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The law of exponential decay for r-adic transformations

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Abstract. Let $([0, 1], \mu_f)$ be a probability space with $d\mu_f = f d\mu, f \in L^1$, and let $T(x) = rx \pmod 1$ for $x \in [0, 1]$. Denote by F_ε the distribution of the random variable given by the formula

$$\xi_\varepsilon(x) = \max\{n\colon\ T^k(x) > \varepsilon,\ k = 0, 1, 2, \ldots, n\}.$$

Then $F_\varepsilon(z/\varepsilon)$ tends to an exponential function as $\varepsilon \to 0$ and the limiting function does not depend on the initial f.

1. Introduction. Consider the transformation of the unit interval into itself $T(x) = rx \pmod 1$ and the family of random variables

$$\xi_\varepsilon(x) = \max\{n\colon\ T^k(x) > \varepsilon,\ k = 0, 1, 2, \ldots, n\}$$

which measure the time we must wait to observe $T^n(x) \notin [0, \varepsilon]$.

The purpose of this paper is an investigation of limiting behaviour of these variables as ε tends to zero. We shall show that the distribution of these random variables tends to an exponential function as $\varepsilon \to 0$. This is proved by showing that the limiting function is a solution of a certain linear differential equation.

A. Lasota and J. A. Yorke dealt with a similar problem in [7]. They considered a family of transformations $T_\varepsilon\colon A \to R^n$, $A \subset R^n$, and a family of random variables which measure the time we must wait to observe $T^n(x) \notin A$. The main result of those authors shows that the limiting distribution of those random variables is also, as in our case, an exponential function. But the proof of their results is based on another idea than that of ours.

In Section 2 we state the main theorem. In Section 3 we prove some necessary lemmas and the main theorem. In Section 4 we state and prove a corollary to the main theorem.

2. The law of exponential decay for r-adic transformations. Denote by μ the Lebesgue measure and by L^1 the space of all integrable functions defined on $[0, 1]$.

Let N denote the set of all natural numbers.

THEOREM 1. *Let T: $[0, 1] \to [0, 1]$ be given by the formula*

$$T(x) = rx(\mathrm{mod}\ 1)$$

where r is a natural number greater than 1 and let

$$\xi_\varepsilon(x) = \max\{n\colon\ T^k(x) > \varepsilon,\ k = 0, 1, \ldots, n-1\}.$$

Then for any $f \in L^1$, $f \geqslant 0$, $\|f\| = 1$,

$$\lim_{\varepsilon \to 0} F_\varepsilon\left(f, \frac{z}{\varepsilon}\right) = 1 - \exp\left(\frac{1-r}{r}\, z\right),$$

where $F_\varepsilon(f, z) = \mu_f\{x\colon\ \xi_\varepsilon(x) < z\}$, $d\mu_f = f d\mu$.

3. Auxiliary lemmas and the proof of Theorem 1. In the proof of Theorem 1 we shall use the following lemmas and theorems.

LEMMA 1. *Let a transformation T: $[0, 1] \to [0, 1]$ satisfies the assumptions of Theorem 1 and let*

$$A_i = \left\{\frac{k_i}{r^n}, \frac{l_i + \alpha_i}{r^n}\right\}, \qquad i = 1, 2, \ldots, m,$$

be a set of intervals with $k_i, l_i \in N$, $l_i + \alpha_i \leqslant r^n$ and $0 \leqslant \alpha_i < 1$. Then for every $\varepsilon \in (0, 1)$ there exists $\theta \in [-1, 1]$ such that

$$\mu\left(\bigcup_{i=1}^{m} A_i \cap T^{-n}([0, \varepsilon])\right) = \mu\left(\bigcup_{i=1}^{m} A_i\right)\varepsilon + \theta\varepsilon \frac{m}{r^n}.$$

Proof. It is easy to see that

$$\mu\left(\left[\frac{k}{r^n}, \frac{k+1}{r^n}\right] \cap T^{-n}([0, \varepsilon])\right) = \frac{1}{r^n}\varepsilon \tag{1}$$

for $k = 0, 1, 2, \ldots, r^n - 1$. Since there exists a sequence $\{\bar{k}_i\}_{i=1}^{m_1} \subset N$ such that $\bar{k}_1 < \bar{k}_2 < \ldots < \bar{k}_{m_1} < r_n^n$ and

$$\bigcup_{i=1}^{m} \left[\frac{k_i}{r^n}, \frac{l_i}{r^n}\right] = \bigcup_{i=1}^{m_1} \left[\frac{\bar{k}_i}{r^n}, \frac{\bar{k}_i + 1}{r^n}\right],$$

from (1) we obtain

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{m} \left[\frac{k_i}{r^n}, \frac{l_i}{r^n}\right] \cap T^{-n}([0, \varepsilon])\right) &= \mu\left(\bigcup_{i=1}^{m_1} \left[\frac{\bar{k}_i}{r^n}, \frac{\bar{k}_i + 1}{r^n}\right] \cap T^{-n}([0, \varepsilon])\right) \\
&= \sum_{i=1}^{m_1} \mu\left(\left[\frac{\bar{k}_i}{r^n}, \frac{\bar{k}_i + 1}{r^n}\right] \cap T^{-n}([0, \varepsilon])\right) = \sum_{i=1}^{m_1} \mu\left(\left[\frac{\bar{k}_i}{r^n}, \frac{\bar{k}_i + 1}{r^n}\right]\right)\varepsilon \\
&= \mu\left(\bigcup_{i=1}^{m_1} \left[\frac{\bar{k}_i}{r^n}, \frac{\bar{k}_i + 1}{r^n}\right]\right)\varepsilon = \mu\left(\bigcup_{i=1}^{m} \left[\frac{k_i}{r^n}, \frac{l_i}{r^n}\right]\right)\varepsilon.
\end{aligned} \tag{2}$$

Let $\gamma_i = 1$ if $\alpha_i > 0$ and $\gamma_i = 0$ if $\alpha_i = 0$. From (2) we have

$$\text{(3)}\quad \mu\Big(\big(\bigcup_{i=1}^{m} A_i\big)\cap T^{-n}([0,\varepsilon])\Big) \geqslant \mu\left(\bigcup_{i=1}^{m}\left[\frac{k_i}{r^n},\frac{l_i}{r^n}\right]\cap T^{-n}([0,\varepsilon])\right)$$

$$= \mu\left(\bigcup_{i=1}^{m}\left(\left[\frac{k_i}{r^n},\frac{l_i+\gamma_i}{r^n}\right]\setminus\left[\frac{l_i}{r^n},\frac{l_i+\gamma_i}{r^n}\right]\right)\cap T^{-n}([0,\varepsilon])\right)$$

$$\geqslant \mu\left(\left(\bigcup_{i=1}^{m}\left[\frac{k_i}{r^n},\frac{l_i+\gamma_i}{r^n}\right]\setminus\bigcup_{i=1}^{m}\left[\frac{l_i}{r^n},\frac{l_i+\gamma_i}{r^n}\right]\right)\cap T^{-n}([0,\varepsilon])\right)$$

$$= \mu\left(\bigcup_{i=1}^{m}\left[\frac{k_i}{r^n},\frac{k_i+\gamma_i}{r^n}\right]\cap T^{-n}([0,\varepsilon])\right)-$$

$$-\mu\left(\bigcup_{i=1}^{m}\left[\frac{l_i}{r^n},\frac{l_i+\gamma_i}{r^n}\right]\cap T^{-n}([0,\varepsilon])\right)$$

$$\geqslant \mu\big(\bigcup_{i=1}^{m} A_i\big)\,\varepsilon-\frac{m}{r^n}\,\varepsilon,$$

$$\text{(4)}\quad \mu\big(\bigcup_{i=1}^{m} A_i\cap T^{-n}([0,\varepsilon])\big)$$

$$\leqslant \mu\left(\bigcup_{i=1}^{m}\left[\frac{k_i}{r^n},\frac{l_i}{r^n}\right]\cap T^{-n}([0,\varepsilon])\right)+\mu\left(\bigcup_{i=1}^{m}\left[\frac{l_i}{r^n},\frac{l_i+\gamma_i}{r^n}\right]\cap T^{-n}([0,\varepsilon])\right)$$

$$\leqslant \mu\left(\bigcup_{i=1}^{m}\left[\frac{k_i}{r^n},\frac{l_i}{r^n}\right]\right)\varepsilon+\frac{m}{r^n}\,\varepsilon \leqslant \mu\big(\bigcup_{i=1}^{m} A_i\big)\,\varepsilon+\frac{m}{r^n}\,\varepsilon.$$

Inequalities (3) and (4) result in the assertion of the lemma.

Let $s(\varepsilon) = \max\{n\in N\colon\ 1/r^n < \varepsilon\}$, and let $[t] = \max\{n\in N\colon\ n \leqslant t\}$ for $t\in R$.

Lemma 2. *If a transformation $T\colon [0, 1]\to[0, 1]$ satisfies the assumptions of Theorem 1, then for any $t > s(\varepsilon)$ there exists $\theta\in[-4, 4]$ such that*

$$\mu\big(\bigcup_{n=0}^{[t+1]} T^{-n}([0,\varepsilon])\big)-\mu\big(\bigcup_{n=0}^{[t]} T^{-n}([0,\varepsilon])\big)$$

$$= \varepsilon\left(1-\frac{1}{r}\right)-\varepsilon\left(1-\frac{1}{r}\right)\mu\big(\bigcup_{n=0}^{[t]-s(\varepsilon)} T^{-n}([0,\varepsilon])\big)+\frac{\theta\varepsilon}{r^{s(\varepsilon)-1}}.$$

Proof. It is obvious that

$$\mu\big(\bigcup_{n=0}^{k+1} T^{-n}([0,\varepsilon])\setminus\bigcup_{n=0}^{k} T^{-n}([0,\varepsilon])\big) = \varepsilon\,(1-1/r)$$

for $k = 0, 1, \ldots, s(\varepsilon)-2$. Hence, in view of the fact that T preserves μ, we have

$$\mu\Big(\bigcup_{n=p}^{p+k+1} T^{-n}([0,\varepsilon]) \setminus \bigcup_{n=p}^{p+k} T^{-n}([0,\varepsilon])\Big) = \varepsilon(1-1/r) \tag{5}$$

for $p \in N$ and $k = 0, 1, \ldots, s(\varepsilon)-2$. We also have

$$\bigcup_{n=q}^{p+k} T^{-n}([0,\varepsilon]) \cap T^{-p-k-1}([0,\varepsilon]) = T^{-p-k}([0,\varepsilon/r]) \tag{6}$$

for $p, q \in N$, $k = 0, 1, \ldots, s(\varepsilon)-2$ and $p \leqslant q \leqslant p+k$.

It is easy to verify that

$$\mu(A \cup B \cup C) - \mu(A \cup B) = \mu(B \cup C \setminus B) - \mu(A \cap C) + \mu(A \cap B \cap C) \tag{7}$$

whenever $\mu(A \cup B \cup C) < \infty$.

Setting

$$A = \bigcup_{n=0}^{k_{t\varepsilon}} T^{-n}([0,\varepsilon]), \quad B = \bigcup_{n=k_{t\varepsilon}+1}^{[t]} T^{-n}([0,\varepsilon]) \quad \text{and} \quad C = T^{-[t+1]}([0,\varepsilon])$$

where $k_{t\varepsilon} = [t]-s(\varepsilon)+1$, from (5), (6), (7) and Lemma 1 we obtain

$$\mu\Big(\bigcup_{n=0}^{[t+1]} T^{-n}([0,\varepsilon])\Big) - \mu\Big(\bigcup_{n=0}^{[t]} T^{-n}([0,\varepsilon])\Big)$$
$$= \varepsilon\left(1-\frac{1}{r}\right) - \mu\Big(\bigcup_{n=0}^{k_{t\varepsilon}} T^{-n}([0,\varepsilon])\Big) - \theta_1\left(\frac{1+r+\ldots+r^{[t]-s(\varepsilon)+1}}{r^{[t+1]}}\right)\varepsilon +$$
$$+ \mu\Big(\bigcup_{n=0}^{k_{t\varepsilon}} T^{-n}([0,\varepsilon])\Big)\frac{\varepsilon}{r} + \theta_2\left(\frac{1+r+\ldots+r^{[t]-s(\varepsilon)+1}}{r^{[t]}}\right)\frac{\varepsilon}{r}$$

where θ_1 and θ_2 belong to $[-1, 1]$. This yields the equality

$$\mu\Big(\bigcup_{n=0}^{[t+1]} T^{-n}([0,\varepsilon])\Big) - \mu\Big(\bigcup_{n=0}^{[t]} T^{-n}([0,\varepsilon])\Big)$$
$$= \varepsilon\left(1-\frac{1}{r}\right) - \mu\Big(\bigcup_{n=0}^{k_{t\varepsilon}} T^{-n}([0,\varepsilon])\Big)\left(1-\frac{1}{r}\right)\varepsilon + (\theta_2-\theta_1)\frac{r^{[t]-s(\varepsilon)+2}-r}{r-1}\cdot\frac{1}{r^{[t+1]}},$$

which completes the proof of the lemma.

A sequence of functions $\{f_n\}_{n=1}^{\infty}$, $f_n\colon [a, b] \to R$ is said to be *quasi-equicontinuous* on $[a, b]$ if for every $\eta > 0$ there exist n_0 and $\delta > 0$ such that

$$|f_n(x)-f_n(y)| < \eta$$

whenever $|x-y| < \delta$, $x, y \in [a, b]$, and $n > n_0$.

We shall need the following generalization of the Arzela theorem.

THEOREM (Arzela). *If a sequence $\{f_n\}_{n=1}^{\infty}$, $f_n\colon [a, b] \to R$, is uniformly bounded on $[a, b]$ and quasi-equicontinuous on $[a, b]$, then*

(i) $\{f_n\}_{n=1}^{\infty}$ *contains a uniformly convergent subsequence* $\{f_{n_j}\}$,

(ii) $\lim_{j\to\infty} f_{n_j}$ *is a continuous function.*

The proof of this theorem is identical with that of the Arzela theorem in its usual formulation.

LEMMA 3. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of continuous piecewise linear functions from* $[a, b]$ *into* R. *If* $\{f_n\}$ *is uniformly convergent to a function* f *and the sequences* $\{f'_{n+}\}_{n=1}^{\infty}$ *and* $\{f'_{n-}\}_{n=1}^{\infty}$ *of the right and left derivatives of the functions* f_n *converge uniformly to a continuous function* $\bar{f}$, *then* f *has a continuous derivative and*

$$\bar{f} = \lim_{n\to\infty} f'_{n+} = \lim_{n\to\infty} f'_{n-} = f'.$$

Proof. The proof of this lemma is a consequence of the continuity of the limiting function $\bar{f}$ and of the inequalities

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} \leqslant \lim_{n\to\infty} \frac{f_n(x_1)-f_n(x_2)}{x_1-x_2},$$

$$\lim_{n\to\infty} \sup_{k>n} \frac{f_k(x_1)-f_k(x_2)}{x_1-x_2}$$

$$\leqslant \lim_{n\to\infty} \sup_{k>n} \Big\{ \sup_{\theta\in[x_1,x_2]} \{\max\{f'_{k+}(\theta), f'_{k-}(\theta)\}\}\Big\} = \sup_{\theta\in[x_1,x_2]} \bar{f}(\theta),$$

$$\frac{f(x_1)-f(x_2)}{x_1-x_2} \geqslant \lim_{n\to\infty} \frac{f_n(x_1)-f_n(x_2)}{x_1-x_2} \geqslant \lim_{n\to\infty} \inf_{k>n} \frac{f_k(x_1)-f_n(x_2)}{x_1-x_2}$$

$$\geqslant \lim_{n\to\infty} \inf_{k>n} \Big\{ \inf_{\theta\in[x_1,x_2]} \{\min\{f'_{k+}(\theta), f'_{k-}(\theta)\}\}\Big\} = \inf_{\theta\in[x_1,x_2]} \bar{f}(\theta).$$

It is easy to verify the following

LEMMA 4. *Let us consider a family of functions* $\{f_\varepsilon\}_{\varepsilon\in(0,1]}$, f_ε: $[0, \infty) \to [0, 1]$. *If there exists a function* f *such that for any sequence* $\varepsilon_n \to 0$ *and for any interval* $[0, a]$, $a < \infty$, *there exists a subsequence* ε_{n_j} *of* ε_n *such that* $f_{\varepsilon_{n_j}} \to f$ *uniformly on* $[0, a]$, *then there exists* $\lim_{\varepsilon\to 0} f_\varepsilon$ *and* f_ε *converges uniformly to* f *on any interval* $[0, a]$, $a < \infty$.

LEMMA 5. *If a transformation* T: $[0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1, *then for* $z \geqslant 0$

$$\lim_{\varepsilon\to 0} F_\varepsilon\left(\frac{z}{\varepsilon}\right) = 1 - \exp\left(\frac{1-r}{r} z\right),$$

where

$$F_\varepsilon\left(\frac{z}{\varepsilon}\right) = \mu\Big(\bigcup_{n=0}^{[z/\varepsilon]} T^{-n}([0, \varepsilon])\Big).$$

Proof. For $z_1 > z_2$ we have

$$F_\varepsilon\left(\frac{z_1}{\varepsilon}\right)-F_\varepsilon\left(\frac{z_2}{\varepsilon}\right)\leqslant \sum_{n=[z_2/\varepsilon]}^{[z_1/\varepsilon]} \mu(T^{-n}([0, \varepsilon]))$$
$$= \varepsilon\cdot\left(\left[\frac{z_1}{\varepsilon}\right]-\left[\frac{z_2}{\varepsilon}\right]\right)\leqslant z_1-z_2+\varepsilon.$$

Therefore, for any sequence ε_n converging to zero the sequence of functions $F_{\varepsilon_n}(z/\varepsilon_n)$ is quasi-equicontinuous on $[0, \infty)$. Hence, from $F_{\varepsilon_n}(z/\varepsilon_n) \leqslant 1$, by the Arzela theorem it follows that for any $\varepsilon_n \to 0$ and any interval $[0, a]$, $a < \infty$, there exists a subsequence ε_{n_j} such that $F_{\varepsilon_{n_j}}(z/\varepsilon_{n_j})$ converges uniformly on $[0, a]$.

Now, let us assume that $F_{\varepsilon_n}(z/\varepsilon_n)$ converges uniformly on $[0, a]$ as $\varepsilon_n \to 0$, for some $a > 0$.

Denote by $G_n(z)$, $n = 1, 2, \ldots$, the function given by

$$G_n(z) = \begin{cases} F_{\varepsilon_n}(z/\varepsilon_n) & \text{if} \quad z/\varepsilon_n \in N, \\ F'_{\varepsilon_n}(z)\,(z-t_{zn})+F_{\varepsilon_n}([z/\varepsilon_n]) & \text{if} \quad z\in(t_{zn}, t_{zn}+\varepsilon_n), \end{cases} \tag{8}$$

where

$$F'_{\varepsilon_n}(z) = \frac{F_{\varepsilon_n}([z/\varepsilon_n]+1)-F([z/\varepsilon_n])}{\varepsilon_n}$$

and $t_{zn} = [z/\varepsilon_n]\,\varepsilon_n$. Since $|F_{\varepsilon_n}(z/\varepsilon_n)-G_n(z)| \leqslant \varepsilon_n$, the sequence $G_n(z)$ converges uniformly to the same function as $F_{\varepsilon_n}(z/\varepsilon_n)$. If $z/\varepsilon_n \notin N$, then from Lemma 2 and from (8) it follows that there exists $\theta_n \in [-4, 4]$ such that

$$G'_n(z) = \left(1-\frac{1}{r}\right)-\left(1-\frac{1}{r}\right)\mu\Big(\bigcup_{m=0}^{k_{zn}} T^{-m}([0, \varepsilon_n])\Big)+\frac{\theta_n}{r^{s(\varepsilon_n)-1}} \tag{9}$$

where $k_{zn} = [z/\varepsilon_n]-s(\varepsilon_n)$. Similarly, if $z/\varepsilon_n \in N$, then there exist $\theta_n^+, \theta_n^- \in [-4, 4]$ such that the right and left derivative of G_n are given by the formulas

$$G'_{n+}(z) = \left(1-\frac{1}{r}\right)-\left(1-\frac{1}{r}\right)\mu\Big(\bigcup_{m=0}^{k_{zn}} T^{-m}([0, \varepsilon_n])\Big)+\frac{\theta_n^+}{r^{s(\varepsilon_n)-1}}, \tag{10}$$

$$G'_{n-}(z) = \left(1-\frac{1}{r}\right)-\left(1-\frac{1}{r}\right)\mu\Big(\bigcup_{m=0}^{k_{zn}} T^{-m}([0, \varepsilon_n])\Big)+\frac{\theta_n^-}{r^{s(\varepsilon_n)-1}}. \tag{11}$$

It is obvious that

$$\mu\Big(\bigcup_{m=k_{zn}+1}^{[z/\varepsilon_n]} T^{-m}([0, \varepsilon_n])\Big) \leqslant \mu\Big(\bigcup_{m=k_{zn}}^{[z/\varepsilon_n]} T^{-m}([0, \varepsilon_n])\Big) \leqslant s(\varepsilon_n)\,\varepsilon_n \leqslant \frac{s(\varepsilon_n)}{r^{s(\varepsilon_n)-1}}. \tag{12}$$

Therefore, since $F_{\varepsilon_n}(z/\varepsilon_n)$ converges uniformly on $[0, a]$, from (9), (10), (11),

(12) and Lemma 3 it follows that the function $F(z) = \lim_{n\to\infty} F_{\varepsilon_n}(z/\varepsilon_n)$ satisfies for $z > 0$ the equation

$$(13)\qquad F'(z) = \left(1-\frac{1}{r}\right) - \left(1-\frac{1}{r}\right) F(z).$$

It is also easy to verify that $F(z)$ satisfies this equation at $z = 0$. Since

$$F(0) = \lim_{n\to\infty} F_{\varepsilon_n}\left(\frac{0}{\varepsilon_n}\right) = \lim_{n\to\infty} \varepsilon_n = 0,$$

from (13) we obtain

$$F(z) = 1-\exp\left(\frac{1-r}{r}\,z\right) \quad \text{for} \quad z\in[0, a].$$

Hence, by Lemma 4 we obtain the assertion of the lemma.

Denote by P the Frobenius–Perron operator given by the formula

$$\int_A Pf\,d\mu = \int_{T^{-1}(A)} f\,d\mu,$$

which is valid for every measurable set $A \subset [0, 1]$.

It is well known that $P\colon L^1 \to L^1$, whenever T is nonsingular.

Proof of Theorem 1. For any $f \geqslant 0$, $f\in L^1$, $\|f\| = 1$, we have

$$(14)\qquad F_\varepsilon(f, z/\varepsilon) = \mu_f\Big(\bigcup_{n=0}^{[z/\varepsilon]} T^{-n}([0, \varepsilon])\Big) = \int_{B_{z,\varepsilon}} f\,d\mu,$$

where $B_{z,\varepsilon} = \bigcup_{n=0}^{[z/\varepsilon]} T^{-n}([0, \varepsilon])$ and

$$(15)\qquad \Big|\int_{B_{z,\varepsilon}} f\,d\mu - \int_{B_{z,\varepsilon}} d\mu\Big| \leqslant \Big|\int_{B_{z,\varepsilon}} f\,d\mu - \int_{B_{z,\varepsilon}\setminus B_{q\varepsilon,\varepsilon}} f\,d\mu\Big| + \Big|\int_{B_{z,\varepsilon}\setminus B_{q\varepsilon,\varepsilon}} f\,d\mu - \int_{B_{z,\varepsilon}\setminus B_{q\varepsilon,\varepsilon}} d\mu\Big| + \Big|\int_{B_{z,\varepsilon}\setminus B_{q\varepsilon,\varepsilon}} d\mu - \int_{B_{z,\varepsilon}} d\mu\Big|$$
$$\leqslant \int_{B_{q\varepsilon,\varepsilon}} f\,d\mu + \Big|\int_{B_{z-q\varepsilon,\varepsilon}} (P^q f - 1)\,d\mu\Big| + \int_{B_{q\varepsilon,\varepsilon}} d\mu,$$

where $B_{q\varepsilon,\varepsilon} = \bigcup_{n=0}^{q} T^{-n}([0, \varepsilon])$, $q\varepsilon < [z]$. Since the sequence $P^q f$ is strongly convergent to the function $g \equiv 1$ (see [6] or [3]), from (14) and (15) we obtain

$$(16)\qquad \lim_{\varepsilon\to 0}\left|F_\varepsilon\left(f, \frac{z}{\varepsilon}\right) - F_\varepsilon\left(1, \frac{z}{\varepsilon}\right)\right| = 0.$$

From (16) and Lemma 5 we obtain the assertion of the theorem.

4. Final remarks. The proof of the following theorem is the same as the proof of Theorem 1.

THEOREM 2. *Let*

$$T(x) = \begin{cases} 2x & \text{if} \quad x \in [0, \frac{1}{2}], \\ 2-2x & \text{if} \quad x \in (\frac{1}{2}, 1], \end{cases}$$

and let $\xi_\varepsilon(x) = \max\{n\colon T^k(x) > \varepsilon,\ k = 0, 1, \ldots, n-1\}$. *Then for any* $f \in L^1$, $f \geqslant 0$, $\|f\| = 1$,

$$\lim_{\varepsilon \to 0} F_\varepsilon(f, z/\varepsilon) = 1 - \exp(-\tfrac{1}{2}z),$$

where $F_\varepsilon(f, z) = \mu_f\{x\colon \xi_\varepsilon(x) < z\}$, $d\mu_f = f d\mu$.

From Theorem 2 we derive

THEOREM 3. *If* $T(x) = 4x(1-x)$ ($T\colon [0, 1] \to [0, 1]$) *and* $\xi_\varepsilon(x) = \max\{n\colon T^k(x) > \varepsilon,\ k = 0, 1, 2, \ldots, n-1\}$, *then for any* $f \in L^1$, $f \geqslant 0$, $\|f\| = 1$,

$$\lim_{\varepsilon \to 0} F_\varepsilon(f, z/\varepsilon^2) = 1 - \exp(-\tfrac{1}{8}\pi^2 z),$$

where $F_\varepsilon(f, z) = \mu_f\{x\colon \xi_\varepsilon(x) < z\}$, $d\mu_f = f d\mu$.

Proof. Set $T_1(x) = 4x(1-x)$ and

$$T_2(x) = \begin{cases} 2x & \text{if} \quad x \in [0, \frac{1}{2}], \\ 2-2x & \text{if} \quad x \in (\frac{1}{2}, 1]. \end{cases}$$

It is well known that if $\varrho(x) = \frac{2}{\pi}\arcsin\sqrt{x}$, then $T_1 = \varrho \circ T_2 \circ \varrho^{-1}$. We have

$$F_\varepsilon\left(f, \frac{z}{\varepsilon^2}\right) = \mu_f\Big(\bigcup_{n=0}^{k_{z\varepsilon}} T_1^{-n}([0, \varepsilon])\Big) = \mu_f\Big(\bigcup_{n=0}^{k_{z\varepsilon}} \varrho \circ T_2^{-n} \circ \varrho^{-1}([0, \varepsilon])\Big) = \int_{B_{z,\varepsilon}} f d\mu,$$

where

$$B_{z,\varepsilon} = \bigcup_{n=0}^{k_{z\varepsilon}} \varrho \circ T_2^{-n} \circ \varrho^{-1}([0, \varepsilon]).$$

and $k_{z\varepsilon} = [z/\varepsilon^2]$. By Theorem 2 and by the mean value theorem we obtain

$$\lim_{\varepsilon \to 0} \int_{B_{z,\varepsilon}} f d\mu = \lim_{\varepsilon \to 0} \int_{\varrho(B'_{z,\varepsilon})} f d\mu = \lim_{\varepsilon \to 0} \int_{B'_{z,\varepsilon}} f(\varrho(x))\,|\varrho'(x)|\,d\mu = 1 - \exp(-\tfrac{1}{8}\pi^2 z)$$

where $B'_{z,\varepsilon} = \bigcup_{n=0}^{k_{z\varepsilon}} T_2^{-n}\varrho^{-1}([0, \varepsilon]) = \bigcup_{n=0}^{k_{z\eta}} T_2^{-n}([0; \eta])$, $\eta = \varrho^{-1}(\varepsilon)$ and $k_{z\eta} = [z/\varrho(\eta)^2]$.

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