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On the uniqueness of initial-value problems for partial differential equations of the first order

by P. Besala (Gdańsk)

Abstract. We prove the uniqueness of solutions of the Cauchy problem for system (1), posed in an unbounded zone, under weaker assumptions concerning functions f^k appearing in (1), viz., the Lipschitz continuity of f^k with respect to variables $u^k_{x_i}$, assumed in [1], is here replaced by Hölder continuity (see inequalities (3) below). We further show by constructing an example that in case of bounded domains the same assumptions on f^k are not sufficient for the Cauchy problem to have at most one solution. This points out a certain curiosity that occurs in initial-value problems of the type considered. At the end we state a theorem on differential inequalities.

1. Let S be the set of points $(t, x) = (t, x_1, \ldots, x_n) \in R^{n+1}$ such that $|t| < a = \text{const} \leqslant \infty$. In the zone S we consider the system

$$u^k_t = f^k(t, x, u, u^k_x) \quad (k = 1, \ldots, N), \tag{1}$$

where $u = (u^1, \ldots, u^N)$, $u^k_x = (u^k_{x_1}, \ldots, u^k_{x_n})$, with the initial conditions

$$u^k(0, x) = \varphi^k(x) \quad (k = 1, \ldots, N). \tag{2}$$

By a solution of problem (1)–(2) in S we mean a function $u(t, x) = (u^1(t, x), \ldots, u^N(t, x))$ which is continuous in S, has first order partial derivatives and satisfies system (1) in $S \cap (t \neq 0)$ together with conditions (2).

Theorem 1. *Assume that functions f^k are defined for $(t, x) \in S$ and for arbitrary values of the remaining arguments and that there exist constants $L \geqslant 0$, $H \geqslant 0$, $0 < \alpha < 1$ such that for $(t, x) \in S$ and any $z = (z^1, \ldots, z^N)$, $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^N)$, $q = (q_1, \ldots, q_n)$, $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_n)$ we have*

$$|f^k(t, x, z, q) - f^k(t, x, \tilde{z}, \tilde{q})| \leqslant L \sum_{j=1}^{N} |z^j - \tilde{z}^j| + H \sum_{i=1}^{n} \max(|q_i - \tilde{q}_i|, |q_i - \tilde{q}_i|^\alpha) \tag{3}$$

$(k = 1, \ldots, N)$. Then problem (1)–(2) admits at most one solution in S.

Proof. Let $u(t, x)$, $v(t, x)$ be two solutions of problem (1)–(2) and denote $w^k(t, x) = u^k(t, x) - v^k(t, x)$. By (1), (3) we have

$$(4)\qquad |w_t^k| \leqslant L \sum_{j=1}^{N} |w^j| + H \sum_{i=1}^{n} \max(|w_{x_i}^k|, |w_{x_i}^k|^\alpha),$$

$$(5)\qquad w^k(0, x) = 0 \quad (k = 1, \ldots, N).$$

It suffices to prove that $w^k(t, x) = 0$ $(k = 1, \ldots, N)$ in that part of S, where $t > 0$, since the case $t < 0$ can be reduced to the previous one by changing the sign of variable t.

Let $b \in (0, a)$. Put (cf. [1])

$$\psi_0(r) = \max_{k} \max_{\substack{0 \leqslant t \leqslant b \\ |x| \leqslant r}} |w^k(t, x)|, \quad \text{where } |x| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}, r \geqslant 0.$$

Let $\psi(r)$ be a function of class C^1, such that $\psi(r) > \psi_0(r), \psi'(r) \geqslant 1$ for $r \geqslant 0$. Evidently such a function exists and we have

$$(6)\qquad |w^k(t, x)| \leqslant \psi(|x|) \quad \text{in } S' = S \cap (0 \leqslant t \leqslant b)\ (k = 1, \ldots, N).$$

For $\varepsilon \in (0, 1)$ and $\beta = (1-\alpha)/\alpha$ we define the auxiliary function

$$h(t, x; \varepsilon) = \varepsilon \exp\big\{\psi\big(\varepsilon^\beta (|x|^2+1)^{1/2} + nHt\big) + (NL+1)t\big\}.$$

Since $\psi(r) \to \infty$ as $r \to \infty$ we have for any fixed $\varepsilon \in (0, 1)$:

$$(7)\qquad \lim_{|x| \to \infty} \frac{\psi(|x|)}{h(t, x; \varepsilon)} = 0$$

uniformly in t. Moreover, h satisfies the inequality

$$(8)\qquad h_t \geqslant (NL+1)h + H \sum_{i=1}^{n} \max(|h_{x_i}|, |h_{x_i}|^\alpha)$$

for $(t, x) \in S'$, $\varepsilon \in (0, 1)$. Indeed, we have

$$(9)\qquad h_t = h(\psi' nH + NL + 1).$$

$$(10)\qquad |h_{x_i}|^\alpha = \big|\varepsilon \exp\{\psi + (NL+1)t\}\, \psi' \varepsilon^\beta x_i (|x|^2+1)^{-1/2}\big|^\alpha$$
$$\leqslant \varepsilon^{(1+\beta)\alpha} \exp\{\psi + (NL+1)t\}\, \psi' = h\psi' \quad \text{and} \quad |h_{x_i}| \leqslant h\psi'.$$

(9), (10) imply (8).

By (6), (7), to each $\varepsilon \in (0, 1)$ there corresponds $\varrho(\varepsilon)$ such that

$$(11)\qquad z^k(t, x) \underset{\text{df}}{=} |w^k(t, x)| - h(t, x; \varepsilon) < 0 \quad (k = 1, \ldots, N)$$

for $0 \leqslant t \leqslant b, |x| \geqslant \varrho$. Keeping ε fixed we shall show that inequalities (11) also hold in the cylinder C: $0 \leqslant t \leqslant b$, $|x| < \varrho$. Now, $h > 0$ and (5) imply (11) for $t = 0$. Therefore, if inequalities (11) were not satisfied in C, there would exist an index k_0 and a point (t^0, x^0) in the interior of C such that

$$(12)\qquad z^{k_0}(t^0, x^0) = 0, \quad z^k(t^0, x^0) \leqslant 0, \quad z^k(t, x) < 0$$
$$\text{in } C \cap (0 \leqslant t < t^0)\ (k = 1, \ldots, N).$$

This implies that the function $|w^{k_0}(t, x)|$, and thereby also $z^{k_0}(t, x)$, has first order derivatives at (t^0, x^0) and that

$$z^{k_0}_{x_i}(t^0, x^0) = 0 \quad (i = 1, \dots, N),\; z^{k_0}_t(t^0, x^0) \geqslant 0. \tag{13}$$

Further, at (t^0, x^0) we have $\big||w^{k_0}|_{x_i}\big| = |w^{k_0}_{x_i}|$ and $|w^{k_0}|_t \leqslant |w^k_t|$. Hence, by (13),

$$|w^{k_0}_{x_i}(t^0, x^0)| = |h_{x_i}(t^0, x^0; \varepsilon)|, \tag{14}$$

$$|w^{k_0}_t(t^0, x^0)| - h_t(t^0, x^0; \varepsilon) \geqslant 0. \tag{15}$$

On the other hand, combining (4), (8), (12) and (14) we find

$$|w^{k_0}_t(t^0, x^0)| - h_t(t^0, x^0; \varepsilon) \leqslant -h(t^0, x^0; \varepsilon) < 0$$

which contradicts (15). Hence inequalities (11) hold true in the whole zone S'. Now letting $\varepsilon \to 0$ we obtain, from (11), $w^k(t, x) = 0$ in S' $(k = 1, \dots, N)$. Since b may be chosen arbitrarily close to a, the proof is complete.

2. It is well known that if in (3) $\alpha = 1$ (i.e. $f^k(t, x, z, q)$ are Lipschitz continuous in z, q) and condition (2) is imposed in a bounded set, the uniqueness holds for solutions defined in a certain bounded domain (see [2], [3]). Now we give an example showing that condition (3), with $\alpha < 1$, does not ensure the uniqueness of the initial-value problem in bounded domains. Namely, consider the equation in one x-variable

$$u_t = |u_x|^\alpha, \quad 0 < \alpha < 1, \tag{16}$$

with the initial condition

$$u(0, x) = 0 \quad \text{for } a < x < b, \tag{17}$$

where at least one of a, b is finite. Since $\big||q|^\alpha - |\tilde{q}|^\alpha\big| \leqslant |q - \tilde{q}|^\alpha$ for arbitrary $q, \tilde{q}$ and $|q - \tilde{q}|^\alpha \leqslant |q - \tilde{q}|$ for $|q - \tilde{q}| \geqslant 1$, therefore (3) is satisfied. Suppose a is finite and let D be any domain containing the segment $I = \{t = 0, a < x < b\}$ and lying in the half-plane $x > x_0$, where $x_0 < a$.

Besides the trivial solution, each function

$$u(t, x) = (1-\alpha)\alpha^{\alpha/(1-\alpha)}|t|^{1/(1-\alpha)}|x - c|^{\alpha/(\alpha-1)}\operatorname{sgn}(t) \tag{18}$$

with parameter $c < x_0$ is a solution of (16)–(17) in D. For some α, (18) is a family of analytic solutions in $D \cup I$.

3. A modification of the proof of Theorem 1 leads to the following theorem on differential inequalities.

THEOREM 2. *Suppose that* $u(t, x) = \big(u^1(t, x), \dots, u^N(t, x)\big)$, $v(t, x) = \big(v^1(t, x), \dots, v^N(t, x)\big)$ *are continuous in the strip* $S = \{(t, x) \in R^{n+1} : 0 \leqslant t < a\}$; *suppose that* u *and* v *have first order partial derivatives and satisfy the inequalities*

$$u^k_t \leqslant f^k(t, x, u, u^k_x), \quad v^k_t \geqslant f^k(t, x, v, v^k_x)$$

for $(t, x) \in S$, $t > 0$ $(k = 1, \ldots, N)$. *Assume* $u^k(0, x) \leqslant v^k(0, x)$ *for arbitrary* x *and* $k = 1, \ldots, N$. *Moreover, assume that functions* $f^k(t, x, z, q)$ *satisfy condition* (3) *and each* f^k *is non-decreasing in the system of variables* $(z^1, \ldots, z^{k-1}, z^{k+1}, \ldots, z^N)$.

Then we have $u^k(t, x) \leqslant v^k(t, x)$ *in* S $(k = 1, \ldots, N)$.

References

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