COLLOQUIUM MATHEMATICUM

VOL. XXI 1970 FASC. 1

ON UNIFORM DINI DERIVATIVES

BY

N. C. MANNA (BURDWAN, INDIA)

It is well known that the discontinuities of a derivative f' are precisely the points of non-uniform differentiability of f(cf. [2] and [4]) and that the set of points of non-uniform differentiability of f is an F_σ and of the first category (cf. [2] and [1]). In the present paper, the above concepts have been studied in a more general way with the help of the notion of uniform Dini derivatives.

Let a function f be defined on an open interval I of which $[a, b]$ is a closed subinterval. Let us further suppose that $D^+f(x)$ is finite for each $x \in [a, b]$. Let

$$\varphi(x, h) = \frac{f(x+h)-f(x)}{h} - D^+f(x).$$

Now for each point $x \in [a, b]$ and for each $\varepsilon > 0$, there is $\delta(x) > 0$ such that $\varphi(x, h) < \varepsilon$ whenever $0 < h < \delta(x)$ and $x+h \in I$. For fixed $\varepsilon > 0$, the function $\delta(x)$ may not have a positive lower bound in $[a, b]$. If, however, $\delta(x)$ has a positive lower bound for every $\varepsilon > 0$, then D^+f is said to be the *uniform right hand upper Dini derivative* of $f(x)$.

A point $\xi \in [a, b]$ is said to be a *point of uniform right-hand upper Dini differentiability* if for each $\varepsilon > 0$ there is a neighbourhood of ξ in which $\delta(x)$ has a positive lower bound. So, a point in every neighbourhood of which $\delta(x)$ has no positive lower bound for some sufficiently small $\varepsilon > 0$ is said to be a point of *non-uniform* right-hand upper Dini differentiability. It is clear that D^+f is uniform or non-uniform at $\xi \in [a, b]$ accordingly as $\overline{\lim}_{(x,h)\to(\xi,0+)} \varphi(x, h) = 0$ or > 0. Taking $w^+(\xi) = \overline{\lim}_{(x,h)\to(\xi,0+)} \varphi(x, h)$ we shall term $w^+(\xi)$ the *measure* of the non-uniformity of D^+f at ξ and the function $w^+(x)$ defining the measure of non-uniformity of D^+f at x is termed the *measure function* of non-uniformity of D^+f.

In a similar manner we introduce the functions $w_+(x)$, $w^-(x)$ and $w_-(x)$ by considering $D_+f(x)$, $D^-f(x)$ and $D_-f(x)$ respectively, in the following way:

$$w_{+}(\xi) = \underline{\operatorname{Lim}}_{(x,h)\to(\xi,0+)} \left\{\frac{f(x+h)-f(x)}{h} - D_{+}f(x)\right\},$$

$$w^{-}(\xi) = \overline{\operatorname{Lim}}_{(x,h)\to(\xi,0-)} \left\{\frac{f(x+h)-f(x)}{h} - D^{-}f(x)\right\},$$

$$w_{-}(\xi) = \underline{\operatorname{Lim}}_{(x,h)\to(\xi,0-)} \left\{\frac{f(x+h)-f(x)}{h} - D_{-}f(x)\right\}.$$

It is clear that if f is differentiable, then ξ is a point of uniform differentiability of f [2] iff all the four Dini derivatives are uniform at ξ. If the right-hand derivative f'_{+} exists, i.e., if $D^{+}f = D_{+}f$, then ξ is a point of uniformity of f'_{+} iff ξ is a point of uniformity of both $D^{+}f$ and $D_{+}f$. Similar is the case for the left-hand derivative f'_{-}, when it exists.

THEOREM 1. *The function $w^{+}(x)$ is upper semicontinuous on $[a, b]$.*

Proof. Let $\xi \in [a, b]$ and let $\varepsilon > 0$ be arbitrary. From the definition of $w^{+}(\xi)$ there is a neighbourhood D of ξ and a $\delta > 0$ such that $\varphi(x, h) < w^{+}(\xi)+\varepsilon/2$ for all $x \in D$ and $0 < h < \delta$. If possible, let there be points x' in every neighbourhood of ξ such that $w^{+}(x') \geqslant w^{+}(\xi)+\varepsilon$. So there is an $x' \in D$ such that $w^{+}(x') > w^{+}(\xi)+\varepsilon/2$ and hence there are points $x \in D$ and h, $0 < h < \delta$, such that $\varphi(x, h) > w^{+}(\xi)+\varepsilon/2$, which is a contradiction. This completes the proof.

COROLLARY 1. *If $w^{+}(x)$ is unbounded from above on $[a, b]$, then there is at least one point ξ, where $w^{+}(\xi) = \infty$.*

For, if $w^{+}(x)$ is unbounded from above on $[a, b]$, there is a point ξ in every neighbourhood of which $w^{+}(x)$ is unbounded and since $w^{+}(x)$ is upper semicontinuous, $w^{+}(\xi) = \infty$.

COROLLARY 2. *The set of points where $w^{+}(x) = \infty$ is closed.*

For, since $w^{+}(x)$ is upper semicontinuous, the set $\{x : x \in [a, b];\ w^{+}(x) \geqslant n\}$ is closed for each positive integer n and hence the set

$$\{x : x \in [a, b];\ w^{+}(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : x \in [a, b];\ w^{+}(x) \geqslant n\}$$

is closed.

THEOREM 2. *If f is continuous at $\xi \in [a, b]$ and if $w^{+}(\xi) = 0$ then $D^{+}f$ is lower semicontinuous at ξ.*

Proof. Since $w^{+}(\xi) = 0$, corresponding to $\varepsilon > 0$ there is a neighbourhood D_1 of ξ and a $\delta > 0$ such that $\varphi(x, h) < \varepsilon/3$ for all $x \in D_1$ and for all h, $0 < h < \delta$. Since $\overline{\operatorname{Lim}}_{h\to 0+}\varphi(\xi, h) = 0$, there is an h_1, $0 < h_1 < \delta$, such that $|\varphi(\xi, h_1)| < \varepsilon/3$. Let

$$\psi(x) = \frac{f(x+h_1)-f(x)}{h_1}.$$

Since $\psi(x)$ is continuous at ξ, there is a neighbourhood D_2 of ξ such that $|\psi(x)-\psi(\xi)| < \varepsilon/3$ for all $x \in D_2$. Let $D = D_1 \cap D_2$. Then D is a neighbourhood of ξ. For $x \in D$ we have

$$\begin{aligned} D^+f(x)-D^+f(\xi) &= D^+f(x)-\frac{f(x+h_1)-f(x)}{h_1}+\frac{f(x+h_1)-f(x)}{h_1}+ \\ &\quad +\frac{f(\xi+h_1)-f(\xi)}{h_1}-D^+f(\xi)-\frac{f(\xi+h_1)-f(\xi)}{h_1} \\ &= -\varphi(x, h_1)+\varphi(\xi, h_1)+\psi(x)-\psi(\xi) \\ &> -\varepsilon/3-\varepsilon/3-\varepsilon/3 = -\varepsilon. \end{aligned}$$

Hence $D^+f(x)$ is lower semicontinuous at ξ.

COROLLARY. *If f is continuous at $\xi \in [a, b]$, f'_+ exists in $[a, b]$ and if $w^+(\xi) = w_+(\xi) = 0$, then f'_+ is continuous at ξ, and hence $f'(\xi)$ exists.*

THEOREM 3. *If f is continuous on I, then the set*

$$\{x : x \in [a, b];\ w^+(x) > \sup_{x \in [a,b]} [D^+f(x)-D_+f(x)]\}$$

is of the first category and it is an F_σ.

Proof. Let $s = \sup_{x \in [a,b]} [D^+f(x)-D_+f(x)]$ and let $\sigma > s$. Then since $w^+(x)$ is upper semicontinuous, the set $\{x : x \in [a, b];\ w^+(x) \geqslant \sigma\}$ is closed. If possible, let this set be dense in some subinterval $[a', b'] \subset [a, b]$ and so $[a', b'] \subset \{x : x \in [a, b];\ w^+(x) \geqslant \sigma\}$. Let $\xi \in [a', b']$. Then $w^+(\xi) \geqslant \sigma$. Choose $\sigma > \sigma' > s$ and a positive null sequence $\{\delta_n\}$. Then since $w^+(\xi) > \sigma'$, there is a point ξ' in some neighbourhood of ξ contained in $[a', b']$ and a number h_1, $0 < h_1 < \delta_1$, such that

$$\frac{f(\xi'+h_1)-f(\xi')}{h_1}-D^+f(\xi') > \sigma'.$$

Therefore we can find an h_2, $0 < h_2 < h_1$, such that

$$\frac{f(\xi'+h_1)-f(\xi')}{h_1}-\frac{f(\xi'+h_2)-f(\xi')}{h_2} > \sigma'.$$

Since

$$\frac{f(x+h_1)-f(x)}{h_1} \quad \text{and} \quad \frac{f(x+h_2)-f(x)}{h_2}$$

are continuous at ξ', there exists a neighbourhood D_1 of ξ' contained in $[a', b']$ such that for every point $x \in D_1$ we have

$$\frac{f(x+h_1)-f(x)}{h_1} - \frac{f(x+h_2)-f(x)}{h_2} > \sigma'.$$

Now $w^+(\xi') \geqslant \sigma$ and hence there is a point $\xi'' \in D_1$ and an h_3, $0 < h_3 < \min\,[\delta_2, h_2]$, such that

$$\frac{f(\xi''+h_3)-f(\xi'')}{h_3} - D^+f(\xi'') > \sigma'.$$

Proceeding as above, we can find a neighbourhood D_2 of ξ'' contained in D_1 such that for every $x \in D_2$ we have

$$\frac{f(x+h_3)-f(x)}{h_3} - \frac{f(x+h_4)-f(x)}{h_4} > \sigma',$$

where $0 < h_4 < h_3$.

Continuing this process we get a decreasing sequence of neighbourhoods $\{D_n\}$ such that for every $x \in D_n$ we have

$$\frac{f(x+h_{2n-1})-f(x)}{h_{2n-1}} - \frac{f(x+h_{2n})-f(x)}{h_{2n}} > \sigma',$$

where $0 < h_{2n} < h_{2n-1} < \min\,[\delta_n, h_{2n-2}]$. We can choose the neighbourhoods D_n in such a way that there is at least one point x_0 which belongs to each D_n and so at this point

$$\frac{f(x_0+h_{2n-1})-f(x_0)}{h_{2n-1}} - \frac{f(x_0+h_{2n})-f(x_0)}{h_{2n}} > \sigma'$$

for all n. So we conclude that

$$D^+f(x_0) - D_+f(x_0) \geqslant \sigma'.$$

But since $x_0 \in [a, b]$, we have $D^+f(x_0) - D_+f(x_0) \leqslant s < \sigma'$. This is a contradiction. Hence we conclude that the set $\{x : x \in [a, b];\ w^+(x) \geqslant \sigma\}$ is closed and nowhere dense. Considering a sequence $\{\sigma_n\}$, $\sigma_n > s$ and $\sigma_n \to s$, we infer that the set

$$\{x : x \in [a, b];\ w^+(x) > s\} = \bigcup_{n=1}^{\infty} \{x : x \in [a, b];\ w^+(x) \geqslant \sigma_n\}$$

is an F_σ-set of the first category.

COROLLARY. *For a continuous function f, if f'_+ exists, then the set where f'_+ is not continuous is a set of the first category and hence f' exists except a set of the first category.*

THEOREM 4. *If f is continuous on I, then a necessary and sufficient condition that D^+f be continuous at a point $\xi \in [a, b]$ is that*

$$\operatorname*{Lim}_{(x,h)\to(\xi,0)} \varphi(x, h) = 0.$$

Proof. To prove the necessity we shall show that for each $x \in [a, b]$ and each $h \neq 0$, $x+h \in I$, one of the following inequalities must be true:

$$\text{(i)} \qquad D^+f(x+\vartheta h) \leqslant \frac{f(x+h)-f(x)}{h} \leqslant D_-f(x+\vartheta h),$$

$$\text{(ii)} \qquad D_+f(x+\vartheta h) \geqslant \frac{f(x+h)-f(x)}{h} \geqslant D^-f(x+\vartheta h),$$

where $0 < \vartheta < 1$.

Let $c \in [a, b]$ and $c+h \in I$. Let

$$\psi(x) = f(x) - \frac{f(c+h)-f(c)}{h} x.$$

Then $\psi(x)$ is continuous in $[c, c+h]$. Also $\psi(c+h) = \psi(c)$. Let M and m be the upper and the lower bounds of ψ on $[c, c+h]$. If $M = m$, then ψ is constant on $[c, c+h]$ and hence the above conclusion remains valid. So, let us suppose that at least one of M and m is different from $\psi(c)$. If $M \neq \psi(c)$, then there is a ϑ, $0 < \vartheta < 1$, such that $\psi(c+\vartheta h) = M$, and hence

$$D^+\psi(c+\vartheta h) \leqslant 0 \leqslant D_-\psi(c+\vartheta h),$$

i.e.

$$D^+f(c+\vartheta h) \leqslant \frac{f(c+h)-f(c)}{h} \leqslant D_-f(c+\vartheta h).$$

Similarly, if $m \neq \psi(c)$, then for some ϑ, $0 < \vartheta < 1$, we have

$$D_+f(c+\vartheta h) \geqslant \frac{f(c+h)-f(c)}{h} \geqslant D^-f(c+\vartheta h).$$

Thus for each $x \in [a, b]$ and h with $x+h \in I$, at least one of the following must be true:

$$\text{(i)} \qquad D^+f(x+\vartheta h) - D^+f(x) \leqslant \frac{f(x+h)-f(x)}{h} - D^+f(x) \leqslant D_-f(x+\vartheta h) - D^+f(x),$$

$$\text{(ii)} \qquad D_+f(x+\vartheta h) - D^+f(x) \geqslant \frac{f(x+h)-f(x)}{h} - D^+f(x) \geqslant D^-f(x+\vartheta h) - D^+f(x).$$

Since $D^+f(x)$ is continuous at ξ, $D^-f(x)$, $D_+f(x)$ and $D_-f(x)$ are also continuous at ξ and

$$D^+f(\xi) = D^-f(\xi) = D_+f(\xi) = D_-f(\xi)$$

(cf. [3], p. 204).

Hence, letting $x \to \xi$, $h \to 0$, we get

$$\mathrm{Lim}\left\{\frac{f(x+h)-f(x)}{h} - D^+f(x)\right\} = 0.$$

The proof of the sufficiency follows from the proof of Theorem 2 if we note that for any $\varepsilon > 0$ there is a neighbourhood D_1 of ξ and a $\delta > 0$ such that $|\varphi(x, h)| < \varepsilon/3$ for $x \in D_1$ and $0 < |h| < \delta$, and hence that

$$|D^+f(x) - D^+f(\xi)| \leqslant |\varphi(x, h_1)| + |\varphi(\xi, h_1)| + |\psi(x) - \psi(\xi)| < \varepsilon$$

whenever $x \in D_1 \cap D_2$.

I am thankful to Dr. S. N. Mukhopadhyay for his kind help in the preparation of the paper.

REFERENCES

[1] A. M. Bruckner and J. L. Leonard, *Derivatives*, American Mathematical Monthly 73 (1966), p. 24–56.
[2] B. K. Lahiri, *On non-uniform differentiability*, ibidem 67 (1960), p. 649–652.
[3] S. Saks, *Theory of the integral*, Warszawa — Lwów 1937.
[4] R. Weinstock, *On continuous differentiability*, American Mathematical Monthly 64 (1957), p. 492–495.

Reçu par la Rédaction le 24. 5. 1968