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COMMON FIXED POINTS AND FIXED EDGES FOR MONOTONE MAPPINGS IN POSETS

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Introduction. One of the most exploited subjects in the field of partially ordered sets (i.e., posets) is the existence of a fixed point of a monotone (increasing or decreasing) mapping of a set into itself. In the literature this question is treated in various forms. The problem of fixed edges is a kind of generalization of the problem of fixed points (see the definition). As for the terminology we assume the notions of an increasing (or isotone) and decreasing (or antitone) to be well known, as well as the notions of a chain, of a complete lattice, of a complete semi-lattice, of a chain complete poset. Here we recall only the definition of a fixed edge (see [2]). Let f be a mapping of a poset P into itself and let $x \leqslant y$ be elements in P. An ordered pair (x, y) is called a *fixed edge* of f if $f(x) = y$ and $f(y) = x$. In this paper we continue the investigations of [1]–[4] concerning the existence of a common fixed point or a fixed edge for a family of isotone or antitone self-mappings of a poset.

Remark. The set of fixed points of a mapping f is denoted by I_f and the set of fixed edges by E_f.

Composed monotone mappings. It is clear that the composition of monotone mappings is again a monotone mapping. In this section we study such compositions from the point of view of fixed points and fixed edges, continuing in this way investigations from [1] and [4].

LEMMA. *Let P be a non-empty set, let f: $P \to P$ be a decomposable mapping, i.e., $f = f_1 f_2 \ldots f_n$ (where $f_i f_j = f_j f_i$ for any $i, j \in \{1, 2, \ldots, n\}$). Then, for any $k \in \{1, 2, \ldots, n\}$, $f_k(I_f) \subset I_f$ and $f_k|I_f$ is a permutation of I_f.*

Proof. For any $k \in \{1, \ldots, n\}$, f_k commutes with f, so

$$f(f_k(x)) = f_k(f(x)) = f_k(x) \quad \text{for any } x \in I_f,$$

hence $f_k(I_f) \subset I_f$.

Let $x, y \in I_f$, $x \neq y$, and $k \in \{1, \ldots, n\}$. Suppose that $f_k(x) = f_k(y)$. Applying $f_1 f_2 \ldots f_{k-1} f_{k+1} \ldots f_n$, we obtain $f(x) = f(y)$ or $x = y$, which is a contradiction.

COROLLARY. *Let P be a non-empty set, let f: $P \to P$ be a decomposable mapping, $f = f_1 f_2 \ldots f_n$ (where $f_i f_j = f_j f_i$ for any $i, j \in \{1, \ldots, n\}$). Then, for any $k \in \{1, \ldots, n\}$ and any $i = 1, 2, \ldots$, $f_k^i(x) \in I_f$ if $x \in I_f$.*

THEOREM 1. *Let P be a poset and let $f\colon P\to P$ be an isotone mapping, a composition of commuting monotone mappings, $f=f_1\ldots f_n$. If $I_f\neq\emptyset$ and $\sup I_f\in I_f$, then there exists $a\in P$ such that*

$$f_{j_1}(a)=\ldots=f_{j_s}(a)$$

for any isotone component of f, and

$$f_{i_1}(a)=\ldots=f_{i_t}(a)$$

for any antitone component of f.

Remark. For the case $f=gh$, where g and h are isotone, this theorem reduces to Theorem 3 of [4].

Proof. 1° Let all f_i be antitone and $a=\sup I_f$. From $a\in I_f$ and $x\leqslant a$ for all $x\in I_f$ it follows that $f_i(a)\leqslant f_i(x)$. Suppose there exists $y\in I_f$ such that $y=f_i(a)$ or $y<f_i(a)$. Then, by the Lemma, there exists $z\in I_f$ such that $y=f_i(z)$, so $y\geqslant f_i(a)$, contradicting the supposition. We conclude that $f_i(a)\leqslant x$ for all $x\in I_f$. In particular, $f_i(a)\leqslant f_j(a)$ for $i\neq j$ ($i,j\in\{1,\ldots,n\}$) by a similar argument. Hence $f_i(a)=f_j(a)$ for all $i,j\in\{1,\ldots,n\}$. The theorem is proved in this case.

2° Let all f_i be isotone. By the Lemma, $f_i(x)\leqslant a$ for all $i\in\{1,\ldots,n\}$. It follows that $f_jf_i(a)\leqslant f_j(a)\leqslant a$, and so

$$f_1f_2\cdots f_n(a)\leqslant f_i(a),$$

which implies $a\leqslant f_i(a)$, hence $f_i(a)=a$ for all $i\in\{1,\ldots,n\}$, proving the theorem also in this case.

3° Let $f_1,f_2,\ldots,f_k$ be antitone and let $f_{k+1},\ldots,f_n$ be isotone. By the argument used in the first part of this proof we have $f_1(a)=f_2(a)=\ldots=f_k(a)$. Since f is isotone, it follows that k is an even number, so $f_1f_2\cdots f_k$ is an isotone mapping. By the Lemma,

$$f_1f_2\cdots f_k(a)\leqslant a.$$

Let $j\in\{k+1,\ldots,n\}$. Then

$$f_jf_1f_2\cdots f_k(a)\leqslant f_j(a)\leqslant a,$$

implying that $a\leqslant f_j(a)$ for all $j\in\{k+1,\ldots,n\}$, which together with $f_j(a)\leqslant a$, argumented in the second part of this proof, gives $f_j(a)=a$. The theorem is completely proved.

Let P_f denote the set $\{x\colon x\leqslant f(x)\}$.

THEOREM 2. *Let P be a complete semi-lattice and let $f=f_1\ldots f_{2n}$ (where all f_i, $i\in\{1,2,\ldots,2n\}$, are monotone and commuting) be an isotone self-mapping of P such that $P_f\neq\emptyset$. Then $P_{f_i}\neq\emptyset$ for all $i\in\{1,2,\ldots,2n\}$. Moreover,*

$$\bigcap_i I_{f_i}\neq\emptyset$$

for f_i isotone, and

$$\bigcap_i E_{f_i} \neq \emptyset$$

for f_i antitone.

Proof. By Tarski's theorem, $I_f \neq \emptyset$, and by the semi-completeness of P, $\sup I_f = a$ exists. By the Lemma, $f_i(x) \in I_f$ for all $x \in I_f$ and all $i = 1, \ldots, 2n$. Hence $f_i(a) \leqslant a$ for all $i = 1, \ldots, 2n$.

Suppose that all f_i are isotone. Then

$$f_1 \ldots f_{i-1} f_{i+1} \ldots f_{2n}(a) \leqslant a,$$

hence $a \leqslant f_i(a)$ or $f_i(a) = a$, which proves that

$$\bigcap_i I_{f_i} \neq \emptyset.$$

If all f_i are antitone, then $f_i(a) \leqslant f_i(x)$ for all $x \in I_f$. Since $f_i | I_f$ is a permutation of I_f, we have $f^2(a) \geqslant f^2(x) \geqslant x$ for all $x \in I_f$. In particular, $f_i^2(a) \geqslant a$. By the Corollary to the Lemma, $f_i^2(a) = a$. Thus $(f_i(a), a)$ is a fixed edge for f_i for all $i \in \{1, \ldots, 2n\}$. But, by Theorem 1, $f_1(a) = \ldots = f_{2n}(a)$, hence

$$\bigcap_i E_{f_i} \neq \emptyset.$$

Let some f_i be isotone and some f_j antitone.

Since f satisfies the condition of Theorem 1, a is a fixed point for isotone components of f, and $f_i(a) = f_j(a)$ for any antitone component of f. As in the second paragraph of this proof we have $f_i^2(a) = a$ for any antitone component of f. It follows that $(f_i(a), a)$ is a common fixed edge for any antitone component of f. The theorem is proved.

A fixed edge theorem. Theorems in [2] assert that antitone self-mappings of a complete lattice have fixed edges. A complete lattice is a poset having the fixed point property and this example can lead to the idea that only antitone self-mappings of posets having the fixed point property must have fixed edges. The following theorem shows that this is not true.

We say that a poset P has the *fixed edge property* if and only if every antitone self-mapping of P has a fixed edge.

THEOREM 3. *Every complete semi-lattice P has the fixed edge property. Moreover, for any antitone self-mapping f of P there exists a minimal (a maximal) element p of P such that $(p, f(p))$ is a fixed edge.*

Proof. We shall prove the following fact: If P is a complete semi-lattice and f an isotone self-mapping of P factorable into two antitone factors, then f has a non-empty set of fixed points, which is a semi-lattice. Let f be isotone and $f = gh$, where g and h are antitone. Consider $h(P)$. Since P is a complete semi-lattice, $\sup h(P) = c$ exists. Since g is also antitone, $g(c) \doteq b \leqslant g(y)$ for all $y \in h(P)$. It follows that $b \leqslant f(x)$ for all $x \in P$. In particular, $f(b) \geqslant b$, and now our assertion follows (see, e.g., the proof of Theorem 1 in [3]).

To prove the theorem, let f be an arbitrary antitone self-mapping of a complete semi-lattice P. Consider f^2. By the proved fact, there exists a non-empty set Q of fixed points for f^2. Let $s = \sup Q$. Then $s \in Q$. By the Lemma, $f|Q$ is a permutation of Q, so it follows that $t = f(s) \leqslant x$ for all $x \in Q$. On the other hand, $f(t) = s$, so (t, s) is a fixed edge for f.

It is clear that t is a minimal element of P such that $(t, f(t))$ is a fixed edge for f. The proof that there exists a maximal p such that $(p, f(p))$ is a fixed edge for f runs as the proof of Theorem 2 in [2].

Weakly commuting families of mappings. Let P be a poset and let F be a family of self-mappings of P. We say that F is a *weakly commuting family* if there exists A, $\emptyset \neq A \subset P$, on which F is commuting and A satisfies the following condition: If $B \subset A$, $B \neq \emptyset$ and $\sup B$ (resp., $\inf B$) exists, then $\sup B \in A$ (resp., $\inf B \in A$). If a family F is weakly commuting on a set A and is not weakly commuting on any set Q strictly containing A, then we say that F is *A-weakly commuting.*

Tarski ([3], Theorem 2) proved that a commuting family of isotone self-mappings of a complete lattice has a non-empty set of common fixed points and this set is also a complete lattice.

The analogous result is not valid for a weakly commuting family of isotone self-mappings of a complete lattice. This is shown by the following example:

EXAMPLE. Let $P = \{0, 1, 2, 3\}$ with the ordinary ordering and let f and g be defined as follows:

$$f\colon\ 0 \to 2,\ 1 \to 2,\ 2 \to 3,\ 3 \to 3,$$

$$g\colon\ 0 \to 1,\ 1 \to 1,\ 2 \to 2,\ 3 \to 2.$$

The mappings f and g commute on $\{0, 1\}$, but there is no common fixed point for both mappings.

A self-mapping f of P is *A-closed* if $x \in A$ implies $f(x) \in A$.

THEOREM 4. *Let P be a complete lattice, $\emptyset \neq A \subset P$, F an A-weakly commuting family of isotone A-closed self-mappings of P, and assume that there exists $a \in P$ such that $\{a, f(a)\}$ is a chain for all $f \in F$. Then the set of common fixed points for F is non-empty and is a complete lattice.*

For the proof see Remark 1 following the next theorem.

THEOREM 5. *Let P be a complete semi-lattice, $\emptyset \neq A \subset P$, let F be an A-weakly commuting family of isotone A-closed self-mappings of P, and assume that there exists $a \in A$ such that $a \leqslant f(a)$ for all $f \in F$. Then the set of common fixed points for F is non-empty and is a complete semi-lattice.*

Proof. Define

$$B = \{x \in A \mid a \leqslant x \leqslant f(x) \text{ and } f(x) \in A \text{ for all } f \in F\}.$$

Since $a \in B$, it follows that B is non-empty. By assumption, $\sup B = c$ exists. Clearly, $a \leqslant x \leqslant g(x) \leqslant c$ for all $x \in B$ and all $g \in F$, hence $c \leqslant g(c)$ for all $g \in F$. Define

$$d = \inf\{f(c) \mid f \in F\}.$$

Evidently, $c \leqslant d$, so for $g \in F$ we have $g(c) \leqslant g(d) \leqslant g(f(c))$. Since $c \in A$, F is commuting on $\{c\}$, so we have

$$g(c) \leqslant g(f(c)) = f(g(c)) \quad \text{or} \quad g(c) \in B \quad \text{for all } g \in F.$$

It follows that $g(c) \leqslant c$, hence $g(c) = c$ for all $g \in F$.

Let M be a non-empty subset of the set of common fixed points for F and let $m = \sup M$. Then every element of M satisfies the condition $x = f(x) \in A$ for all $f \in F$, and applying the above procedure we arrive at the conclusion that m is also a common fixed point for F. So the set of common fixed points is a complete semi-lattice.

Remark 1. A similar proof can be applied for Theorem 4 in the case of $a \leqslant f(a)$. If $a \geqslant f(a)$, then this proof can be applied for the lattice P with the inverse ordering.

THEOREM 6. *Let P be a chain complete poset, $\emptyset \neq A \subset P$, let F be an A-weakly commuting family of A-closed isotone self-mappings of P such that for some $a \in A$ the set $\{a, f(a)\}$ is a chain for all $f \in F$. Then there exists a maximal common fixed point for F.*

Proof. Suppose $a \leqslant f(a)$ for some $a \in A$ and all $f \in F$. (The case of $a \geqslant f(a)$ is treated similarly.) Define

$$B = \{x \in A \mid a \leqslant x \leqslant f(x) \text{ for all } f \in F\}.$$

Since $a \in B$, B is non-empty. Let C be a maximal chain in B. Clearly, C is non-empty, and since P is chain complete, $\sup C = c$ exists. We see at once that $a \leqslant x \leqslant f(x) \leqslant f(c)$ for all $x \in C$ and all $f \in F$. Hence $a \leqslant c \leqslant f(c)$ for all $f \in F$ and the family F is commuting on $\{c\}$. If $g \in F$, then

$$f(c) \leqslant f(g(c)) = g(f(c)) \quad \text{or} \quad f(c) \in B.$$

But C is a maximal chain and c is the supremum of C, so we must have $f(c) \leqslant c$, hence $f(c) = c$ for all $f \in F$. Now, if $b \geqslant c$ and $f(b) = b$ for all $f \in F$, then $b \in B$. If $b > c$, then $C \cup \{b\}$ is again a chain, which contradicts the maximality of C. Thus $b = c$ and c is a maximal common fixed point for the family F.

Weakly commuting families of antitone self-mappings of posets. Now we pass to a weakly commuting family F of antitone self-mappings of a poset, searching for common fixed edges and maximal fixed edges for F.

Even in a finite chain C the fact that a family F is an A-weakly commuting family of self-mappings of C does not imply the existence of a common fixed edge for F. This is shown by the following example:

EXAMPLE. Let $P = \{0, 1, 2, 3\}$ with the natural ordering and let f and g be defined as follows:

$$f\colon\ 0 \to 3,\ 1 \to 2,\ 2 \to 2,\ 3 \to 1,$$

$$g\colon\ 0 \to 3,\ 1 \to 1,\ 2 \to 1,\ 3 \to 2.$$

We see that f and g are commuting on $\{0\}$ but there is no common fixed edge. Thus we assume that the family F is (besides being A-weakly commuting) also A-closed.

THEOREM 7. *Let P be a complete lattice, $\emptyset \neq A \subset P$, and let F be a weakly commuting family of antitone A-closed self-mappings of P. Then there exists a common fixed edge for F. Moreover, there exists a maximal common fixed edge for F.*

Proof. Since $a = \inf A$ and $b = \sup A$ both exist and F is a closed family, it follows that the family

$$F' = \{fg \mid f, g \in F\}$$

of self-mappings of P is a weakly commuting family of isotone closed self-mappings of P. Hence the set I of common fixed points for F' is non-empty. Let $u = \inf I$ and $v = \sup I$. From Theorem 4 it follows that $u, v \in I$, $u \leqslant v$. By the Lemma, $f \mid I$ is a permutation of I for all $f \in F$, hence $f(u) = \sup I = v$ and $f(v) = \inf I = u$ for all $f \in F$ or (u, v) is a common fixed edge for F.

Let

$$B = \{x \in P \mid (x, f(x)) \text{ is a common fixed edge for } F\}.$$

Evidently, $B \neq \emptyset$. Let C be a maximal chain in B and

$$u = \sup C, \qquad v = \inf\{f(x) \mid x \in C\}.$$

Let $x, y \in C$, $x \leqslant y$. Then $f(y) \leqslant f(x)$ or $x \leqslant y \leqslant f(y)$, hence $x \leqslant f(y)$. If $y \leqslant x$, then $x \leqslant f(x) \leqslant f(y)$, so again $x \leqslant f(y)$. We have $x \leqslant f(y)$ for any $x, y \in C$, which implies $u \leqslant v$. From $u \leqslant f(x)$ we have $f^2(x) = x \leqslant f(u)$ for all $x \in C$ and all $f \in F$, so $u \leqslant f(u)$. From $v \leqslant f(x)$ we see that $f(v) \geqslant u$. Consider the complete lattice $[u, v]$. Let $z \in [u, v]$. Then $u \leqslant f(v) \leqslant f(z) \leqslant f(u) \leqslant v$, so the restriction of f on $[u, v]$ is a self-mapping of $[u, v]$ for all $f \in F$. Since $A \cap [u, v] \neq \emptyset$, it follows (according to the first part of the proof) that there exists a common fixed edge (say $(w, f(w))$) for F and $w \in [u, v]$. But then $C \cup \{w\}$ is a chain in B, which contradicts the maximality of C. We must conclude that $u \in C$, $(u, f(u))$ is a common fixed edge for F, and u is a maximal element of P such that $(u, f(u))$ is a common fixed edge for F.

THEOREM 8. *Let P be a complete semi-lattice, $\emptyset \neq A \subset P$, and F be an A-weakly commuting family of antitone self-mappings of P. Then there exists a common fixed edge for F. Moreover, there exists a maximal common fixed edge for P.*

Proof. Let $a = \sup A$ and take a fixed $f \in F$. Since F is A-closed, it follows that $(fg)(a) \leqslant a$ for any $g \in F$. The last relation implies $f(a) \leqslant (fg)f(a)$ or there exists $b \in A$ (in fact, $b = f(a)$) such that $b \leqslant (fg)(b)$ for all $g \in F$. Theorem 5 implies that the set I of common fixed points for the family $F_f = \{fg \mid g \in F\}$ of isotone self-mappings of P is non-empty and forms a semi-lattice with the same ordering as P. Let $v = \sup I$, $u = \inf I$ (u exists because the set I is bounded from below by the element b). It is clear that $b \leqslant u$. We shall show that $u \in I$. Suppose that $u \notin I$ and consider the complete lattice $[b, u]$. Define

$$C = \{x \in [b, u] \mid x \leqslant (fg)(x)\}.$$

C is non-empty since $b \in C$. Let $w = \sup C$. If $w < u$, then (by Tarski's theorem) $w \in I$, which is in contradiction with the definition of u. It follows that $w = u$. As in the proof of Tarski's theorem we conclude that $u \leqslant (fg)(u)$. If $u < fg(u)$, then there exists $x \in I$ such that $u < x < fg(u)$, which implies

$$(fg)(u) \leqslant (fg)(x) = x.$$

The obtained contradiction proves that $(fg)(u) = u$.

Since $g|I$ is a permutation of I for all $g \in F$, we conclude that $g(v) \leqslant g(x)$ for all $x \in I$ and all $g \in F$, or $g(v) = u$. From a similar argument we conclude that $g(u) = v$. Hence (u, v) is a common fixed edge for F.

The proof that there exists a maximal common fixed edge for F can be achieved as in the proof of Theorem 7.

Linked families of mappings.

DEFINITION. Let P be a poset, let F be a family of isotone self-mappings of P, and let G be a family of antitone self-mappings of P. Let I_F denote the set of common fixed points for F, and let E_G denote the set of common fixed edges for G. We say that F and G are *linked families* if

1° $g(I_F) \subset I_F$ for all $g \in G$;

2° $(x, y) \in E_G$ implies $(f(x), f(y)) \in E_G$ for all $f \in F$.

(A self-mapping f of P satisfying the condition 2° is said to be *edge preserving*.)

THEOREM 9. *Let F and G be two commuting linked families of self-mappings of a complete semi-lattice P. Then there exists at least one $c \in P$ such that c is a common fixed point for F and $(c, f(c))$ is a common fixed edge for G.*

Proof. Suppose that $E_G = \emptyset$ and let $(x, g(x))$ be a common fixed edge for G, where $g \in G$. Take an arbitrary $f \in F$. Then $(f(x), f(g(x)))$ is a fixed edge for g, since f is edge preserving. But the edge for g whose first coordinate is $f(x)$ must be $g(f(x))$, so $gf(x) = fg(x)$. The same conclusion is obtained if x is the second coordinate of the considered edge. Now let $I_F \neq \emptyset$, $x \in I_F$, and let $f \in F$ and $g \in G$ be arbitrary. Then $f(g(x)) = g(x)$, since F and G are linked. It is also true that $g(f(x)) = g(x)$, since $x \in I_F$. It follows that the restrictions of elements of F and G commute on I_F as well as on the first or second coordinate of elements of E_G.

By Theorem 8, E_G is non-empty and is bounded from below. Hence

$$\inf\{x \mid (x, y) \in E_G\} = d$$

exists and it is easily seen that $(d, g(d))$ is a fixed edge for all $g \in G$ (a common fixed edge for G). Consider the complete lattice $T = [d, g(d)]$. For any $x \in T$ and all $f \in F$ we have $f(d) \leqslant f(x) \leqslant fg(d)$, since f is isotone. But f is also edge preserving, and d is the least element of P such that $(d, g(d))$ is a common fixed edge for G, so $d \leqslant f(d)$ and $f(g(d)) \leqslant g(d)$. Besides $fg(d) = gf(d)$. It follows that for all $f \in F$ the restriction of f on T is a self-mapping of T. Define $f^* = f \mid T$. The family $F^* = \{f^* \mid f \in F\}$ is a commuting family of isotone self-mappings of the complete lattice T. By Theorem 2 of [3], the set I_{F^*} of common fixed points for F^* is non-empty and is a complete lattice. It follows that $\inf I_{F^*} = m$ and $\sup I_{F^*} = n$ exist in T. According to the first paragraph of this proof it is true that $fg = gf$ on I_{F^*} for any $f \in F$ and any $g \in G$. We shall put

$$g^* = g \mid I_{F^*} \quad \text{and} \quad G^* = \{g^* \mid g \in G\}.$$

According to Theorem 3 in [2], the minimal fixed edge for G^* on I_{F^*} exists. Denote it by (u, v). It is true that $u \leqslant v$, $g^*(u) = v$ and $g^*(v) = u$ for all $g^* \in G^*$, and also $g(u) = v$, $g(v) = u$ for all $g \in G$. But u and v are fixed points for F^*, and therefore for F. The theorem is proved.

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