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On the continuous dependence of solutions of some functional equations on given functions

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In the present paper we shall prove that, under suitable assumptions, the continuous solutions $\varphi(x)$ of the functional equation

$$\varphi[f(x)] = g(x)\varphi(x) + F(x) \tag{1}$$

depend in a continuous manner on the given functions $f(x)$, $g(x)$, $F(x)$. In the cases $g(x) \equiv 1$ or $g(x) \equiv -1$ this problem has been investigated in [3] and for the more general equation

$$\varphi(x) = H\big(x, \varphi[f(x)]\big) \tag{2}$$

in [4] (under different assumptions).

The functions $f(x)$, $g(x)$, $F(x)$ will be subjected to some of the following conditions:

(i) The function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b)$, $a < f(x) < x$ in (a, b), $f(a) = a$.

(ii) The function $g(x)$ is continuous in $\langle a, b)$, $g(x) \neq 0$ in $\langle a, b)$.

(iii) The function $F(x)$ is continuous in $\langle a, b)$.

(iv) There exists a constant ϑ, $0 < \vartheta < 1$, such that the inequality

$$|f(x) - a| \leqslant \vartheta |x - a| \tag{3}$$

holds in $\langle a, a+\delta) \subset \langle a, b)$ for some $\delta > 0$.

(v) There exist constants $M > 0$ and $\mu > 0$ such that

$$|g(x) - 1| \leqslant M|x - a|^{\mu} \tag{4}$$

in $\langle a, a+\delta)$

(vi) There exist a constant θ, $0 < \theta < 1$, and a bounded function $B(x)$ such that the inequalities

$$|F(x)| \leqslant B(x), \tag{5}$$

$$B[f(x)] \leqslant \theta B(x) \tag{6}$$

hold in $\langle a, a+\eta)$, $\eta > 0$.

We put

$$G_n(x) = \prod_{\nu=0}^{n-1} g[f^\nu(x)], \qquad n = 1, 2, \ldots \tag{7}$$

We assume that

(A) The limit

$$G(x) = \lim_{n\to\infty} G_n(x) \tag{8}$$

exists, $G(x)$ is continuous in $\langle a, b)$, and $G(x) \neq 0$ in $\langle a, b)$.

The following result is known (cf. [1]):

Lemma 1. *If hypotheses* (i)-(vi) *are fulfilled, then equation* (1) *has a one-parameter family of continuous solutions in* $\langle a, b)$ *given by the formula*

$$\varphi(x) = \bar{\varphi}(x) + \frac{c}{G(x)}, \tag{9}$$

where

$$\bar{\varphi}(x) = -\sum_{n=0}^{\infty} \frac{F[f^n(x)]}{G_{n+1}(x)}, \tag{10}$$

and $G(x)$ (1) *is defined by* (8) *and* $c = \varphi(a)$.

Lemma 2. *Let* $a_n^k(x)$, $n, k = 0, 1, 2, \ldots$, *be real functions defined in an interval* I, $a_n^k(x) > 0$ *in* I, *and for every* $k = 0, 1, 2, \ldots$

$$\lim_{n\to\infty} a_k^n(x) = a^k(x) > 0 \quad \textit{for } x \in I \tag{11}$$

uniformly in I. *Moreover, if there exist functions* $A^{(k)}(x)$ *such that*

$$|\ln a_n^k(x)| \leqslant A^k(x), \qquad n, k = 0, 1, 2, \ldots, \tag{12}$$

and the series $\sum_{k=0}^{\infty} A^{(k)}(x)$ *converges uniformly in* I, *then*

$$\prod_{k=0}^{\infty} a_n^k(x) = a_n(x), \qquad n = 0, 1, 2, \ldots, \tag{13}$$

and

$$\prod_{k=0}^{\infty} a^k(x) = a(x) \tag{14}$$

uniformly in I *and*

$$\lim_{n\to\infty} a_n(x) = a(x) \tag{15}$$

uniformly in I.

(1) Hypotheses (i), (ii), (iv) and (v) imply that case (A) occurs (cf. [1]).

Proof. According to (12) the products (13) and (14) uniformly converge in I. It suffices to prove (cf. [2], § 29) that

$$\lim_{n\to\infty}\Big(\sum_{k=0}^{\infty}\ln a_n^k(x)\Big) = \sum_{k=0}^{\infty}\lim_{n\to\infty}\big(\ln a_n^k(x)\big) \tag{16}$$

uniformly in I. We have

$$\Big|\sum_{k=0}^{\infty}\ln a_n^k(x)-\sum_{k=0}^{\infty}\ln a^k(x)\Big| \leqslant \sum_{k=0}^{N-1}|\ln a_n^k(x)-\ln a^k(x)|+\sum_{k=N}^{\infty}|\ln a_n^k(x)|+\sum_{k=N}^{\infty}|\ln a^k(x)| \leqslant \sum_{k=0}^{N-1}|\ln a_n^k(x)-\ln a^k(x)|+2\sum_{k=N}^{\infty}A^{(k)}(x).$$

Since the series $\sum_{k=0}^{\infty}A^{(k)}(x)$ converges uniformly in I, for every $\varepsilon>0$ there exists an index N such that

$$\sum_{k=N}^{\infty}A^{(k)}(x)\leqslant \tfrac{1}{4}\varepsilon \quad \text{for } x\in I. \tag{17}$$

Moreover, we have

$$|\ln a_n^k(x)-\ln a^k(x)|\leqslant\frac{\varepsilon}{2N} \quad \text{for } n>N_1,\ k=0,1,\dots,N-1,\ x\in I. \tag{18}$$

From (17) and (18) we obtain (16), which completes the proof.

THEOREM. *Let us assume that the functions* $f_n(x)$, $g_n(x)$, $F_n(x)$, $n=0,1,\dots$, *fulfil hypotheses* (i)-(vi) *with common* ϑ, δ, M, μ, $B(x)$, θ, η *and*

$$\lim_{n\to\infty}f_n(x)=f(x),\quad \lim_{n\to\infty}g_n(x)=g(x),\quad \lim_{n\to\infty}F_n(x)=F(x) \tag{19}$$

uniformly in $\langle a,d\rangle$ *for every* $d\in(a,b)$ *and further*

(20) $f(x)$ *is strictly increasing in* $\langle a,b)$, $a<f(x)<x$ *in* (a,b), $g(x)\neq 0$ *in* $\langle a,b)$, *and* $f(x)$ *fulfils* (6).

Moreover, if

(21) *the functions* $g_n(x)$, $n=0,1,\dots$, *are monotonic in* $\langle a,b)$ *and the sequences* $\{f_n(x)\}$ *and* $\{g_n(x)\}$ *are monotonic for every fixed* $x\in\langle a,b)$, *then*

$$\lim_{n\to\infty}\varphi_n(x)=\varphi(x) \tag{22}$$

uniformly in $\langle a, b)$, *where* $\varphi(x)$ *is the solution of equation* (1) *and* $\varphi_n(x)$ *is the solution of the equation*

$$\varphi[f_n(x)] = g_n(x)\varphi(x)+F_n(x)$$

such that $\varphi_n(a) = c = \varphi(a)$, $n = 0, 1, 2, \ldots$

Proof (cf. also proof of Theorem 3 in [3]). Let us suppose that the sequence $\{g_n(x)\}$ is increasing, the functions $g_n(x)$ are increasing, and the sequence $\{f_n(x)\}$ is decreasing (in the other seven cases the proof runs similarly).

It is readily seen that the functions $f(x), g(x), F(x)$ also fulfil hypotheses (i)-(vi), whence, in view of Lemma 1, solutions $\varphi(x)$ and $\varphi_n(x)$ exist and are given by the formulas

$$\varphi(x) = \bar{\varphi}(x)+\frac{c}{G(x)}, \quad \varphi_n(x) = \bar{\varphi}_n(x)+\frac{c}{G^{(n)}(x)},$$

where

$$\bar{\varphi}(x) = -\sum_{v=0}^{\infty}\frac{F[f^v(x)]}{G_{v+1}(x)}, \quad \bar{\varphi}_n(x) = -\sum_{v=0}^{\infty}\frac{F_n[f_n^v(x)]}{G_{v+1}^{(n)}(x)},$$

$$G_v(x) = \prod_{n=0}^{v-1} g[f^n(x)], \quad G(x) = \lim_{n\to\infty} G_v(x),$$

$$G_v^{(n)}(x) = \prod_{m=0}^{v-1} g_n[f_n^m(x)], \quad G^{(n)}(x) = \lim_{v\to\infty} G_v^{(n)}(x).$$

According to (21) and (v)

$$1 \leqslant g_n[f_n^k(x)] \leqslant g_n[f_1^k(x)] \leqslant g[f_1^k(x)], \quad k = 0, 1, 2, \ldots, \tag{23}$$

and hence

$$0 \leqslant \ln g_n[f_n^k(x)] \leqslant \ln g[f_1^k(x)] \quad \text{for } n, k = 0, 1, 2, \ldots, x \in \langle a, b).$$

The product $\prod_{v=0}^{\infty} g[f_1^v(x)]$ converges uniformly in every interval $\langle a, d\rangle$, $d \in (a, b)$ (cf. [1], Theorem 2) and hence also the series $\sum_{v=0}^{\infty} \ln g[f_1^v(x)]$ converges uniformly in $\langle a, d\rangle$. According to Lemma 2, we have

$$\lim_{n\to\infty} G^{(n)}(x) = G(x) \tag{24}$$

uniformly in $\langle a, d\rangle$ for every $d \in (a, b)$.

In view of (21), (23), and (v) we obtain

$$1 \leqslant G_v^{(n)}(x), \quad 1 \leqslant G_v(x), \quad n = 0, 1, 2, \ldots, \ v = 1, 2, \ldots, x \in \langle a, d\rangle. \tag{25}$$

Evidently, $\lim_{x\to a} B(x) = 0$ [2]. Consequently, for an arbitrary $\varepsilon > 0$ we can choose $\eta_1 > 0$, $\eta_1 < \eta$, such that

(26) $$B(x) < \frac{\varepsilon(1-\theta)}{8} \quad \text{for } x \in (a, a+\eta_1).$$

Since $f(x)$ is monotonic and $\lim_{v\to\infty} f^v(x) = a$, we can find an index N_1 such that

(27) $$f^v(x) \in (a, a+\eta_1) \quad \text{for } x \in (a, d\rangle, v \geqslant N_1.$$

We shall prove that

(28) $$f_n^v(x) \in (a, a+\eta_1) \quad \text{for } x \in (a, d\rangle, n, v > N_2.$$

In fact, there exists an index K such that

$$f^K(d) < a+\eta_1/2.$$

Next

$$f_n^K(d) < f^K(d)+\eta_1/2 \quad \text{for } n > N_2 > K,$$

whence

$$f_n^k(x) < f_n^k(d) < f_n^K(d) < f^K(d)+\frac{\eta_1}{2} < a+\frac{\eta_1}{2}+\frac{\eta_1}{2}$$

for $n, k > N_2$, $x \in (a, d\rangle$.

We have

$$|\bar{\varphi}_n(x)-\bar{\varphi}(x)| \leqslant \sum_{v=0}^{N-1} \left| \frac{F_n[f_n^v(x)]}{G_{v+1}^{(n)}(x)} - \frac{F[f^v(x)]}{G_{v+1}(x)} \right| + \\ + \sum_{v=N}^{\infty} \left| \frac{F_n[f_n^v(x)]}{G_{v+1}^{(n)}(x)} \right| + \sum_{v=N}^{\infty} \left| \frac{F[f^v(x)]}{G_{v+1}(x)} \right|,$$

where $N = \max\{N_1, N_2\}$. We have by (2), (3), (27) and (28) for $x \in (a, d\rangle$ and $n, v \geqslant N$

(29) $$|F_n[f_n^v(x)]| \leqslant \theta^{v-N} B[f_n^N(x)], \quad |F[f^v(x)]| \leqslant \theta^{v-N} B[f^N(x)].$$

From (19) we obtain

(30) $$\left| \frac{F_n[f_n^v(x)]}{G_{v+1}^{(n)}(x)} - \frac{F[f^v(x)]}{G_{v+1}(x)} \right| < \frac{\varepsilon}{4N}.$$

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[2] We can find a function $H(x)$ increasing and bounded in $\langle a, a+\eta)$ and such that $B(x) \leqslant H(x)$, $H[f(x)] \leqslant \theta H(x)$ in $\langle a, a+\eta)$. The proof of the above fact does not differ from that given in [5] (theorem 8).

for $x \in \langle a, d\rangle, n > N_3 > N, v = 0, 1, 2, \ldots, N-1$, whence

$$\sum_{v=0}^{N-1} \left| \frac{F_n[f_n^v(x)]}{G_{v+1}^{(n)}(x)} - \frac{F[f^v(x)]}{G_{v+1}(x)} \right| < \frac{\varepsilon}{4} \quad \text{for } x \in \langle a, d\rangle, n > N_3.$$

According to (25), (26) and (29) we have

$$\sum_{v=N}^{\infty} \left| \frac{F_n[f_n^v(x)]}{G_{v+1}^{(n)}(x)} \right| \leqslant \sum_{v=N}^{\infty} \theta^{v-N} B[f_n^N(x)] \leqslant \frac{\varepsilon}{8}$$

and analogously

$$\sum_{v=N}^{\infty} \left| \frac{F[f^v(x)]}{G_{v+1}(x)} \right| \leqslant \frac{\varepsilon}{8},$$

whence

$$|\bar{\varphi}_n(x) - \bar{\varphi}(x)| < \varepsilon/2 \quad \text{for } x \in (a, d\rangle, n > N_3.$$

Since $\bar{\varphi}_n(a) = \bar{\varphi}(a) = 0$, we have

$$|\bar{\varphi}_n(x) - \bar{\varphi}(x)| < \varepsilon/2 \quad \text{for } x \in \langle a, d\rangle, n > N_3. \tag{31}$$

From (24) and (31) we obtain

$$|\varphi_n(x) - \varphi(x)| \leqslant |\bar{\varphi}_n(x) - \bar{\varphi}(x)| + \left| \frac{c}{G^{(n)}(x)} - \frac{c}{G(x)} \right| < \varepsilon$$

for $x \in \langle a, d\rangle$ and $n > M > N_3$, which was to be proved.

References

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