ON LOCALLY $m$-CONVEX ALGEBRAS OF TYPE ES

BY

NGUYEN VAN KHUE (WARSZAWA)

All algebras in this note are assumed to be commutative with unit element $e$. A locally $m$-convex algebra is a topological algebra whose topology can be determined by a family of submultiplicative seminorms. A locally $m$-convex algebra $F$ is called an ES-algebra (shortly $F \in ES$) if for every subalgebra $E$ of $F$ every continuous multiplicative linear functional on $E$ can be extended to such a functional on $F$.

A characterization of $B_0$-algebras of type ES has been established by Želazko [5]. The aim of this note is to study characterizations of locally $m$-convex algebras of type ES.

Let $\mathfrak{Q}$ be a class of locally $m$-convex algebras and let $Q \in \mathfrak{Q}$. The algebra $Q$ is said to have the extension property with respect to $\mathfrak{Q}$ (shortly $Q \in EP(\mathfrak{Q})$) if for every closed subalgebra $E$ of $F$, $E \in \mathfrak{Q}$, every continuous homomorphism from $E$ into $Q$ can be extended to a continuous homomorphism from $F$ into $Q$.

In Section 1, using a method of Želazko [5], we give characterizations of ES-algebras. In Section 2 we prove that $C[z_1, \ldots, z_n] \in ES$ if and only if $n = 1$, where $C[z_1, \ldots, z_n]$ denotes the locally $m$-convex algebra of complex polynomials in $n$ variables equipped with the compact-open topology. Section 3 is devoted to prove that a semi-simple Fréchet–Montel ES-algebra $Q$ has the extension property with respect to the class of all metric ES-algebras if and only if $Q$ is isomorphic to $C^m$ for some $m \leq \infty$.

1. Characterizations of ES-algebras. For each locally $m$-convex algebra $F$ we denote by $S(F)$ the set of all continuous submultiplicative seminorms $p$ on $F$ satisfying $p(e) = 1$. Let $p \in S(F)$. Put $F(p) = F/p^{-1}(0)$. Then $F(p)$ is a normed algebra with the norm $p$. If $\pi(p)$ denotes the canonical projection of $F$ onto $F(p)$, then the spectrum $\sigma(x)$ of an element $x \in F$ is given by

$$\sigma(x) = \bigcup \{\sigma_p(x) : p \in S(F)\},$$

where $\sigma_p(x)$ denotes the spectrum of $\pi(p)x$ in the completion $(F(p))\hat{\cdot}$ of $F(p)$.
A locally \( m\)-convex algebra \( F \) is called a \( Q\)-algebra if the set of all invertible elements \( G(F) \) of it is open.

**Theorem 1.1.** Let \( F \) be a locally \( m\)-convex algebra satisfying one of the following two conditions:

(Q) \([F, \tau] \) is a \( Q\)-algebra under a topology stronger than the original one,

(C) \([F, \tau] \) is sequentially complete under a topology stronger than the original one.

Then the following conditions are equivalent:

(i) \( F \in ES \),

(ii) \( F(p) \in ES \) for every \( p \in S(F) \),

(iii) for every \( x \in F \) and for every \( p \in S(F) \) the spectrum \( \sigma_p(x) \) is totally disconnected.

**Proof.** (a) Assume first that \( F \) satisfies (Q).

(i) \( \Rightarrow \) (iii): For a contradiction, there exist an \( x \in F \) and a \( p \in S(F) \) such that \( \sigma_p(x) \) contains a continuum \( K \). Let \( \alpha, \beta \in K \), \( \alpha \neq \beta \) and let us write \( y = (x - \alpha \epsilon)/(\beta - \alpha) \). Then \( \sigma_p(y) \) contains a continuum \( \bar{K} \) with \( 0, 1 \in \bar{K} \).

\[
P_n(\lambda) = 1 + (i\lambda)/1! + \ldots + (i\lambda)^n/n!.
\]

Since \( \sigma_p(P_n(y)) = P_n(\sigma_p(y)) \), it follows that \( 1, P_n(1) \in K_n \), where \( K_n = P_n(\bar{K}) \). Take \( N \) such that \( \arg P_N(1) \neq 0 \) and \( Z_N = P_N(y) \in G([F, \tau] \bar{\tau}) \). By the connectedness, it follows that \( K_N \) contains \( \bar{z}_1 \) with \( \arg \bar{z}_1 = 2\pi/n \) for some positive integer \( n \). Let \( E \) be a subalgebra of \( F \) generated by \( (e, Z_N^\bar{\tau}) \). Each \( u \in E \) is of the form \( u = P(Z_N^\bar{\tau}) \), where \( P \in C[z] \), and we put

\[
f(P(Z_N^\bar{\tau})) = P(0);
\]

Obviously \( f \) is a multiplicative linear functional on \( E \). Since \( \sigma_p(Z_N^\bar{\tau}) \) separates the complex plane between 0 and \( \infty \), by the maximum principle we have

\[
\left| f(P(Z_N^\bar{\tau})) \right| = |P(0)| \leq \max \left\{ |P(\lambda)| : \lambda \in \sigma_p(Z_N^\bar{\tau}) \right\}
\]

\[
= \max \left\{ |\lambda| : \lambda \in \sigma_p(P(Z_N^\bar{\tau})) \right\} \leq p(P(Z_N^\bar{\tau})).
\]

So \( f \) can be extended by continuity onto the whole of \([F, \tau] \bar{\tau}\). On the other hand, \( f \) cannot be extended to a multiplicative linear functional on \([F, \tau] \bar{\tau}\), since \( f(Z_N^\bar{\tau}) = 0 \) and \( Z_N \in G([F, \tau] \bar{\tau}) \).

(iii) \( \Rightarrow \) (ii): Let \( E \) be a closed subalgebra of \( F(p) \), where \( p \in S(F) \) and \( x \in E \). Since \( \sigma_p(x) \) does not contain a continuum, we have \( \sigma_E(x) = \sigma_p(x) \) [3], where \( \sigma_E(x) \) denotes the spectrum of \( x \) in \( E \). Thus the spectrum of every element in \( E \) is disconnected. Applying Lemma 2 in [5] we have \( \Gamma(E) = \mathfrak{M}(E) \), where \( \Gamma(E) \) (resp. \( \mathfrak{M}(E) \)) denotes the Šilov boundary (resp. the space of all non-zero multiplicative linear functionals on \( E \) equipped with the weak topology) of \( E \). Since every multiplicative linear functional on \( E \) belonging to \( \Gamma(E) \) can be extended to such a functional on \( F \), we get the implication (iii) \( \Rightarrow \) (ii).
(ii) \(\Rightarrow\) (i) is trivial.

(b) Now assume that \(F\) satisfies condition (C). Then in notations of (a) we replace \(y\) by \(\frac{\pi}{2} \frac{x - e}{\beta - x}\) and \(Z_N\) by \(e^{iy}\) and by an argument similar as in (a) we get

\[ (i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i). \]

Theorem is proved.

**Corollary 1.2** ([5]). A locally \(m\)-convex \(B_0\)-algebra is an ES-algebra if and only if spectrum of any of its elements contains no continuum.

**Proof.** By category arguments, Corollary 1.2 is an immediate consequence of Theorem 1.1.

**Corollary 1.3.** Let \(F\) be a locally \(m\)-convex ES-algebra satisfying either of conditions (Q) and (C) in Theorem 1.1 and let \(f\) be a continuous homomorphism of \(F\) onto \(E\). Then \(E \in ES\).

**Proof.** To prove Corollary 1.3 it suffices to check that \(E(q) \in ES\) for every \(q \in S(E)\). Let \(q \in S(E)\). By the continuity of \(f\), there exists \(p \in S(F)\) such that \(q(f(x)) \leq p(x)\) for every \(x \in F\). Thus \(f\) induces naturally a continuous homomorphism \(\tilde{f}: F(p) \to E(q)\) such that \(\text{Im} \tilde{f} = E(q)\). Since \(\sigma_q(y) \leq \sigma_p(x)\) holds for every \(x \in F(p)\), \(\tilde{f}(x) = y\), by Theorem 1.1 it follows that \(E(q) \in ES\).

**Corollary 1.4.** Let \(F_\alpha\), \(x \in \Omega\) be locally \(m\)-convex algebras satisfying either of conditions (Q) and (C) in Theorem 1.1. Then \(F = \prod \{F_\alpha: \alpha \in \Omega\} \in ES\).

**Proof.** Let \(p \in S(F)\). Take a finite system \((p_1, \ldots, p_n)\) of \(p_\alpha \in S(F_\alpha)\) such that \(F(p) = F_{\alpha_1}(p_1) \oplus \ldots \oplus F_{\alpha_n}(p_n)\). By Theorem 1.1 and since

\[ \sigma_{F(p)}(x) = \bigcup_{j=1}^n \sigma_{F_\alpha}(p_{\alpha_j}) \quad \text{for every } x = (x_1, \ldots, x_n) \in \prod_{j=1}^n F_{\alpha_j}(p_{\alpha_j}), \]

it follows that \(F(p) \in ES\). Hence \(F \in ES\).

**Remark 1.5.** There exists a normed algebra \(A \in ES\) such that \(\hat{A} \notin ES\).

Let \(A\) be the Cantor set in \(C\) and let \(\text{Lip}^p(A)\) denote the Banach algebra of all Lipschitz functions of order \(x\) on \(A\). Then \(\text{Lip}^p(A) \in ES\) ([6]) and hence, by 1.3, \(\text{Im} \theta \in ES\), where \(\theta\) denotes the canonical homomorphism of \(\text{Lip}^p(A)\) into \(C(A)\). On the other hand, since each continuum in \(C\) is a continuous image of \(A\) and \(\text{Im} \theta = C(A)\) then by [6], or (iii) of Theorem 1.1 we have \(C(A) \notin ES\).

2. **An example of ES-algebras.** Let \(C[z_1, \ldots, z_n]\) be the locally \(m\)-convex algebra of complex polynomials in \(n\) variables equipped with the compact-open topology. We prove the following

**Proposition 2.1.** \(C[z_1, \ldots, z_n] \in ES\) if and only if \(n = 1\).

**Proof.** Let \(A\) be a subalgebra of \(C[z]\) and \(f\) a continuous multiplicative linear functional on \(A\). Take a sequence \(\{q_k\}_{k=1}^\infty\) in \(A\) such that
\( \varphi_1 \neq \text{const} \) and \( \{\varphi_k\} \) is dense in \( A \). For each \( k \) consider the map \( \theta_k : C \to C^* \) given by the formula

\[
\theta_k(z) = (\varphi_1(z), \ldots, \varphi_k(z)) \quad \text{for} \quad z \in C.
\]

Put

\[
\theta_\infty(z) = (\varphi_1(z), \ldots, \varphi_k(z), \ldots) \quad \text{for} \quad z \in C.
\]

Since \( \varphi_1 \neq \text{const} \), \( \theta_k \) and \( \theta_\infty \) are proper, i.e. \( \theta_k^{-1}(K) \) and \( \theta_\infty^{-1}(K) \) are compact for every compact set \( K \) in \( C^* \) and \( C^\infty \) respectively. Hence, by a theorem of Remmert ([1], Theorem 5, p. 162), \( \text{Im} \theta_k \) is an analytic set in \( C^* \). Observe that \( \text{Im} \theta_\infty \) is closed in \( C^\infty \), where \( C^\infty \) is equipped with the product topology. Since every holomorphic function on an analytic set in \( C^* \) can be extended to a holomorphic function on \( C^* \) ([1], Theorem 18, p. 245) it follows that

\[
(2.1) \quad \text{Im} \theta_k = V(Ker \hat{\theta}_k)
\]

where \( \hat{\theta}_k \) denotes the homomorphism from \( \Theta(C^k) \), the space of holomorphic functions on \( C^k \), equipped with the compact open topology, into \( \Theta(C) \) induced by \( \theta_k \) and

\[
V(Ker \hat{\theta}_k) = \{ \omega \in C^k : \varphi(\omega) = 0 \text{ for } \varphi \in \text{Ker} \hat{\theta}_k \}.
\]

Since \( f \hat{\theta}_k \) is a continuous multiplicative linear functional on \( C[z_1, \ldots, z_k] \), we have

\[
f \hat{\theta}_k(\varphi) = \varphi(\omega) \quad \text{for} \quad \varphi \in C[z_1, \ldots, z_k]
\]

where \( \omega = (f(\varphi_1), \ldots, f(\varphi_k)) \in V(Ker \hat{\theta}_k) \). Hence, by (2.1), we have \( \omega = \theta_k z^k \) for some \( z^k \in C \). Since

\[
(f(\varphi_1), \ldots, f(\varphi_k)) = \omega = \theta_k z^k = (\varphi_1(z^k), \ldots, \varphi_k(z^k)),
\]

we have

\[
\theta_\infty(z^k) = (\varphi_1(z^k), \ldots, \varphi_k(z^k), \varphi_{k+1}(z^k), \ldots) = (f(\varphi_1), \ldots, f(\varphi_k), \varphi_{k+1}(z^k), \ldots)
\]

and hence

\[
\lim_k \theta_\infty(z^k) = \omega = (f(\varphi_1), \ldots, f(\varphi_k), \ldots).
\]

On the other hand, since \( \text{Im} \theta_\infty \) is closed in \( C^\infty \), we have \( \omega = \theta_\infty(z^0) \) for some \( z^0 \in C \). Hence, setting \( \tilde{f}(\varphi) = \varphi(z^0) \) for \( \varphi \in C[z] \), we get the required extension of \( f \).

Let \( n \geq 2 \). Consider the subalgebra \( A \) of \( C[z_1, \ldots, z_n] \) generated by \( z_1 \) and \( z_1 z_2 \). Since \( z_1 \) and \( z_1 z_2 \) are algebraically independent, the formula

\[
f(\varphi) = \sum_{j=0}^\infty a_{0j},
\]
where
\[ \varphi(z_1, z_2) = \sum_{i \geq 1} a_{i0} z_1^i + \sum_{i,j \geq 1} a_{ij} z_1^i (z_1 z_2)^j + \sum_{j \geq 0} a_{0j} (z_1 z_2)^j \in A, \]
defines a multiplicative linear functional \( f \) on \( A \). Since for every compact set \( K \) in \( C^n \) the seminorm
\[ p(\varphi) = \sup_{(z_1, \ldots, z_n) \in K} \sum_{i_1, \ldots, i_n \geq 0} |a_{i_1 \ldots i_n} z_1^{i_1} \cdots z_n^{i_n}| \]
is continuous on \( \Theta(C^n) \), it follows that \( f \) is continuous. On the other hand \( f \) cannot be extended to a continuous multiplicative linear functional \( \tilde{f} \) on \( C[z_1, \ldots, z_n] \), since otherwise
\[ 0 = \tilde{f}(z_1) \quad \text{and} \quad 1 = \tilde{f}(z_1 z_2) = \tilde{f}(z_1) \tilde{f}(z_2) = 0. \]
The proposition is proved.

Let \( V \) be an irreducible algebraic subset of \( C^n \). By \( C(V) \) we denote the locally \( m \)-convex algebra \( C[z_1, \ldots, z_n]/I[V] \), where \( I[V] = \{ \varphi \in C[z_1, \ldots, z_n] : \sigma(\varphi) = 0 \} \). It is known that if \( k = \dim V \), then there exists a surjective map \( \pi : V \to C^k \) such that \( \hat{\pi}(C[z_1, \ldots, z_k]) \subseteq C[V] \) and \( C[V] \) is integrally dependent on \( \hat{\pi}(C[z_1, \ldots, z_k]) \), i.e. every \( u \in C[V] \) is a solution of some equation
\[ X^n + a_1 X^{n-1} + \ldots + a_n = 0, \]
where \( a_1, \ldots, a_n \in \hat{\pi}(C[z_1, \ldots, z_k]). \]
This implies that \( \hat{\pi} \) is an embedding of \( C[z_1, \ldots, z_k] \) into \( C[V] \). Thus by Proposition 2.1, \( C[V] \in ES \) if \( k \geq 2 \).

Now consider the irreducible algebraic subset \( V \) of \( C^2 \) given by
\[ V = \{(z_1, z_2) \in C^2 : z_1 z_2 - 1 = 0 \}. \]

By the maximum principle, the homomorphism \( \pi : C[z_1] \to C[V] \), induced by the canonical projection \( \pi : (z_1, z_2) \mapsto z_1 \), is an embedding. Thus the multiplicative linear functional \( f \) on \( \hat{\pi}(C[z_1]) \) given by \( f(\hat{\pi} \sigma) = \sigma(0) \) is continuous. The functional \( f \) cannot be extended to a multiplicative linear functional \( \tilde{f} \) on \( C[V] \), since \( f(\hat{\pi} z_1) = 0 \) and \( \hat{\pi} z_1 \) is invertible in \( C[V] \).

Thus we get the following

**Corollary 2.2.** Let \( V \) be an irreducible algebraic subset of \( C^n \) of dimension \( \geq 2 \). Then \( C[V] \notin ES \).

Moreover, there exists an irreducible algebraic subset \( V \) of \( C^2 \) of dimension 1 such that \( C[V] \notin ES \).

**Remark 2.3.** By Theorem 1.1, \( C[z](p) \notin ES \) for some \( p \in S(C[z]) \). Hence by Proposition 2.1 it follows that the conditions (i) and (ii) in Theorem 1.1 are not equivalent without one of the hypotheses (Q) and (C).

3. **Locally \( m \)-convex algebras having the EP(MES).** By MES we denote the class of all metric ES-algebras. In this section we prove the following

**Theorem 3.1.** Let \( Q \) be a semi-simple Fréchet–Montel ES-algebra. Then
$Q$ has the $EP\text{(MES)}$ if and only if $Q$ is isomorphic to the algebra $C^m$ for some $m \leq \infty$.

We need the following

**Lemma 3.2** ([2], Lemma 1.4). Let $F$ be a Fréchet–Montel space. If there exists a continuous linear map from $\prod_{j=1}^{i} B_j$ onto $F$, where $B_j$ are Banach spaces, then $F$ is isomorphic to $C^m$ for some $m \leq \infty$.

**Proof of Theorem 3.1.** The fact that if $Q \cong C^m$, then $Q$ has the $EP\text{(MES)}$ follows from the relations

$$C^n \subset ES \quad \text{for every } n \geq 1.$$

Now assume that $Q$ is a Fréchet–Montel algebra having the $EP\text{(MES)}$. We prove that $\tilde{Q} = Q/\text{Rad } Q$ is isomorphic to the algebra $C^m$ for some $m \leq \infty$.

(a) Take an increasing sequence $\{p_n\}$ in $S(Q)$ determining the topology of $Q$. Theorem 1.1 implies that $Q_n = Q(p_n) \subset ES$ for every $n \geq 1$ and hence, by Corollary 1.4, $\tilde{Q} = \prod_{n=1}^{\infty} Q_n \subset ES$. Consider the canonical embedding of $Q$ into $\tilde{Q}$

$$0x = \{\pi_n x\}_{n=1}^\infty \quad \text{for } x \in Q$$

where $\pi_n = \pi(p_n)$. By hypothesis, there exists a continuous homomorphism $P: \tilde{Q} \to Q$ such that $P0 = \text{id}$. Thus by Lemma 3.2 we get

$$\dim Q(p) < \infty \quad \text{for every } p \in S(Q). \quad (3.1)$$

(b) Let us consider the canonical homomorphism $\bar{\omega}: \tilde{Q} \to E$, where $E = \prod_{n=1}^{\infty} C(\mathfrak{M}(Q_n))$, given by the formula

$$\bar{\omega}(x + \text{Rad } Q) = \{\beta_n \pi_n x\} \quad \text{for } x + \text{Rad } Q \in \tilde{Q},$$

where $\beta_n: Q_n \to C(\mathfrak{M}(Q_n))$ are canonical homomorphisms. Note that $\beta_n$ is surjective for every $n \geq 1$ and

$$\begin{cases}
\text{Rad } \prod_{n=1}^{\infty} Q_n = \bigcap_{n=1}^{\infty} \text{Rad } Q_n, \\
P(\text{Rad } \prod_{n=1}^{\infty} Q_n) \subset \text{Rad } Q. \quad (3.2)
\end{cases}$$

By (3.2) and by the openness of the homomorphism $\beta = (\beta_n): \tilde{Q} \to E$ it follows that the formula

$$Pz = Pu + \text{Rad } Q \quad \text{for } z \in E,$$

where $u \in \tilde{Q}$, $\beta u = z$, defines a continuous homomorphism $\bar{P}: E \to \tilde{Q}$ such that $\bar{P}0 = \text{id}$. Thus $\tilde{Q}$ can be considered as a closed subalgebra of $E$. 
(c) Let \( \{ \bar{E}_n \} \) be an increasing sequence in \( S(E) \) determining the topology of \( E \) such that \( \mathfrak{M}(E(\bar{E}_n)) \) separates points of \( E(\bar{E}_n) \) for every \( n \geq 1 \). Then \( \mathfrak{M}(\bar{Q}(\bar{E}_n)) \), where \( \bar{E}_n = \bar{E}_n \mid \bar{Q} \), separates points of \( \bar{Q}(\bar{E}_n) \) for every \( n \geq 1 \). Then by (3.1) we have
\[
\bar{Q} = \lim \bar{Q}(\bar{E}_n) \cong \lim C(\mathfrak{M}(\bar{Q}(\bar{E}_n))) \cong C^m
\]
for some \( m \leq x \).

The theorem is proved.

Acknowledgement. I wish to express my gratitude to Professor W. Żelazko for his guidance, remarks and constant encouragements during the preparation of this paper.

REFERENCES


INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSAWA, POLAND

Reçu par la Rédaction le 15. 07. 1981