-1)$-times.

A little trouble arises with multipliers, if we use Theorems 1 and 2 so many times. The total factor denoted by M_2, which is required when Theorem 2 is applied m times, is given by the formula

$$M_2 = \prod_{j=1}^{m} a_{n_j,j}^{(n_j-\frac{1}{2})a_j(a_j-1)}.$$

The leading coefficient of $F_j x$ is $b_j = a_{n_j,j}^{a_j}$ and the total factor, denoted by M_1, which is needed when Theorem 1 is used $(m-1)$-times, is the following product:

$$M_1 = \prod_{j=1}^{m} b_j^{\sigma - n_j a_j}, \quad \text{where } \sigma = \sum_{j=1}^{m} n_j a_j.$$

It is easier to formulate the next theorem now, remembering how the eliminant E and the wronskian W have been defined earlier.

THEOREM 3. *Let* $f_j x \equiv \sum_{i=0}^{n_j} a_{i,j} D^i x$ $(a_{i,j} \in K_1 \subseteq K)$. *If*

(i) $f_j^{\varkappa} x_{\varkappa,p,j} = 0$ ($\varkappa = 1, \ldots, a_j$; $p = 1, \ldots, n_j$; $j = 1, \ldots, m$),

(ii) *linear combinations* $h_{\varkappa,j} x \equiv \sum_{r=0}^{n_j-1} d_{\varkappa,r,j} D^r x$ ($d_{\varkappa,r,j} \in K_1$, $\varkappa = 2, \ldots, a_j$; $j = 1, \ldots, m$) *exist such that the coefficients* $d_{\varkappa,0,j}, \ldots, d_{\varkappa,n_j-1,j}$ *satisfy the*

n_j linear algebraic equations $h_{\varkappa,j} f_j^{\varkappa-2} x_{\varkappa-1,p,j} = f_j^{\varkappa-1} x_{\varkappa,p,j}$ $(p = 1, \ldots, n_j)$ for any combination $h_{\varkappa,j}$ $(\varkappa = 2, \ldots, a_j)$,

(iii) *$f_1^{a_1} \ldots f_s^{a_s} x = 0$ for $x = x_{\varkappa,p,j}$ $(\varkappa = 1, \ldots, a_j;$ $p = 1, \ldots, n_j;$ $j = 1, \ldots, s)$ for each fixed $s = 1, \ldots, m-1$,*
then

$$M_1 M_2 \cdot W(X_1^1, \ldots, X_1^{a_1}, \ldots, X_m^1, \ldots, X_m^{a_m})$$
$$= \prod_{\mu=2}^{m} E(f_\mu^{a_\mu}, f_1^{a_1} \ldots f_{\mu-1}^{a_{\mu-1}}) \cdot \prod_{j=1}^{m} \Big\{ \prod_{\varkappa=2}^{a_j} E^{a_j-\varkappa+1}(f_j, h_{\varkappa,j}) \cdot W^{a_j}(X_j^1) \Big\},$$

where

$$X_j^\varkappa = (x_{\varkappa,1,j}, \ldots, x_{\varkappa,n_j,j}), \qquad M_1 = \prod_{j=1}^{m} a_{n_j,j}^{a_j(\sigma-n_j a_j)},$$

$$\sigma = n_1 a_1 + \ldots + n_m a_m, \qquad M_2 = \prod_{j=1}^{m} a_{n_j,j}^{(n_j-1)a_j(a_j-1)}.$$

Remark 1. Assumption (iii) is fulfilled identically when $K_1 = K_0$ or $K_1 = C_0$, i.e., for any ring or field of constant coefficients.

Remark 2. If the ring K_1 is a field, it is possible to divide the equation $f_j x = 0$ by $a_{n_j,j}$ and when this is done for $j = 1, \ldots, m$, the leading coefficients become units, and so the total multiplier $M_1 M_2$ is equal to 1.

10. The linear independence of the elements $x_{1,1,j}, \ldots, x_{1,n_j,j}$ is assumed for any $j = 1, \ldots, m$ separately and the problem is to prove the linear independence of the elements $x_{\varkappa,p,j}$ $(\varkappa = 1, \ldots, a_j;$ $p = 1, \ldots, n_j;$ $j = 1, \ldots, m)$ by Theorem 3. If the expressions $f_j x$ $(j = 1, \ldots, m)$ are of the first order, the product of wronskians is a product of powers of the solutions. All the solutions are different from zero, being linearly independent, and they are not divisors of zero on the strength of assumption (A). Thus in this particular case only the eliminants show whether the wronskian

$$W(X_1^1, \ldots, X_1^{a_1}; \ldots; X_m^1, \ldots, X_m^{a_m}) \tag{16}$$

is zero or not. Similarly in more general cases, where the expressions $f_j x$ are of order higher than one, the wronskians are not divisors of zero if condition (A) is assumed. A difficulty arises when it is intended to show that the wronskian $W(x_{1,1,j}, \ldots, x_{1,n_j,j})$ is not equal to zero provided that the linearly independent elements $x_{1,1,j}, \ldots, x_{1,n_j,j}$ satisfy the equation $f_j x = 0$. A proof is not possible without an additional condition or an additional information on the ring K. However, even if we are justified in drawing the conclusion that a wronskian of linearly independent elements is different from zero and cannot be a divisor of zero, still the eliminants (and only the eliminants) show whether wronskian (16) vanishes or not.

11. An example, given by J. Mikusiński [6], will be presented here in order to point out that a wronskian of linearly independent elements can be zero. Polynomials of 2 variables u and v

$$\sum_{i,j=0}^{n} c_{i,j} u^i v^j,$$

having coefficients $c_{i,j}$ from a number field, constitute the ring with ordinary addition and multiplication. The operation D is defined as follows:

$$D \sum_{i,j=0}^{n} c_{i,j} u^i v^j = \sum_{i,j=0}^{n} (i+j) c_{i,j} u^i v^j.$$

For example, the equation $Dx - x = 0$ has 2 linearly independent solutions $x_1 = u$, $x_2 = v$, "the derivatives" are $Du = u$, $Dv = v$. There is no uniqueness theorem for this ring of polynomials.

Equation $D^2 x - 2Dx + x = 0$ also has the elements $x_1 = u$, $x_2 = v$ as solutions and the wronskian

$$W(x_1, x_2) = \begin{vmatrix} x_1, & x_2 \\ Dx_1, & Dx_2 \end{vmatrix} = \begin{vmatrix} u, & v \\ u, & v \end{vmatrix} = 0$$

although the elements $x_1 = u$, $x_2 = v$ are linearly independent.

The same example may be used to show that an element t satisfying the equation $Dt = 1$ does not necessarily exist in a ring K. For the polynomials of 2 variables and the operation D defined above, conditions (3) and (4) are fulfilled. The polynomials of degree zero are constants but an element t such that $Dt = 1$ does not exist.

Wronskian (16) contains the elements $x_{\varkappa,p,j}$ with $\varkappa > 1$ and no relation between $x_{2,p,j}, \ldots, x_{a_j,p,j}$ and $x_{1,p,j}$ is established till now. This is the reason why the linear independence of the elements $x_{\varkappa,p,j}$ $(\varkappa = 1, 2, \ldots, a_j)$ is connected not only with wronskians of the elements $x_{1,p,j}$ $(\varkappa = 1)$ but first of all with eliminants. A relation between $X_j^{\varkappa}$ $(\varkappa = 2, \ldots, a_j)$ and $X_j^1 = (x_{1,1,j}, \ldots, x_{1,n_j,j})$ can be fixed in the case of constant coefficients when an element t such that $Dt = 1$ is used.

12. Now a field C_0 of constant elements is taken as the ring K_1 and it is assumed that in the ring K an element t exists such that $Dtx = t \cdot Dx + x$ for any $x \in K$. If $fx_0 = 0$, then the equation $f^k x = 0$ is satisfied by the elements $t^{\varkappa} x_0$ $(\varkappa = 0, 1, \ldots, k-1)$ (see [4], p. 229). I quote the proof:

Let $f^{(1)}x \overset{\mathrm{df}}{=} \sum_{r=1}^{n} ra_r D^{r-1}x$ denote the algebraic dervative of the expression $fx \equiv a_n D^n x + \ldots + a_1 Dx + a_0 x$. The equalities

$$(17) \qquad (f^k x)^{(1)} = kf^{(1)}f^{k-1}x, \quad f(tx) = tfx + f^{(1)}x$$

are true for constant coefficients. Therefore

$$f^{k+1}(t^k x) = tf^{k+1}(t^{k-1}x) + (k+1)f^{(1)}f^k(t^{k-1}x)$$

and the equalities $f^k(t^\varkappa x_0) = 0$ for $\varkappa = 0, 1, \ldots, k-1$ are obtained by mathematical induction.

By substituting $x_{\varkappa,p} = t^{\varkappa-1}x_{1,p}$ in Theorem 2, we can easily evaluate $h_\varkappa x = (\varkappa-1)f^{(1)}x$ by (17), remembering the conditions $f^\nu x_{\varkappa,p} = 0$ for $\nu \geqslant \varkappa$. In fact,

$$\begin{aligned} f^{\varkappa-1}x_{\varkappa,p} &= f^{\varkappa-1}(tx_{\varkappa-1,p}) = tf^{\varkappa-1}x_{\varkappa-1,p} + (f^{\varkappa-1})^{(1)}x_{\varkappa-1,p} \\ &= (f^{\varkappa-1})^{(1)}x_{\varkappa-1,p} = (\varkappa-1)f^{(1)}f^{\varkappa-2}x_{\varkappa-1,p}. \end{aligned}$$

Hence

$$h_\varkappa x = (\varkappa-1)f^{(1)}x, \quad E(f, h_\varkappa) = (\varkappa-1)^n \cdot E(f, f^{(1)}),$$

$$\begin{aligned} \prod_{\varkappa=2}^{k} E^{k-\varkappa+1}(f, h_\varkappa) &= \prod_{\varkappa=2}^{k} (\varkappa-1)^{n(k-\varkappa+1)} E^{k-\varkappa+1}(f, f^{(1)}) \\ &= [1^{k-1}\cdot 2^{k-2}\cdot \ldots \cdot (k-1)]^n \cdot [E(f, f^{(1)})]^{1+2+\ldots+(k-1)} \\ &= [1!2!\cdot \ldots \cdot (k-1)!]^n \cdot [E(f, f^{(1)})]^{\frac{k(k-1)}{2}}. \end{aligned}$$

The eliminants $E(f, g)$ are the resultants of the polynomials $f(w)$ and $g(w)$. Let the leading coefficients be unit coefficients, i.e., monic polynomials are taken as $f_j(w)$ $(j = 1, \ldots, m)$. By using the identity $E(f, gh) = E(f, g)\cdot(E(f, h)$ many times the following theorem is obtained from Theorem 3:

THEOREM 3^a. *If*

(i) $f_j x \equiv \sum_{r=0}^{n_j} a_{rj} D^r x = 0$ *for* $x = x_{p,j}$ $(p = 1, \ldots, n_j;\ j = 1, \ldots, m)$, $a_{r,j} \in C_0 \subseteq K$, $a_{n_j,j} = 1$ *for* $j = 1, \ldots, m$,

(ii) *an element* $t \in K$ *exists such that* $Dtx = tDx + x$ *for any* $x \in K$, *then*

$$W(X_1^0, \ldots, X_1^{a_1-1}; \ldots; X_m^0, \ldots, X_m^{a_m-1})$$

$$= \prod_{1 \leqslant \nu < \mu \leqslant m} E^{a_\mu a_\nu}(f_\mu, f_\nu) \cdot \prod_{j=1}^{m} \{[1!2!\ldots(a_j-1)!]^{n_j} \cdot [E(f_j, f_j^{(1)})]^{\frac{a_j(a_j-1)}{2}} \cdot W^{a_j}(X_j^0)\},$$

where

$$X_j^\varkappa = (t^\varkappa x_{1,j}, \ldots, t^\varkappa x_{n_j,j}), \quad \varkappa = 0, \ldots, a_j - 1;\ j = 1, \ldots, m.$$

13. The superposition $Fx = F_1F_2 \ldots F_m x$ corresponds to a factorization of the polynomial $F(w)$, constant coefficients of which are taken from a field C_0. The field of operators is not closed algebraically [9]. However, if the operational equation (2) has been obtained from the partial differential equation (1), the polynomial $f(w) = a_m w^m + \ldots + a_1 w + a_0$ is a product of linear factors $(w - w_j)$ only ([4], p. 242–244), and if the equation $fx \equiv a_m D^m x + \ldots + a_0 x = 0$ has m linearly independent solutions, then each of the equations $Dx = w_j x$ $(j = 1, \ldots, m)$ has a solution $x_j \neq 0$. This follows from the theorem on the uniqueness of solutions (see [1]).

In the particular case where the polynomial $F(w)$ of degree n contains only linear factors, i.e.,

$$F(w) = c \cdot \prod_{j=1}^{m} (w - w_j)^{a_j},$$

the resultant of the polynomials $f_\mu(w) = w - w_\mu$ and $f_\nu(w) = w - w_\nu$ is equal to $(w_\mu - w_\nu)$. If the product $x_1^{a_1} \ldots x_m^{a_m}$ of the solutions of the equations $Dx = w_j x$ $(j = 1, \ldots, m)$ is different from zero, then the following result [5] is obtained from Theorem 3^a:

$$\begin{vmatrix} (\omega_1^0)^{(0)}, & \ldots, (\omega_1^0)^{(a_1-1)}, & \ldots, (\omega_m^0)^{(0)}, & \ldots, (\omega_m^0)^{(a_m-1)} \\ (\omega_1^1)^{(0)}, & \ldots, (\omega_1^1)^{(a_1-1)}, & \ldots, (\omega_m^1)^{(0)}, & \ldots, (\omega_m^1)^{(a_m-1)} \\ \cdots & \cdots & \cdots & \cdots \\ (\omega_1^{n-1})^{(0)}, & \ldots, (\omega_1^{n-1})^{(a_1-1)}, & \ldots, (\omega_m^{n-1})^{(0)}, & \ldots, (\omega_m^{n-1})^{(a_m-1)} \end{vmatrix}$$

$$= \prod_{\mu=1}^{m} [1!2! \ldots (a_\mu - 1)!] \cdot \prod_{1 \leqslant \varkappa < \mu \leqslant m} (\omega_\mu - \omega_\varkappa)^{a_\varkappa a_\mu},$$

where

$$(\omega^\sigma)^{(0)} = \omega^\sigma \ (\sigma = 0, 1, \ldots),\ (\omega^0)^{(\nu)} = 0 \ (\nu = 1, 2, \ldots),$$
$$(\omega^\sigma)^{(\nu)} = \sigma(\sigma - 1) \ldots (\sigma - \nu + 1)\,\omega^{\sigma - \nu} \ (\sigma, \nu = 1, 2, \ldots)$$

and the natural numbers $a_1, a_2, \ldots, a_m$ satisfy the condition $a_1 + a_2 + \ldots$ $\ldots + a_m = n$.

The determinant on the left-hand side is a simplified wronskian of the solutions of the equation $Fx = 0$ after multiplying by the product $x_1^{a_1} \ldots x_m^{a_m}$ (see the beginning of the proof of Theorem 2).

14. A polynomial $F(w)$, with the coefficients from the field C_0 can be factorized,

$$F(w) = \prod_{j=1}^{m} f_j^{a_j}(w).$$

Let the polynomials $f_j(w)$ $(j = 1, \ldots, m)$ be irreducible over this field C_0. By Theorem 3^a the wronskian of the elements $t^\varkappa x_{p,j}$ $(\varkappa = 0, 1, \ldots \ldots, a_j-1;\ p = 1, \ldots, n_j;\ j = 1, \ldots, m)$ is the product of the resultants and discriminants of the polynomials $f_j(w)$ and of the powers of the wronskians $W(x_{1,j}, \ldots, x_{n_j,j})$. The resultants of the polynomials which are relatively prime (with respect to one another) are different from zero. In spite of the irreducibility of the polynomials in the field C_0, it may happen, however, that some discriminants of the polynomials $f_j(w)$ may vanish the case of a field having a positive characteristic. This is because the algebraic derivative $f_j^{(1)}(w)$ can be identically zero (all coefficients are zeros) in the case of positive characteristic.

15. If we want to infer the linear independence of the elements $t^\varkappa x_{p,j}$ $(\varkappa = 0, 1, \ldots, a_j-1;\ p = 1, \ldots, n_j;\ j = 1, \ldots, m)$ from the linear independence of the elements $x_{1,j}, \ldots, x_{n_j,j}$ for any $j = 1, \ldots, m$ by Theorem 3^a, we must prove that the wronskians $W(x_{1,j}, \ldots, x_{n_j,j})$ are different from zero and they are not divisors of zero. The trouble may be avoided if we do not use wronskians at all. The next theorem needs only the idea of linear independence and some algebraic theorems.

THEOREM 4. *A linear space K over a field C_0 of constants is considered. If*

(i) *operations t and D satisfy the conditions*

$$t(ax+by) = atx+bty, \qquad D(ax+by) = a\cdot Dx+b\cdot Dy,$$
$$Dtx = tDx+x \quad \textit{for any } a, b \in C_0,\ x, y \in K,$$

(ii) *the polynomials $f_j(w)$ $(j = 1, \ldots, m)$, over the field C_0, are relatively prime (with respect to one another) and no $f_j(w)$ has a common factor with its derivative,*

(iii) $f_j x \equiv \sum_{r=0}^{n_j} a_{r,j} D^r x = 0$ $(a_{r,j} \in C_0,\ x \in K)$ *for* $x = x_{p,j}$ $(p = 1, \ldots, p_j;$ $j = 1, \ldots, m)$,

(iv) *the elements $x_{1,j}, \ldots, x_{p_j,j}$ are linearly independent over the field C_0 for any $j = 1, 2, \ldots, m$,*

then the elements $t^\varkappa x_{p,j}$ $(\varkappa = 0, 1, \ldots, a_j-1;\ p = 1, \ldots, p_j;\ j = 1, \ldots, m)$ are linearly independent over C_0 for arbitrary natural numbers a_j $(j = 1, \ldots \ldots, m)$.

Proof. If the equations $fx \equiv a_k D^k x + \ldots + a_0 x = 0$ and $gx \equiv b_n D^n x + \ldots + b_0 x = 0$ have a common solution x_0, then $x_0 \cdot E(f, g) \equiv 0$, where $E(f, g)$ denotes the resultant of the corresponding polynomials f, g. This follows from the system of equations (11).

Suppose that the elements $t^\varkappa x_{p,j}$ ($\varkappa = 0, 1, \ldots, a_j - 1$; $p = 1, \ldots, p_j$; $j = 1, \ldots, m$) are linearly dependent, i.e.,

$$\sum_{j=1}^{m} \sum_{p=1}^{p_j} \sum_{\varkappa=0}^{a_j-1} c_{\varkappa,p,j} \cdot t^\varkappa x_{p,j} = 0$$

and not all $c_{\varkappa,p,j}$ are zeros. Let $c_{\varkappa_0,p_0,1} \neq 0$. The equations $f_1^{a_1} x = 0$ and $f_2^{a_2} \ldots f_m^{a_m} x = 0$ have the common solution

$$x_0 = \sum_{p=1}^{p_1} \sum_{\varkappa=0}^{a_1-1} c_{\varkappa,p,1} \cdot t^\varkappa x_{p,1} = -\sum_{j=2}^{m} \sum_{p=1}^{p_j} \sum_{\varkappa=0}^{a_j-1} c_{\varkappa,p,j} \cdot t^\varkappa x_{p,j}$$

and $E(f_1^{a_1}, f_2^{a_2} \ldots f_m^{a_m}) \cdot x_0 = 0$. The resultant $E(f_1^{a_1}, f_2^{a_2} \ldots f_m^{a_m})$ does not equal to zero because the polynomials $f_1^{a_1}(w)$ and $f_2^{a_2}(w) \ldots f_m^{a_m}(w)$ are relatively prime, and so the equality

$$\sum_{p=1}^{p_1} \sum_{\varkappa=0}^{a_1-1} c_{\varkappa,p,1} \cdot t^\varkappa x_{p,1} = 0$$

is obtained. We write $x_{p,1} = x_p$, $p_1 = n$, $a_1 = k$, $c_{\varkappa,p,1} = c_{\varkappa,p}$, $f_1(w) = f(w)$ in order to simplify notation. Now

$$x_0 = \sum_{\varkappa=0}^{k-1} \sum_{p=1}^{n} c_{\varkappa,p} \cdot t^\varkappa x_p = 0, \qquad c_{\varkappa_0,p_0} \neq 0.$$

If $c_{k-1,p} \neq 0$ for a certain subscript p ($p = 1, \ldots, n$), we put $k_0 = k$, and if $a_{k-1,p} = 0$ for $p = 1, \ldots, n$, we choose the number k_0 in such a way that $c_{\varkappa,p} = 0$ for $\varkappa \geqslant k_0$, $p = 1, \ldots, n$ but $c_{k_0-1,p} \neq 0$ for a certain subscript p.

The element

$$y_0 = c_{k_0-1,1} x_1 + \ldots + c_{k_0-1,n} x_n$$

is different from zero. This follows from the linear independence of the elements $x_1, \ldots, x_n$.

The equation $f^{k_0-1} x = 0$, corresponding to the polynomial $g(w) = f^{k_0-1}(w) = b_s w^s + \ldots + b_1 w + b_0$, is satisfied not only by the elements

$t^{\varkappa}y_0$ ($\varkappa = 0, 1, \ldots, k_0-2$) but also by the element $t^{k_0-1}y_0$ because

$$t^{k_0-1}y_0 = -\sum_{\varkappa=0}^{k_0-2}\sum_{p=1}^{n} c_{\varkappa,p}\cdot t^{\varkappa}x_p.$$

Let $g^{(\nu)}(w)$ denote the ν-th derivative of the polynomial $g(w)$, $t^0x = x$, $\binom{p}{\nu}\cdot t^{p-\nu}x = 0$ for $\nu > p$. Using the condition $Dtx = tDx+x$, we prove the following identity

$$\sum_{j=0}^{s} b_j D^j(t^p y_0) = \sum_{j=0}^{s} b_j \sum_{\nu=0}^{j}\binom{j}{\nu}\cdot\binom{p}{\nu}\cdot\nu!\cdot t^{p-\nu}D^{j-\nu}y_0$$

$$= \sum_{j=0}^{s} b_j \sum_{\nu=0}^{j}\frac{j!}{(j-\nu)!}\cdot\binom{p}{\nu}\cdot t^{p-\nu}D^{j-\nu}y_0$$

$$= \sum_{\nu=0}^{s}\binom{p}{\nu}\cdot t^{p-\nu}\sum_{j=\nu}^{s}\frac{j!}{(j-\nu)!}b_jD^{j-\nu}y_0 = \sum_{\nu=0}^{s}\binom{p}{\nu}t^{p-\nu}g^{(\nu)}y_0.$$

Putting successively $p = 1, 2, \ldots, k_0-1$, we see that the element y_0 satisfies not only the equation $fx = 0$ but also the equation $g^{(k_0-1)}x = 0$, corresponding to the polynomial

$$g^{(k_0-1)}(w) = [f^{(1)}(w)]^{k_0-1}+Q(w)\cdot f(w),$$

where $Q(w)$ is a polynomial. J. Mikusiński proved [8] the following theorem: "*If the equations $fx = 0$ and $gx = 0$ have a common solution $x_0 \neq 0$, then the polynomials $f(w)$ and $g(w)$ have a common divisor of positive degree*". According to this theorem, the polynomials $f(w)$ and $[f^{(1)}(w)]^{k_0-1}$ must have a common factor of positive degree, which contradicts assumption (ii).

A theorem, very similar to Theorem 4, was proved by J. Mikusiński [8] by another method, assuming $p_j = n_j$, $x_{p,j} = D^{p-1}x_j$. The number of solutions of the equation $fx = 0$ is not necessarily equal to the order of the expression fx in the proof of Theorem 4.

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