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ON CONDITIONS UNDER WHICH ISOMETRIES HAVE BOUNDED ORBITS

BY

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1. Introduction. In [4] Kirk asked the following question (P 706): If f is an isometry of a G-space M on itself and if some subsequence of $\{f^n(z)\}$, $z \in M$, is bounded, then is the sequence $\{f^n(z)\}$ bounded?

The purpose of this paper is to give a positive answer to this question. Furthermore, we shall show that in a more general situation than that of the above question, it follows necessarily that the sequence $\{f^n(z)\}$ is bounded.

In Section 2 we give some more notation and discuss elementary properties of some classes of spaces, needed in the next sections. The principal tool used to prove the main results is Lemma 3.1 (Section 3). This lemma gives a sufficient condition for a noncontractive mapping f (see Section 4 for the definition) to have a bounded sequence of iterates of a point. The main results of the present paper, concerning noncontractive mappings and nonexpansive mappings, are proved in Sections 4 and 5.

Throughout this paper, for a nonempty subset A of a metric space M and for number $\eta > 0$, we put

$$\varrho(z, A) = \inf\{\varrho(z, a) \colon a \in A\} \quad \text{for every } z \in M,$$

$$K(A, \eta) = \{z \in M \colon \varrho(z, A) < \eta\},$$

$$\delta(A) = \sup\{\varrho(a, b) \colon a, b \in A\},$$

and cl A denotes the closure of A. If $A = \{z\}$, then we let $K(z, \eta) = K(A, \eta)$.

2. Preliminary concepts and remarks.

2.1. Definition. Let M be a metric space and let z_0 be a point of M. Then

(a) M is said to have *property* $P(\eta, \delta)$ at z_0 (where $\eta, \delta > 0$) if there exists a finite set $F \subset M$ such that $K(z_0, \eta) \subset K(F, \delta)$;

(b) M is said to have *property* $Q(\delta)$ at z_0 (where $\delta > 0$) if there exists an unbounded sequence $\{\eta_n\}$ of positive numbers such that for every n the space M has property $P(\eta_n, \delta\eta_n)$ at z_0.

The following remark is immediate:

2.2. Remark. If a metric space M has property $P(\eta, \delta)$ at z_0, then for each $z \in K(z_0, \eta)$ and for all η_1, δ_1 such that

$$0 < \eta_1 \leqslant \eta - \varrho(z_0, z) \quad \text{and} \quad \delta_1 \geqslant \delta$$

M has property $P(\eta_1, \delta_1)$ at z.

2.3. Lemma. *Let A be a subset a metric space M. If $F \subset M$ is a finite set such that $A \subset K(F, \delta)$, then there exists a finite set $F_1 \subset A$ such that $A \subset K(F_1, 2\delta)$.*

Proof. We can assume that $A \neq \emptyset$ and, replacing F by a suitable subset, that $F = \{x_1, \ldots, x_k\}$ is such that $\varrho(x_i, A) < \delta$, $i = 1, \ldots, k$. Thus, for every $i = 1, \ldots, k$, there exists $a_i \in A$ with $\varrho(a_i, x_i) < \delta$. Hence the set $F_1 = \{a_1, \ldots, a_k\}$ is the required one.

2.4. Proposition. *Let M be a metric space and let $z_0 \in M$.*

(a) *If M has property $P(\eta, \delta)$ (respectively, $Q(\delta)$) at z_0, then each subspace of M containing z_0 has property $P(\eta, 2\delta)$ (respectively, $Q(2\delta)$) at z_0.*

(b) *If M has property $Q(\delta)$ at z_0, then for every $\delta_1 > \delta$ the space M has property $Q(\delta_1)$ at every point.*

Consequently, if M has property $Q(\delta)$ at z_0, then for every $\delta_1 > \delta$ each subspace of M has property $Q(2\delta_1)$ at each of its points.

Proof. (a) follows from Lemma 2.3.

(b) Let $\{\eta_n\}$ be a sequence of positive numbers such that $\lim \eta_n = \infty$ and such that, for every n, M has property $P(\eta_n, \delta\eta_n)$ at z_0. If $z \in M$ and $\delta_1 > \delta$, then there exists an integer n_0 such that

$$\eta_n > \varrho(z_0, z) \quad \text{and} \quad \delta\eta_n < \delta_1 (\eta_n - \varrho(z_0, z))$$

for every $n \geqslant n_0$. Setting $\eta_n^* = \eta_{n_0+n} - \varrho(z_0, z)$, it follows by Remark 2.2 that for each n the space M has property $P(\eta_n^*, \delta_1 \eta_n^*)$ at z. Since the sequence $\{\eta_n^*\}$ is unbounded, this shows that M has property $Q(\delta_1)$ at z. Thus the proof is complete.

2.5. Definition (cf. [1]). A metric space M is said to be *finitely totally bounded* (respectively, *finitely compact*) if each bounded subset of M is totally bounded (respectively, if each bounded and closed subset of M is compact).

2.6. Remarks. (a) For a metric space M and a number $\varepsilon > 0$, we consider the following conditions:

$(1)_\varepsilon$ For each $\eta > 0$ the space M has property $P(\eta, \varepsilon)$ at every point.

$(2)_\varepsilon$ For each bounded set $A \subset M$ there exists a finite set $F \subset A$ such that $A \subset K(F, \varepsilon)$.

$(3)_\varepsilon$ There exists a set $F \subset M$ such that (i) $K(F, \varepsilon) = M$, and (ii) each bounded subset of F is finite.

$(4)_\varepsilon$ There exists a countable decomposition of M, $M = A_0 \cup A_1 \cup \ldots$, such that (i) $\delta(A_n) < \varepsilon$ for every n, and (ii) for each bounded set $A \subset M$ there exists $n \geqslant 0$ with $A \subset A_0 \cup \ldots \cup A_n$.

$(5)_\varepsilon$ For each bounded infinite set $A \subset M$ there exist two distinct points $a, b \in A$ with $\varrho(a, b) < \varepsilon$.

Then we have the following relations:
1° If M satisfies $(1)_\varepsilon$, then it satisfies $(2)_{2\varepsilon}$.
2° If M satisfies $(2)_\varepsilon$, then it satisfies $(3)_\varepsilon$.
3° If M satisfies $(3)_\varepsilon$, then it satisfies $(4)_{2\varepsilon}$.
4° If M satisfies $(4)_\varepsilon$, then it satisfies $(5)_\varepsilon$.
5° If M satisfies $(5)_\varepsilon$, then it satisfies $(1)_\varepsilon$.

Indeed, 1° follows from Lemma 2.3.

2° Fix $z_0 \in M$ and a number $r > 0$. Setting

$$B_1 = K(z_0, r) \quad \text{and} \quad B_n = K(z_0, nr) \backslash K(z_0, (n-1)r)$$

for each integer $n \geqslant 2$, we see that condition $(2)_\varepsilon$ implies that for each $n \geqslant 1$ there exists a finite set $F_n \subset B_n$ with $B_n \subset K(F_n, \varepsilon)$. The set $F = \bigcup_{n=1}^{\infty} F_n$ satisfies (i) and (ii) of $(3)_\varepsilon$.

3° Consider a set $F \subset M$ satisfying (i) and (ii) of $(3)_\varepsilon$. Then F is countable. Setting $F = \{x_0, x_1, \ldots\}$, we infer that the sets $A_n = K(x_n, \varepsilon)$, $n = 0, 1, \ldots$, satisfy (i) of $(4)_{2\varepsilon}$. In order to verify (ii), let $A \subset M$ be bounded. Then so is $B = K(A, \varepsilon)$; hence $n = \max\{i \geqslant 0: x_i \in B\}$ is finite and $A \subset A_0 \cup \ldots \cup A_n$, as desired.

4° is immediate.

5° Let $z_0 \in M$ and let $\eta > 0$ be given. Consider a maximal set $F \subset K(z_0, \eta)$ such that $\varrho(x, y) \geqslant \varepsilon$ for every two distinct points $x, y \in F$. Then $K(z_0, \eta) \subset K(F, \varepsilon)$ and, since F is bounded, condition $(5)_\varepsilon$ implies that F is finite. This means that M has property $P(\eta, \varepsilon)$ at z_0, as desired.

(b) Let M be a metric space. If M satisfies one of the conditions $(1)_\varepsilon$–$(5)_\varepsilon$ for some $\varepsilon > 0$, then for every $\delta > 0$ it has property $Q(\delta)$ at every point.

Indeed, it follows from $(1)_\varepsilon$ that for every $\delta > 0$ and each $\eta > \varepsilon/\delta$ the space M has property $P(\eta, \delta\eta)$ at every point. Hence the assertion follows from the relations 1°–5°.

(c) Using the relations 1°–5°, we infer that for a metric space M the following statements are equivalent:

(i) M is finitely totally bounded.

(ii) M satisfies one of the conditions $(1)_\varepsilon$–$(5)_\varepsilon$ for every $\varepsilon > 0$.

(iii) Every bounded sequence of points of M contains a Cauchy subsequence.

(iv) The completion of M is finitely compact.

(d) From Remarks (b) and (c) we infer the following:

(i) If a metric space M is finitely compact, then it is finitely totally bounded.

(ii) If a metric space M is finitely totally bounded, then for every $\delta > 0$ it has property $Q(\delta)$ at every point.

3. Basic lemma.

3.1. Lemma. *Let $A = \{x_0, x_1, \ldots\}$ be a sequence of points of a metric space M such that*

(a) $\varrho(x_{n+1}, x_{m+1}) \geqslant \varrho(x_n, x_m)$ *for all n, $m \geqslant 0$.*

Assume that there exists a number $\eta > 0$ such that

$(\mathrm{b})_\eta$ *the set $\{i \geqslant 0\colon \varrho(x_i, x_0) < \eta\}$ is infinite;*

$(\mathrm{c})_\eta$ *the subspace $A \subset M$ has property $P(\eta, \eta/2)$ at x_0.*

Then A is bounded.

Proof. By $(\mathrm{c})_\eta$, there exists a finite sequence $i_1, i_2, \ldots, i_r$ of nonnegative integers such that

$$K(x_0, \eta) \cap A \subset \bigcup_{j=1}^{r} K(x_{i_j}, \eta/2). \tag{6}$$

It follows from $(\mathrm{b})_\eta$ and (6) that for some $s \in \{i_1, i_2, \ldots, i_r\}$ the set $\{i \geqslant 0\colon \varrho(x_i, x_s) < \eta/2\}$ is infinite. However, by (a), $\varrho(x_{i-s}, x_0) \leqslant \varrho(x_i, x_s)$ for each $i \geqslant s$, and we infer that

$$\text{the set } \{i \geqslant 0\colon \varrho(x_i, x_0) < \eta/2\} \text{ is infinite.} \tag{7}$$

Now, by (7), we may choose an integer i_0 such that

$$i_0 \geqslant \max\{1, i_1, i_2, \ldots, i_r\}, \tag{8}$$

$$\varrho(x_{i_0}, x_0) < \eta/2. \tag{9}$$

Put

$$T = \bigcup_{j=1}^{i_0} K(x_j, \eta).$$

In order to prove that A is bounded, it suffices to show that $A \subset T$.

Let $n \geqslant 0$ be given. If $n \leqslant i_0$, then obviously $x_n \in T$. Assume that $n > i_0$. By (7), there is $n_0 \geqslant n$ such that $\varrho(x_{n_0}, x_0) < \eta/2$. Thus, by (9), $\varrho(x_{n_0}, x_{i_0}) < \eta$; hence by (a) we get

$$\varrho(x_{n_0-j}, x_{i_0-j}) \leqslant \varrho(x_{n_0}, x_{i_0}) < \eta \quad \text{for } 0 \leqslant j \leqslant i_0.$$

Therefore, we have

$$x_j \in T \quad \text{for } n_0 - i_0 \leqslant j \leqslant n_0 \tag{10}$$

and $x_{n_0-i_0} \in K(x_0, \eta) \cap A$. It follows from (6) that there exists $s \in \{i_1, i_2, \ldots, i_r\}$ such that $\varrho(x_{n_0-i_0}, x_s) < \eta/2$. Hence, by (a), we get

$$(11) \qquad \varrho(x_{n_0-i_0-j}, x_{s-j}) < \eta/2 \quad \text{for } 0 \leqslant j \leqslant \min\{s, n_0-i_0\}.$$

If $s \geqslant n_0-i_0$, then, by (10), (11), (8), and the definition of T, we have $x_j \in T$ for $0 \leqslant j \leqslant n_0$.

If $s < n_0-i_0$, then setting $n_1 = n_0-i_0-s$, by (8) we have $0 < n_1 < n_0$, and using (10), (11), and the definition of T we infer that $x_j \in T$ for $n_1 \leqslant j \leqslant n_0$ and that $\varrho(x_{n_1}, x_0) < \eta/2$.

Therefore, replacing n_0 by n_1 and continuing in this fashion, we finally obtain $x_j \in T$ for each j, $0 \leqslant j \leqslant n_0$. In particular, $x_n \in T$. This shows that $A \subset T$ and the proof of Lemma 3.1 is complete.

3.2. Remark. In the preceding lemma, condition $(\mathrm{b})_\eta$ may be replaced by each of the following:

$(\mathrm{b})_\eta^*$ for some $n \geqslant 0$ the set $\{i \geqslant 0\colon \varrho(x_i, x_n) < \eta\}$ is infinite;

$(\mathrm{b})_\eta^{**}$ the set $\{j-i\colon \varrho(x_i, x_j) < \eta\}$ is infinite.

Indeed, it is clear that $(\mathrm{b})_\eta$ implies $(\mathrm{b})_\eta^*$ and $(\mathrm{b})_\eta^*$ implies $(\mathrm{b})_\eta^{**}$. However, by (a), $\varrho(x_{j-i}, x_0) \leqslant \varrho(x_i, x_j)$ for every $i \leqslant j$; hence $(\mathrm{b})_\eta$ follows from $(\mathrm{b})_\eta^{**}$.

4. Conditions under which noncontractive mappings have bounded orbits.

4.1. Definition. A mapping f (not necessarily continuous) of a metric space M into itself is said to be *noncontractive* if $\varrho(f(x), f(y)) \geqslant \varrho(x, y)$ for all $x, y \in M$.

From Lemma 3.1 we have the following

4.2. THEOREM. *Let f be a noncontractive mapping of a metric space M into itself and let z_0 be a given point of M. Assume that there exists a number $\eta > 0$ such that*

(a) *$\{f^n(z_0)\}$ contains a subsequence which lies in $K(z_0, \eta)$;*

(b) *M has property $P(\eta, \eta/4)$ at z_0.*

Then the sequence $\{f^n(z_0)\}$ is bounded.

Proof. Let $A = \{x_0, x_1, \ldots\}$, where $x_n = f^n(z_0)$ for $n = 0, 1, \ldots$ It suffices to verify that A satisfies conditions (a), $(\mathrm{b})_\eta$, $(\mathrm{c})_\eta$ of Lemma 3.1.

Condition (a) of Lemma 3.1 is a consequence of the fact that f is noncontractive; $(\mathrm{b})_\eta$ follows from (a) above; and $(\mathrm{c})_\eta$ follows from (b) above and (a) of Proposition 2.4. Thus the proof is complete.

The following theorem is an immediate consequence of Theorem 4.2 and Definition 2.1.

4.3. THEOREM. *Let f be a noncontractive mapping of a metric space M into itself and let z_0 be a given point of M. Assume that*

(a) *$\{f^n(z_0)\}$ contains a bounded subsequence;*

(b) *M has property $Q(1/4)$ at z_0.*

Then the sequence $\{f^n(z_0)\}$ *is bounded.*

In view of the discussion in Section 2, we have

4.4. Corollary. *Theorem* 4.3 *remains true if* (b) *is replaced by any of the following conditions*:

(b_1) *there is* δ, $0 < \delta < 1/4$, *such that* M *has property* $Q(\delta)$ *at some point*;

(b_2) *there is* $\varepsilon > 0$ *such that* M *satisfies one of the conditions* $(1)_\varepsilon$–$(5)_\varepsilon$;

(b_3) M *is finitely totally bounded*;

(b_4) M *is finitely compact.*

Proof. It follows from Remarks 2.6 that (b_i) implies (b_{i-1}) for $i = 2, 3, 4$. By (b) of Proposition 2.4, condition (b_1) implies (b) of Theorem 4.3. This completes the proof.

Since each isometry is a noncontractive mapping, we have the following particular case of Theorem 4.3.

4.5. Corollary. *Let* f *be an isometry of a finitely compact metric space* M *into itself such that for some* $z_0 \in M$ *the sequence* $\{f^n(z_0)\}$ *contains a bounded subsequence. Then for every* $z \in M$ *the sequence* $\{f^n(z_0)\}$ *is bounded.*

4.6. Remark. Since every G-space is finitely compact (cf. [1]), Corollary 4.5 yields the positive answer to Kirk's question.

5. Conditions under which nonexpansive mappings have bounded orbits.

5.1. Definition. A mapping f of a metric space M into itself is said to be *nonexpansive* if $\varrho(f(x), f(y)) \leqslant \varrho(x, y)$ for all $x, y \in M$.

5.2. Definition. Let f be a mapping of a metric space M into itself. Then the set M^f is defined by

$$M^f = \bigcup \{w_f(x):\ x \in M\},$$

where

$$w_f(x) = \bigcap_{n \geqslant 0} \operatorname{cl} \{f^i(x):\ i \geqslant n\},$$

and M^f is called the *f-closure* of M (cf. [2] and [3]).

The following facts are proved in [2]. We include the proofs for the sake of completeness.

5.3. Lemma. *Let* f *be a nonexpansive mapping of a metric space* M *into itself. If* $z_0 \in M^f$, *then*

(a) $z_0 \in w_f(z_0)$;

(b) f *maps the set* $\{f^n(z_0)\}$ *isometrically into itself.*

Proof. (a) Let $\varepsilon > 0$ be given. Since $z_0 \in M^f$, there exists $x \in M$ such that $z_0 \in w_f(x)$. Hence there exist integers n, m such that $0 \leqslant n < m$, $\varrho(f^n(x), z_0) < \varepsilon/2$, and $\varrho(f^m(x), z_0) < \varepsilon/2$. Since f^{m-n} is also nonexpansive, we obtain

$$\begin{aligned} \varrho(f^{m-n}(z_0), z_0) &\leqslant \varrho(f^{m-n}(z_0), f^m(x)) + \varrho(f^m(x), z_0) \\ &< \varrho(z_0, f^n(x)) + \varepsilon/2 < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrary, this shows that $z_0 \in w_f(z_0)$.

(b) It follows from (a) above that a sequence $\{n_i\}$ $(1 < n_1 < n_2 < \ldots)$ of integers exists so that

$$\lim_{i \to \infty} f^{n_i}(z_0) = z_0.$$

Since f is nonexpansive, for all $x, y \in \{f^n(z_0)\}$ we have

$$\varrho(f^{n_i}(x), f^{n_i}(y)) \leqslant \varrho(f(x), f(y)) \leqslant \varrho(x, y),$$

and (since f is continuous)

$$\lim_{i \to \infty} \varrho(f^{n_i}(x), f^{n_i}(y)) = \varrho(x, y).$$

Hence $\varrho(f(x), f(y)) = \varrho(x, y)$ for all $x, y \in \{f^n(z_0)\}$.

This completes the proof.

As a consequence of Lemmas 3.1 and 5.3 we obtain the following

5.4. THEOREM. *Let f be a nonexpansive mapping of a metric space M into itself. If there exist a point $z_0 \in M^f$ and a number $\eta > 0$ such that M has property $P(\eta, \eta/4)$ at z_0, then for every $z \in M$ the sequence $\{f^n(z)\}$ is bounded.*

Proof. Let $A = \{x_0, x_1, \ldots\}$, where $x_n = f^n(z_0)$ for $n = 0, 1, \ldots$ In order to prove that for every $z \in M$ the sequence $\{f^n(z)\}$ is bounded it is sufficient to show that A satisfies conditions (a), (b)$_\eta$, and (c)$_\eta$ of Lemma 3.1.

Conditions (a) and (b)$_\eta$ of Lemma 3.1 follow from (b) and (a) of Lemma 5.3, respectively. Condition (c)$_\eta$ is a consequence of (a) of Proposition 2.4 and the fact that M has property $P(\eta, \eta/4)$ at z_0. Thus the proof is complete.

Observe that every nonexpansive mapping of a metric space M can be extended to a nonexpansive mapping of the completion $\bar{M}$ of M. Thus we get the following

5.5. COROLLARY. *Let f be a nonexpansive mapping of a metric space M into itself. Assume that for some $z_0 \in M$ the sequence $\{f^n(z_0)\}$ contains a Cauchy subsequence and that*

(a) *for each $z \in \bar{M}$ there is a number $\eta_z > 0$ such that $\bar{M}$ has property $P(\eta_z, \eta_z/4)$ at z.*

Then for every $z \in M$ the sequence $\{f^n(z)\}$ is bounded.

The following result is an immediate consequence of Corollary 5.5 and Remark (c) of Section 2.6.

5.6. THEOREM. *Let f be a nonexpansive mapping of a finitèly totally bounded metric space M into itself. If for some $z_0 \in M$ the sequence $\{f^n(z_0)\}$ contains a bounded subsequence, then for every $z \in M$ the sequence $\{f^n(z)\}$ is bounded.*

6. Final remarks. 1. It follows from Lemma 3.1 that the assumptions made in many of the results of Sections 4 and 5 for the space M may be replaced by the corresponding (weaker) assumptions for the subspace $\{f^n(z_0)\}$. For example, in Theorem 4.3 it suffices to assume that $\{f^n(z_0)\}$ has property $Q(1/2)$ at z_0, in Theorem 5.4 it suffices to assume that there is a number $\eta > 0$ such that $\{f^n(z_0)\}$ has property $P(\eta, \eta/2)$ at z_0, and in Theorem 5.6 it suffices to assume that $\{f^n(z_0)\}$ is finitely totally bounded.

2. It is interesting to note that condition (a) of Corollary 5.5 follows from any of the following conditions:

(a_1) there exist numbers η and δ such that $0 < \delta < \eta/4$ and that M has property $P(\eta, \delta)$ at every point;

(a_2) there is a number $\varepsilon > 0$ such that M satisfies one of the conditions $(1)_\varepsilon$-$(5)_\varepsilon$;

(a_3) there is a number $\delta > 0$ such that $0 < \delta < 1/4$ and that M has property $Q(\delta)$ at some point.

3. There exists an isometry f of a separable metric space M such that for some $z_0 \in M$ the sequence $\{f^n(z_0)\}$ is unbounded and contains a convergent subsequence. The following example is due to Edelstein (see [3], Theorem 2.1).

Example. Let $f\colon l_2 \to l_2$, where l_2 is the space of all sequences $\{x_n\}$ of complex numbers with

$$\sum_{n=1}^{\infty} |x_n| < \infty \quad \text{and} \quad \|x\| = \Big(\sum_{n=1}^{\infty} |x_n|^2\Big)^{1/2},$$

be defined by

$$f(\{x_n\}) = \{y_n\},$$

where $y_n = \exp\{2\pi i/n!\}(x_n - 1) + 1$ for every $n = 1, 2, \ldots$ Then f is an isometry. If

$$n_k = \frac{1}{2} \sum_{j=1}^{k} (2^j k)!,$$

then $\|f^{n_k}(0)\| \to \infty$ (as $k \to \infty$), while if $n_k = k!$, then $\|f^{n_k}(0)\| \to 0$ (as $k \to \infty$).

It should be noted that (by a theorem of Riesz) a Banach space B satisfies the following condition:

there exist numbers η and δ such that $0 < \delta < \eta$ and that B has property $P(\eta, \delta)$ at some point if and only if B is finite dimensional.

REFERENCES

[1] H. Busemann, *The geometry of geodesics*, New York 1955.

[2] M. Edelstein, *On non-expansive mappings*, Proceedings of the American Mathematical Society 15 (1964), p. 689-695.

[3] — *On non-expansive mappings of Banach spaces*, Proceedings of the Cambridge Philosophical Society 60 (1964), p. 439-447.
[4] W. A. Kirk, *On conditions under which local isometries are motions*, Colloquium Mathematicum 22 (1971), p. 229-233.

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