COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 1

AN EXTENSION OF THE STONE REPRESENTATION FOR ORTHOMODULAR LATTICES

BY

W. H. GRAVES AND S. A. SELESNICK (BATON ROUGE, LOUISIANA)

Orthomodular lattices are non-distributive generalizations of Boolean lattices. They have been suggested as the appropriate generalization of the Boolean lattice of propositions in classical mechanics to the algebra of propositions in quantum mechanics and have been studied in this context. Orthomodular lattices also arise in operator theory: the self-adjoint idempotent elements in a W^*-algebra form an orthomodular lattice. These elements play an important part in the decomposition theory of these algebras.

In this note we show how any orthomodular lattice may be represented as the lattice of global sections of a certain sheaf of orthomodular lattices over the Stone space of the center of the lattice. In the case where the lattice is distributive, hence Boolean, then it coincides with its center and our main result (Theorem 3.8) reduces to the Stone representation.

Although similar results have been obtained in a related ring-theoretic context (cf. [8]) it seems profitable to have a description of the lattice theoretic mechanism in the uncoordinatized case. A general treatment of this type of representation for a class of universal algebras appears in [3]. Our construction is independent of both of these.

Applications to dimension lattices will be considered in a later paper.

1. Orthomodular lattices. We recall some basic facts about orthomodular lattices. Our main reference is [6], though we depart slightly from the notation of that paper (see also [1]).

1.1. Definition. An *orthocomplemented set* is a partially ordered set P containing a universal lower bound 0, a universal upper bound 1, and a unary operation $a \mapsto a'$ ($a \in P$) called *orthocomplementation* which, for any $a, b \in P$, satisfies

(1) $a \leqslant b$ implies $b' \leqslant a'$,

(2) $(a')' = a$,

(3) $a \wedge a' = 0$ and $a \vee a' = 1$.

If the supremum of any family $\{a_i\}_{i\in I}$ in P exists, then so also does the infimum of the family $\{a_i'\}_{i\in I}$ and $(\bigvee a_i)' = \bigwedge a_i'$.

The symmetric binary relation $\perp$ on P, defined by $a \perp b$ iff $a \leqslant b'$ or, equivalently, $a \perp b$ iff $b \leqslant a'$, is called the *orthogonality relation* or the orthocomplementation. If $\bigvee a_i$ exists for a family $\{a_i\}_{i\in I}$ in P, and if $b \perp a_i$ for all $i \in I$ and some $b \in P$, then $b \perp \bigvee a_i$.

1.2. Definition. An orthocomplemented set P in which every finite subset of pairwise orthogonal elements has a supremum is called *orthomodular* if $a \leqslant b$ imples $a' \wedge b$ exists and $b = (a' \wedge b) \vee a$.

1.3. Lemma. *Let P be an orthocomplemented lattice. Then, for any $a, b \in P$, the following are equivalent:*

(1) *P is orthomodular,*

(2) *$a \leqslant b$ and $b \wedge a' = 0$ imply $a = b$,*

(3) *$a \leqslant c$ and $b \perp c$ imply $a = (a \vee b) \wedge b'$.*

Proof. (1) $\Rightarrow$ (3). If $a \leqslant c$ and $b \perp c$, then $b \leqslant c' \leqslant a'$. Hence, by (1), $a' = (a' \wedge b') \vee b$ which implies (3).

(3) $\Rightarrow$ (2). If $a \leqslant b$ and $b \wedge a' = 0$, then $b \perp b'$ implies $a = (a \vee b') \wedge \wedge b = (a' \wedge b)' \wedge b = 1 \wedge b = b$.

(2) $\Rightarrow$ (1). If $a \leqslant b$, then $a \vee (b \wedge a') \leqslant b$. But $b \wedge [a \vee (b \wedge a')]' = b \wedge [a' \wedge (b \wedge a')'] = (b \wedge a') \wedge (b \wedge a')' = 0$. Thus $b = a \vee (b \wedge a')$ by (2).

1.4. Definition. Let P be an orthomodular set with $a, b \in P$. Then, we say that *a commutes* with b, written $a \leftrightarrow b$, if $a = (a \wedge b) \vee (a \wedge b')$.

1.5. Lemma. *Let L be an orthomodular lattice (or OML). Then*

(1) *$\leftrightarrow$ is a symmetric reflexive binary relation on L,*

(2) *$a \perp b$ implies $a \leftrightarrow b$,*

(3) *0 and 1 commute with every $a \in L$.*

Proof. (1) To show $\leftrightarrow$ is symmetric suppose $a = (a \wedge b) \vee (a \wedge b')$. Since L is orthomodular, $b = (a \wedge b) \vee [b \wedge (a \wedge b)']$. Thus it suffices to show that $b \wedge (a' \vee b') = b \wedge a'$. Now, $a = (a \wedge b) \vee (a \wedge b')$ implies $a' = (a' \vee b') \wedge (a' \vee b)$, and so

$$\begin{aligned} b \wedge (a' \vee b') &= [b \wedge (a' \vee b)] \wedge (a' \vee b') \\ &= b \wedge [(a' \vee b) \wedge (a' \vee b')] = b \wedge a', \end{aligned}$$

as required.

(2) $a \perp b$ implies $a \leqslant b'$, and so $a \wedge b \leqslant b \wedge b' = 0$. Thus $a \wedge b = 0$, and so $(a \wedge b) \vee (a \wedge b') = 0 \vee a = a$.

(3) is immediate from (2).

1.6. Definition. Let L be a lattice. If the two distributive laws $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ and $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ are satisfied for all permutations of a triple (a, b, c) of elements L, then (a, b, c) is called a *distributive triple.*

The following theorem summarizes Theorems 2, 3 and 7 of [6]:

1.7. THEOREM. *Let L be an OML. Then*

(1) $a \leftrightarrow b$ *iff* $a \wedge b = a \wedge (b \vee a')$,

(2) $a \leftrightarrow b$ *implies* $a' \leftrightarrow b$ *and* $a' \leftrightarrow b'$,

(3) $a_1 \leftrightarrow b, a_2 \leftrightarrow b$ *imply* $a_1 \vee a_2 \leftrightarrow b, a_1 \wedge a_2 \leftrightarrow b$ *and* (a_1, a_2, b) *is a distributive triple.*

If L is complete, then

(4) *if* $a_i \leftrightarrow b$ *for some family* $\{a_i\}_{i \in I}$, *then* $\bigvee a_i \leftrightarrow b$, $\bigwedge a_i \leftrightarrow b$, $\bigvee (a_i \wedge b) = (\bigvee a_i) \wedge b$, *and* $\bigwedge (a_i \vee b) = (\bigwedge a_i) \vee b$.

1.8. Definitions. (1) A subset M of an OML L is called an *OM sublattice* of L if M is a sublattice of L containing 0 and 1, and it is closed under complementation.

(2) Let L be an OML. For a subset $M \subset L$, the set $M^c \equiv \{a \in L \mid a \leftrightarrow b$ for all $b \in M\}$ is called the *commutant* of M. The set $ZL \equiv L^c$ is called the *center* of L. If $ZL = \{0, 1\}$, L is said to be *irreducible.*

The following theorem summarizes Theorem 6 of [6] and its corollaries:

1.9. THEOREM. *Let L be an OML. If $M \subset L$, then M^c is an OM sublattice of L. If L is complete, then so is M^c. The center ZL is a Boolean lattice which is complete and infinitely distributive if L is complete.*

1.10. Definition. A function $h\colon L \to N$, where L and N are OML's, is called an *OML map* or *morphism* if

(1) $h(a \wedge b) = h(a) \wedge h(b)$,

(2) $h(a') = h(a)'$.

If L and N are complete OML's, then h is said to be a *complete* morphism if

(3) $h(\bigwedge a_i) = \bigwedge h(a_i)$ for any family $\{a_i\}_{i \in I}$ in L.

If h is an OML map, then $h(0) = h(a \wedge a') = h(a) \wedge h(a') = h(a) \wedge \wedge h(a)' = 0$, $h(1) = h(0') = h(0)' = 1$, and $h(a \vee b) = h[(a' \wedge b')'] = h(a' \wedge b')' = h(a) \vee h(b)$. Similarly, complete OML maps preserve arbitrary suprema. It is easy to see that bijective OML maps are isomorphisms.

1.11. Definition. Let L be an OML. A set $I \subset L$ is called an *ideal* of L if

(1) $0 \in I$,

(2) $a, b \in I$ implies $a \vee b \in I$,

(3) $a \in I, b \leqslant a$ imply $b \in I$.

A set $J \subset L$ is called a *filter* if

(4) $1 \in J$,

(5) $a, b \in J$ implies $a \wedge b \in J$,

(6) $a \in J$, $b \geqslant a$ imply $b \in J$.

Let $S' = \{a' \mid a \in S\}$ for any set $S \subset L$. If I is an ideal of L, then I' is a filter. If $h: L \to N$ is an OML map, then

$$\ker h = \{a \in L \mid h(a) = 0\}$$

is an ideal of L.

2. Subdirect decomposition and the Stone representation.

2.1. Definition. Let L be an OML. For any maximal ideal m of ZL we define a binary relation $\sim_m$ by $a \sim_m b$ iff there exists a $c \in m'$ such that $a \wedge c = b \wedge c$. Let $[a]_m = \{b \in L \mid b \sim_m a\}$ and let $L_m = \{[a]_m | a \in L\}$ for any $a \in L$.

2.2. Proposition. *With notation as above,*

(1) $a \in m$ *implies* $[a]_m = [0]_m$ *and* $a \in m'$ *implies* $[a]_m = [1]_m$. *Thus, if* $L = ZL$, *we may write* $L_m = \{0, 1\}$, *since* ZL *is a Boolean lattice.*

(2) L_m *is an OML and* $[\ ]_m: L \to L_m$ *is an OML morphism.*

Proof. (1) $a \in m$ implies $a' \in m'$ and $a \wedge a' = 0 \wedge a'$, so $[a]_m = [0]_m$. $a \in m'$ implies $[a]_m = [1]_m$, since $1 \wedge a = a \wedge a$.

(2) If $a \sim_m b$ and $e \sim_m d$, then $a \wedge c_1 = b \wedge c_1$ and $e \wedge c_2 = d \wedge c_2$ for some $c_1, c_2 \in m'$. Let $c = c_1 \wedge c_2 \in m'$ so that $a \wedge c = b \wedge c$ and $e \wedge c = d \wedge c$. Then $(a \wedge e) \wedge c = (b \wedge d) \wedge c$ and

$$\begin{aligned}(a \vee e) \wedge c &= (a \wedge c) \vee (e \wedge c) && \text{from 1.7 (3)}\\ &= (b \wedge c) \vee (d \wedge c)\\ &= (b \vee d) \wedge c && \text{from 1.7 (3),}\end{aligned}$$

so the operations defined by $[a]_m \wedge [e]_m = [a \wedge e]_m$ and $[a]_m \vee [e]_m = [a \vee e]_m$ are well defined and give to L_m the structure of a lattice. If $a \sim_m b$, then $a \wedge c = b \wedge c$ for some $c \in m'$. Hence, by 1.7 (3),

$$a' \wedge c = (a' \vee c') \wedge c = (a \wedge c)' \wedge c = (b \wedge c)' \wedge c = b' \wedge c.$$

Thus the operation $[a]_m \mapsto [a']_m$ is well defined and is easily verified to be an orthocomplementation. If $[a]_m \leqslant [b]_m$, then $a \wedge c \leqslant b \wedge c$ for some $c \in m'$. Hence

$$\begin{aligned}b \wedge c &= (a \wedge c) \vee \{(b \wedge c) \wedge (a \wedge c)'\}\\ &= (a \wedge c) \vee \{(b \wedge c) \wedge (a' \vee c')\}\\ &= (a \wedge c) \vee (b \wedge c \wedge a') \quad \text{by 1.7 (3).}\end{aligned}$$

So

$$\begin{aligned}[b]_m &= [b \wedge c]_m \\ &= [(a \wedge c) \vee (b \wedge a' \wedge c)]_m = [a \wedge c]_m \vee [b \wedge a' \wedge c]_m \\ &= [a]_m \vee ([b]_m \wedge [a]_m')\end{aligned}$$

establishing the orthomodularity of L_m.

We recall a formulation of the Stone representation theorem:

2.3. THEOREM (cf. [1] and [5]). *Let B be a Boolean lattice and let $\mathscr{M}_B$ be the set of (proper) maximal ideals. (An ideal m of B is maximal iff $a \notin m$ implies $a' \in m$ for all $a \in B$.) The map $u\colon B \to 2^{\mathscr{M}_B}$ defined by $u(a) = \{m \in \mathscr{M}_B \mid a \notin m\}$ is an isomorphism (into) and the image of B is a basis of open and closed sets for a compact Hausdorff topology on $\mathscr{M}_B$. For any $m \in \mathscr{M}_B$, the neighbourhood basis at m of open and closed sets is $\{u(a) \mid m \notin u(a)\} = \{u(a) \mid a \notin m'\} = u(m')$.*

We can motivate the construction of the next section by the following considerations. Suppose L is an OML and denote the set of maximal ideals of ZL by $\mathscr{M}_L$. Let $\tilde{\mathscr{L}} = \dot{\bigcup}_{m \in \mathscr{M}_L} L_m$ and let $p\colon \tilde{\mathscr{L}} \to \mathscr{M}_L$ be the natural projection. Then we can write

$$\prod_{m \in \mathscr{M}_L} L_m = \{s\colon \mathscr{M}_L \to \tilde{\mathscr{L}} \mid (ps)(m) = m \text{ for all } m \in \mathscr{M}_L\}.$$

We have a map $\hat{}\colon L \to \prod_{\mathscr{M}_L} L_m$ defined by $\hat{a}(m) = [a]_m$ for $a \in L$ and $m \in \mathscr{M}_L$ which is an OML map when the product is equipped with pointwise operations. If $c \in ZL$, then $\hat{c}(m) = [0]_m$ if $c \in m$ and $\hat{c}(m) = [1]_m$ if $c \notin m$ by 2.2(1). Thus $\hat{c}$ can be regarded as the characteristic function of $u(c)$ and it follows from the Stone representation that $\widehat{ZL}$ is isomorphic to ZL. Moreover, as part (1) of Lemma 2.4 shows, $\hat{}$ is in fact injective.

Subdirect decompositions of this type have, of course, been considered by many writers (for the case of continuous geometries see, for example, [7]; for von Neumann lattices, see [9]). The significance of the result we shall prove in the next section (Theorem 3.8) is that it gives an *explicit* description of $\hat{L}$.

The following lemma is crucial:

2.4. LEMMA. (1) *Suppose $a \in ZL$ and $c, d \in L$ such that $c \leqslant a$, $d \leqslant a$, and $[c]_m = [d]_m$ for all m such that $a \notin m$. Then $c = d$.*

(2) *If $a_i \in ZL$ and $s_i \leqslant a_i$ for $i = 1, \ldots, n$ ($s_i \in L$), and if $s_i \wedge a_j = s_j \wedge a_i$ for all i, j, then there exists $s \leqslant \bigvee a_i$ such that $s \wedge a_j = s_j$ for all j.*

Proof. (1) The hypotheses imply that, for every $m \in u(a)$, there exists an $h_m \in m'$ such that $c \wedge h_m = d \wedge h_m$. Moreover, $m \in u(a)$ implies

$m \in u(h_m)$ so that $u(a) \subset \bigcup_{m \in u(a)} u(h_m)$ and since $u(a)$ is compact, we can choose a finite number of these h_m's, $\{h_i\}_1^n$ say, such that

$$u(a) = \bigcup_i u(h_i) \bigcap u(a) = \bigcup_i u(h_i \wedge a) = u\big(\bigvee_i (h_i \wedge a)\big).$$

Thus

$$a = \bigvee_i (h_i \wedge a) \qquad (2.3)$$

and

$$c = c \wedge a = c \wedge \bigvee_i (h_i \wedge a) = \bigvee_i (c \wedge h_i \wedge a) \qquad (1.7\ (3))$$

$$= \bigvee_i (d \wedge h_i \wedge a) = d \wedge \bigvee_i (h_i \wedge a) = d \wedge a = d.$$

(2) Let $s = \bigvee_i s_i$. Then

$$s \wedge a_j = \big(\bigvee_i s_i\big) \wedge a_j = \bigvee_i (s_i \wedge a_j) \qquad (1.7\ (3))$$

$$= \bigvee_i (s_j \wedge a_i) = s_j \wedge \bigvee_i a_i = s_j.$$

3. Sheaves and pre-sheaves.

3.1. Definitions. (1) A (downward) *directed set* is a partially ordered set I such that, for any pair i, j of elements of I, there is a $k \in I$ such that $k \leqslant i$ and $k \leqslant j$. A subset $J \subset I$ which is itself a directed set with respect to the inherited partial order and with the property that, for each $i \in I$, there exists a $k \in J$ such that $k \leqslant i$ will be called *cofinal.*

(2) A family $\{L_i\}_{i \in I}$ of OML's, where I is a directed set, is a *direct system* of OML's if

(a) $i \leqslant j$ implies there exists an OML map $\pi_{ij}\colon L_j \to L_i$,

(b) $\pi_{ii}\colon L_i \to L_i$ is the identity map,

(c) $\pi_{ij}\pi_{jk} = \pi_{ik}$ if $i \leqslant j \leqslant k$.

(3) A *direct limit* of a direct system $\{\pi_{ij}\colon L_j \to L_i\}_{i,j \in I}$ of OML's is a family $\{\pi_i\colon L_i \to N\}_{i \in I}$ of OML maps which satisfies

(a) $i \leqslant j$ implies $\pi_i \pi_{ij} = \pi_j$,

(b) if $\{f_i\colon L_i \to P\}_{i \in I}$ is any other family of OML's satisfying (a), then there exists a unique OML map $h\colon N \to P$ such that $f_i = h\pi_i$.

It follows from (3), by a familiar argument, that a direct limit, should it exist, is unique up to OML isomorphism. We denote it by $\varinjlim_{i \in I} L_i$. Suppose L is an OML, and let $L_c = [0, c] = \{a \in L \mid 0 \leqslant a \leqslant c\}$ for any $c \in ZL$. Then L_c is an OML (complete if L is); for $c, d \in ZL$ with $c \leqslant d$, $\pi_{cd}\colon L_d \to L_c$, given by $\pi_{cd}(a) = a \wedge c$, L_c is an OML morphism (complete if L is), and the family $\{\pi_{cd}\colon L_d \to L_c\}_{c,d\in m'}$ for $m \in \mathscr{M}_L$ is a direct system of OML maps.

3.2. PROPOSITION.

$$\varinjlim_{c\in m'} L_c \cong L_m.$$

Proof. Define $\pi_c\colon L_c \to L_m$ by $\pi_c(a) = [a]_m$. Then

(a) $c \leqslant d$ implies $\pi_c\pi_{cd}(a) = \pi_c(a \wedge c) = [a \wedge c]_m = [a]_m = \pi_d(a)$;

(b) if $\{f_c\colon L_c \to P\}_{c\in m'}$ is any other family of OML maps satisfying (a), $h\colon L_m \to P$ is defined by $h([a]_m) = f_1(a)$. This is well defined since $[a]_m = [b]_m$ implies $a \wedge c = b \wedge c$ for some $c \in m'$. But then $f_1(a) = f_{a\wedge c}\pi_{a\wedge c,1}(a) = f_{b\wedge c}\pi_{b\wedge c,1}(b) = f_1(b)$. Evidently, $f_c = h\pi_c$ and h is unique with this property.

3.3. Definition. A *sheaf* of OML's is a triple $(\mathscr{S}, p, X)$, where $\mathscr{S}$ and X are topological spaces, $p\colon \mathscr{S} \to X$ is a local homeomorphism and the following conditions are satisfied:

(1) For all $x \in X$, $p^{-1}(x)$ has the structure of an OML.

(2) If $\mathscr{S}\nabla\mathscr{S}$ denotes the set $\{(s,t)\in \mathscr{S}\times\mathscr{S} \mid p(s) = p(t)\}$ equipped with the topology inherited from $\mathscr{S}\times\mathscr{S}$, then the maps

(i) $\vee\colon \mathscr{S}\nabla\mathscr{S} \to \mathscr{S}$,

(ii) $\wedge\colon \mathscr{S}\nabla\mathscr{S} \to \mathscr{S}$,

(iii) $'\colon \mathscr{S} \to \mathscr{S}$,

defined in the obvious way, are continuous.

The OML $p^{-1}(x)$ we call the *stalk* over x. For any subset $W \subset X$, a continuous function $\sigma\colon W \to \mathscr{S}$ with the property that $p\sigma = 1_W$ (or, equivalently, that $\sigma(x) \in p^{-1}(x)$ for all $x \in W$) is called a *section* over W. We call a section over X *global*. The set of all sections over W we denote by $\Gamma(W, \mathscr{S})$. In virtue of 3.3 (2), it is immediate that $\Gamma(W, \mathscr{S})$ has the structure of an OML with respect to pointwise operations.

In this section we show that every OML can be isomorphically represented as the OML of global sections of a sheaf of OML's over $\mathscr{M}_L$, the stalk over each $m \in \mathscr{M}_L$ being the OML L_m. Indeed, we show that the OML of sections over any basic neighbourhood $u(a)$ is isomorphic with $[0, a]$.

We obtain the sheaf in question by generating it from a pre-sheaf.

3.4. Definition. Let X be a topological space. A *pre-sheaf* of OML's on X is an assignment to each open $U \subset X$ of an OML $\mathscr{L}(U)$ such that if V is open and $V \subset U$, we have an OML map $\varrho_{V,U}\colon \mathscr{L}(U) \to \mathscr{L}(V)$ (restriction) such that $W \subset V \subset U$ implies $\varrho_{W,V}\varrho_{V,U} = \varrho_{W,U}$. (We take $\mathscr{L}(\emptyset)$ to be the trivial OML.)

For example, if $\mathscr{S}$ is a sheaf of OML's, the assignment to each open $U \subset X$ of the OML $\Gamma(U, \mathscr{S})$ together with restriction (of sections) constitutes a pre-sheaf of OML's. This example is canonical in the sense that every pre-sheaf $\mathscr{L}$ of OML's for which $\varinjlim_{x \in U} \mathscr{L}(U)$ exists for all $x \in X$ generates a sheaf $\tilde{\mathscr{L}}$ of OML's such that, for all open $U \subset X$, there is a morphism θ_U: $\mathscr{L}(U) \to \Gamma(U, \tilde{\mathscr{L}})$ commuting with restrictions. We proceed to describe this construction. As a set

$$\tilde{\mathscr{L}} = \dot{\bigcup_{x \in X}} \varinjlim_{x \in U} \mathscr{L}(U).$$

Let p: $\tilde{\mathscr{L}} \to X$ denote the obvious projection. Then, for each open set $U \subset X$ and each $s \in \mathscr{L}(U)$, we have a map $\hat{s}$: $U \to \tilde{\mathscr{L}}$ where $\hat{s}(x)$ equals the class of s in $\varinjlim_{x \in U} \mathscr{L}(U)$; evidently, $p\hat{s} = 1_U$. Give to $\tilde{\mathscr{L}}$ the finest topology making the functions $\{\hat{s} \mid s \in \mathscr{L}(U)$, open $U \subset X\}$ continuous. Then

3.5. PROPOSITION. (1) *$\tilde{\mathscr{L}}$ is a sheaf of OML's on X.*

(2) *The map θ_U: $\mathscr{L}(U) \to \Gamma(U, \tilde{\mathscr{L}})$ given by $\theta_U(s) = \hat{s}$ is a morphism for each open $U \subset X$ commuting with restrictions.*

This is standard sheaf theory — we refer to [2] or [4] for proofs. (These works consider sheaves and pre-sheaves of groups, rings and other algebraic objects; however, it is quite clear that the construction works also for pre-sheaves of OML's.)

3.6. PROPOSITION (cf. [2] and [4]). *The map θ_U: $\mathscr{L}(U) \to \Gamma(U, \tilde{\mathscr{L}})$ is an isomorphism iff the following conditions hold:*

(i) *If $U = \bigcup_a U_a$ with U_a open in X, and $s, t \in \mathscr{L}(U)$ are such that $\varrho_{U_a, U}(s) = \varrho_{U_a, U}(t)$ for all a, then $s = t$.*

(ii) *Let $\{U_a\}$ be a family of open sets in X and let $U = \bigcup_a U_a$. If $s_a \in \mathscr{L}(U_a)$ are such that*

$$\varrho_{U_a \cap U_\beta, U_a}(s_a) = \varrho_{U_a \cap U_\beta, U_\beta}(s_\beta),$$

then there exists an element $s \in \mathscr{L}(U)$ such that

$$\varrho_{U_a, U}(s) = s_a \quad \textit{for all } a.$$

Let L be an OML. We proceed to build a pre-sheaf of OML's on $\mathscr{M}_L$, the maximal ideal space of ZL. Recalling that $\{u(a)\}_{a \in ZL}$ is a base for the topology on $\mathscr{M}_L$, define $\mathscr{L}(U)$ for any open $U \subset \mathscr{M}_L$ by

$$\mathscr{L}(U) = \Big\{f \in \prod_{m \in U} L_m \,\Big|\, \exists \{a_i\}_{i \in I} \subset ZL, \text{ and } s_i \in [0, a_i] \text{ such that }$$
$$U = \bigcup_i u(a_i) \text{ and } m \in u(a_i) \text{ implies } f(m) = [s_i]_m\Big\}$$

with pointwise operations. If $U \subset V$, define $\varrho_{U,V}\colon \mathscr{L}(V) \to \mathscr{L}(U)$ by restriction in the obvious way. Then $\mathscr{L}$ is a pre-sheaf of OML's on $\mathscr{M}_L$ and it is easily verified that $\mathscr{L}$ satisfies the conditions of 3.6. Hence, $\theta_U\colon \mathscr{L}(U) \cong \Gamma(U, \tilde{\mathscr{L}})$ for any open set $U \subset \mathscr{M}_L$. In particular, $\mathscr{L}(u(a)) \cong \Gamma(u(a), \tilde{\mathscr{L}})$ for any $a \in ZL$. The usefulness of this construction results from the following

3.7. PROPOSITION. *For each $a \in ZL$ define $\varphi_a\colon [0, a] \to \mathscr{L}(u(a))$ by $\varphi_a(c)(m) = [c]_m$ for $m \in u(a)$, $c \in [0, a]$. Then*

(i) *for $a, b \in ZL$ with $a \leqslant b$, the following diagram commutes:*

$$\begin{tikzcd} \mathscr{L}(u(b)) \arrow[r, "\varrho_{u(a),u(b)}"] & \mathscr{L}(u(a)) \\ {[0, b]} \arrow[u, "\varphi_b"] \arrow[r, "\pi_{ab}"'] & {[0, a]} \arrow[u, "\varphi_a"'] \end{tikzcd}$$

(ii) *$\varphi_a\colon [0, a] \to \mathscr{L}(u(a))$ is an isomorphism for all $a \in ZL$.*

Proof. (i) For $m \in u(a)$, $c \in [0, b]$,

$$(\varrho_{u(a),u(b)}\varphi_b)(c)(m) = (\varphi_b(c))(m) = [c]_m = [c \wedge a]_m \quad \text{since } a \in m'$$
$$= \varphi_a(c \wedge a)(m) = (\varphi_a \pi_{ab})(m).$$

So the diagram commutes.

(ii) To show that φ_a is injective suppose $\varphi_a(c) = \varphi_a(d)$ for $c, d \in [0, a]$. That is, $[c]_m = [d]_m$ for all $m \in u(a)$. Lemma 2.4(1) then implies that $c = d$.

To show that φ_a is surjective suppose $f \in \mathscr{L}(u(a))$. Then, since $u(a)$ is compact, there is a finite number of elements $a_i \in ZL$ and $s_i \in [0, a_i]$ such that $u(a) = \bigcup_i u(a_i)$ and $f(m) = [s_i]_m$ if $m \in u(a_i)$. But then, for all $m \in u(a_i) \cap u(a_j) = u(a_i \wedge a_j)$, $f(m) = [s_i]_m = [s_i \wedge a_i \wedge a_j]_m$ since $a_i \wedge a_j \in m'$ $= [s_i \wedge a_j]_m$ since $s_i \in [0, a_i]$.

Similarly, $f(m) = [s_j \wedge a_i]_m$, and so $[s_i \wedge a_j]_m = [s_j \wedge a_i]_m$ for all $m \in u(a_i \wedge a_j)$. Thus, by Lemma 2.4 (1), $s_i \wedge a_j = s_j \wedge a_i$ and, by Lemma 2.4 (2), there exists an $s \in [0, a]$ such that $s \wedge a_i = s_i$. Hence, for $m \in u(a_i)$,

$$\varphi_a(s)(m) = [s]_m = [s \wedge a_i]_m = [s_i]_m = f(m)$$

as required.

3.8. THEOREM. (i) *The stalk over m in the sheaf $\tilde{\mathscr{L}}$ is isomorphic with L_m.*

(ii) *For $a \in ZL$, $[0, a] \cong \Gamma(u(a), \tilde{\mathscr{L}})$ via the map $c \mapsto \hat{c}$, where $\hat{c}(m) = [c]_m$.*

In particular, with $a = 1$, we have $L \cong \Gamma(\mathscr{M}_L, \tilde{\mathscr{L}})$.

Proof. (i) The stalk over m in $\tilde{\mathscr{L}}$ is $\varinjlim_{m \in U} \mathscr{L}(U)$. Now, the set of basic neighbourhoods of m is cofinal in the directed set of all neighbourhoods of m and it is not hard to see that a limit taken over a cofinal subset is isomorphic to the limit taken over the whole set. Thus

$$\begin{aligned} \varinjlim_{m \in U} \mathscr{L}(U) &\cong \varinjlim_{m \in u(a)} L_a \qquad (3.7(\text{i}),\ (\text{ii})) \\ &= L_m \qquad (3.2). \end{aligned}$$

(ii) This is immediate from 3.7 (i) and 3.6.

If $L = ZL$, then it is immediate from 2.2 (i) that $\tilde{\mathscr{L}}$ is the constant sheaf $\mathscr{M}_L \times \{0, 1\}$ (with $\{0, 1\}$ discretely topologized) and $\Gamma(\mathscr{M}_L, \tilde{\mathscr{L}})$ can be identified with the lattice of continuous functions on $\mathscr{M}_L$ with values in $\{0, 1\}$. Thus we recover the Stone representation.

REFERENCES

[1] G. Birkhoff, *Lattice theory*, Providence 1967.

[2] G. Bredon, *Sheaf theory*, New York 1967.

[3] S. D. Comer, *Representations by algebras of sections over Boolean spaces*, Pacific Journal of Mathematics 38 (1971), p. 29-38.

[4] R. Godement, *Topologie algébrique et théorie des faisceaux*, Paris 1958.

[5] P. R. Halmos, *Lectures on Boolean algebras*, New York 1963.

[6] S. S. Holland, *A Radon-Nikodym theorem in dimension lattices*, Transactions of the American Mathematical Society 108 (1963), p. 66-87.

[7] T. Iwamura, *On continuous geometries I*, Japanese Journal of Mathematics 19 (1944), p. 57-71.

[8] R. S. Pierce, *Modules over commutative regular rings*, Providence 1967.

[9] A. Ramsey, *Decompositions of von Neumann lattices*, Journal of Mathematical Analysis and Applications 20 (1967), p. 480-506.

LOUISIANA STATE UNIVERSITY
BATON ROUGE, LOUISIANA 70803

Reçu par la Rédaction le 12. 6. 1971