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ON THE RADIUS OF UNIVALENCE OF BOUNDED FUNCTIONS

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It is the purpose of this note to prove the following

THEOREM. *Let $f(z)$ be an analytic function bounded by* 1 *in the unit disc,* $|z| < 1$, *and normalized so that* $f(0) = 0, f'(0) = a$, $0 < |a| = a < 1$. *Let* $z_0 \neq 0$ *be a zero of* $f(z)$ *in the unit disc such that* $f(z_1) = 0$, $|z_1| < 1$ *imply* $|z_0| \leqslant |z_1|$. *Then* $f(z)$ *is univalent in the disc*

$$|z| < \varrho = \Big(a + r_0 + r_0^2 - r_0 a - \big((a + r_0 + r_0^2 - r_0 a)^2 - 4r_0^2 a^2\big)^{1/2}\Big)/2r_0 a,$$

where $r_0 = |z_0|$.

Proof. (A) Let

$$g(z) = f(z)/\big((z - z_0)/(1 - \bar{z}_0 z)\big) = -(a/z_0)z + \ldots$$

It is easily seen that $g(z)$ is analytic and bounded by 1 in the unit disc. By Schwarz lemma [2], p. 165, it follows that

$$|g'(0)| = a/r_0 \leqslant 1.$$

(B) Let

$$h(z) = f(z)/z\big((z - z_0)/(1 - \bar{z}_0 z)\big) = -(a/z_0) + \ldots$$

Again $h(z)$ is analytic and bounded by 1 in the unit disc. It is known (cf. [2], p. 167) that

$$|h(z)| \geqslant (|h(0)| - r)/(1 - |h(0)|r) = (a - rr_0)/(r_0 - ar) \geqslant 0,$$

where $|z| = r < a$.

The right-hand side of this inequality is a non-negative number, since $r < a$ and of the result of part (A). Thus for $r < a$, we get

$$|f(z)| = |z|\,|(z - z_0)/(1 - \bar{z}_0 z)|\,|h(z)| \geqslant r\big((r_0 - r)/(1 - rr_0)\big)\big((a - rr_0)/(r_0 - ar)\big).$$

(C) To prove the Theorem, it is sufficient to show that we cannot have $f(z_1) = f(z_2)$, $z_1 \neq z_2$, $|z_1| = |z_2| = r < \varrho$. To this end, assume $f(z_1) = f(z_2) = b$, $z_1 \neq z_2$, $|z_1| = |z_2| = r$. Then there exists a regular function $g(z)$ such that $|g(z)| \leqslant 1$ in the unit disc and satisfying the following equality:

$$\big(f(z)-b\big)/\big(1-\bar{b}f(z)\big) = \big((z-z_1)/(1-\bar{z}_1 z)\big)\big((z-z_2)/(1-\bar{z}_2 z)\big)g(z).$$

Consequently, $|g(0)| = |b|/r^2 \leqslant 1$, i.e., $|b| \leqslant r^2$; see also [2], p. 171. From this and the inequality at the end of part (B), we get

$$r^2 \geqslant |b| = |f(z_1)| \geqslant r\big((r_0-r)/(1-rr_0)\big)\big((a-rr_0)/(r_0-ar)\big).$$

Hence the radius of univalence for $f(z)$ is at least as large as the smallest positive number satisfying the equality

$$r = \big((r_0-r)/(1-rr_0)\big)\big((a-rr_0)/(r_0-ar)\big),$$

which can be written as

$$(r-1)(r^2+r+1-pr) = 0,$$

where $p = (r_0^2+r_0+a)/r_0 a$.

Thus

$$\varrho = 2^{-1}\big(p-1-\big((p-1)^2-4\big)^{1/2}\big),$$

which proves the Theorem.

Remarks. We remark here that if $r_0 = a$, then $\varrho = \varrho_0 = a^{-1}\big(1-(1-a^2)^{1/2}\big)$, which is the corresponding radius of univalence for the class of bounded functions normalized as in the Theorem. The radius ϱ_0 is the largest possible for such class as can be seen by considering the bounded function $f_0(z) = z(a-z)/(1-\bar{a}z)$. This result is due to *E.* Landau. Our Theorem represents an improvement on Landau's for the individual bounded functions since $\varrho \geqslant \varrho_0$.

For the subclass of bounded functions which is zero free-except at $z = 0$, the problem of obtaining the radius of univalence has been raised recently [1].

REFERENCES

[1] W. O. Egerland, *Problem 5758,* American Mathematical Monthly 77 (1970), p. 873.

[2] Zeev Nehari, *Conformal mapping,* New York 1952.

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