ON $\Gamma$-REGULAR GRAPHS

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0. We accept the terminology and definitions from the book by Harary (1). In particular, by a graph we mean a pair $G = (V; X)$, where $V$ is a non-empty set called the set of vertices and $X$ is a set of 2-element subsets of $V$ called the set of edges. We do not require $V$ to be finite. Two vertices $v_1$ and $v_2$ are called adjacent if $\{v_1, v_2\} \in X$; in that case we write $v_1 \leftrightarrow v_2$. By a subgraph of $G$ we mean any of the graphs $(V_0; X_0)$, where $\emptyset \neq V_0 \subseteq V$ and $X_0 = \{\{v_1, v_2\} \in X : v_1, v_2 \in V_0\}$. A sequence $v_1, \ldots, v_n$ of different elements of $V$ is called a simple chain from $v_1$ to $v_n$ if $n = 1$ or $n > 1$ and $\{v_i, v_{i+1}\} \in X$ for $1 \leq i < n$. A graph $G$ is called connected if for any two vertices $v, v' \in V$ there exists a simple chain from $v$ to $v'$. A maximal connected subgraph of $G$ is called a component of $G$. The graph $G$ is called bipartite if $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$, $V_1 \neq \emptyset \neq V_2$, and $\{v_1, v_3\} \in X \Rightarrow v_1 \in V_1, v_3 \in V_2$, or $v_1 \in V_2, v_3 \in V_1$. For $v \in V$ we write

$$\Gamma(v) = \{u : u \leftrightarrow v, u \in V\}.$$ 

The number $q(v) = |\Gamma(v)|$ will be called the degree of $v$. A graph $G = (V; X)$ is called $k$-regular ($k \geq 0$) if $q(v) = k$ for each $v \in V$.

In this paper we study a more general notion of regularity (2). Namely, for $v \in V$ we define

$$q_r(v) = \begin{cases} \sum_{u \in \Gamma(v)} q(u) & \text{if } \Gamma(v) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a graph $G = (V; X)$ is $m$-$\Gamma$-regular ($m \geq 0$ and $m$ is an integer) if $q_r(v) = m$ for each $v \in V$. Let us say that a bipartite graph

(1) F. Harary, Graph theory, Addison-Wesley, 1969.
(2) The same notion has been independently introduced also in S. Rama Chadran, Nearly regular graphs and their reconstruction, Graph Theory Newsletter 8 (1978), p. 3 (Note of the Editors).
$(V_1 \cup V_2; X)$ is $(k, s)$-regular if for each $v \in V_1$ we have $\varrho(v) = k$ and for each $u \in V_2$ we have $\varrho(u) = s$.

In this paper we give a characterization of $m$-$\Gamma$-regular graphs.

For a real number $r$ we denote by $[r]$ the integer part of $r$.

1. Observe first that if $G = (V; X)$ is a finite graph, then

$$(i) \quad \sum_{v \in \Gamma(v)} \varrho(v) = \sum_{v \in V} \varrho^2(v).$$

Indeed, since $v \in \Gamma(u)$ for any $u \in \Gamma(v)$, counting the left-hand side of (i) we take the number $\varrho(v)$ as many times as many elements the set $\Gamma(v)$ contains, i.e. $\varrho(v)$ times. From (i) we get

$$(ii) \quad \text{if a graph } G = (\{v_1, \ldots, v_n\}; X) \text{ is } m$-$\Gamma$-regular, then

$$m = \frac{\varrho^2(v_1) + \ldots + \varrho^2(v_n)}{n}.$$

A vertex $v$ of a graph $G = (V; X)$ is called $\Gamma$-regular if for each $u$, $w \in \Gamma(v)$ we have $\varrho(u) = \varrho(w)$. If $v$ is not $\Gamma$-regular, we say that $v$ is non-$\Gamma$-regular.

**Lemma 1.** If a graph $G = (V; X)$ is $m$-$\Gamma$-regular for some $m > 0$ and there exists a non-$\Gamma$-regular vertex $v \in V$ such that $\varrho(v) = k$, then there exists a non-$\Gamma$-regular vertex $v' \in V$ such that $\varrho(v') = k' < k$.

**Proof.** Since $v$ is non-$\Gamma$-regular, we have $k > 0$. Moreover, it follows that $1 < k < m$. In fact, if $k = 1$, then $v$ is regular; if $k = m$, then for any $w \in \Gamma(v)$ we have $\varrho(w) = 1$ and $v$ is regular again. Put $s = [m/k]$. Then

$$(1) \quad ks \leq m,$$

and if $q$ is a positive integer, then

$$(2) \quad k(s + q) > m.$$

Let $\Gamma(v) = \{v_1, \ldots, v_{\varrho(v)}\}$. Since $G$ is $m$-$\Gamma$-regular, we have

$$(3) \quad \varrho(v_1) + \ldots + \varrho(v_{\varrho(v)}) = m.$$

But $v$ is non-$\Gamma$-regular, so there exists $v_i \in \Gamma(v)$ such that $\varrho(v_i) = s + q$ for some positive integer $q$. In fact, if $\varrho(v_j) = s$ for $j = 1, \ldots, \varrho(v)$, then $v$ is regular — a contradiction. If $\varrho(v_j) \leq s$ for $j = 1, \ldots, \varrho(v)$ and $\varrho(v_i) < s$ for some $t \in \{1, \ldots, \varrho(v)\}$, then by (1) we get a contradiction with (3). Put

$$k' = \min \{\varrho(w): w \in \Gamma(v_i)\}.$$
Let $v' \in \Gamma(v_i)$ be a vertex of $V$ such that $\varrho(v') = k'$. Since $v \in \Gamma(v_i)$, $\varrho(v_i) = s + q$, $\varrho(v) = k$, so by (2) and the $m$-$\Gamma$-regularity of $G$ we obtain $k' < k$. We have also

$$k' \leq \frac{m - k}{s + q - 1}.$$  \hfill (4)

In fact, let $\Gamma(v_i) = \{v, w_1, \ldots, w_{s+q-1}\}$. Then

$$\varrho(v) + \varrho(w_1) + \ldots + \varrho(w_{s+q-1}) = m$$

and

$$k' \leq \min \{\varrho(v), \varrho(w_1), \ldots, \varrho(w_{s+q-1})\} \leq \min \{\varrho(w_1), \ldots, \varrho(w_{s+q-1})\}$$

$$\leq \frac{\varrho(w_1) + \ldots + \varrho(w_{s+q-1})}{s + q - 1} = \frac{m - k}{s + q - 1}.$$

We have further

$$\frac{m - k}{s + q - 1} (s + q) < m.$$  \hfill (5)

In fact, assume

$$\frac{m - k}{s + q - 1} (s + q) \geq m.$$  

Then $-k(s+q) \geq -m$ and $k(s+q) \leq m$, which contradicts (2). By (4) and (5) we have

$$k'(s+q) \leq \frac{m - k}{s + q - 1} (s + q) < m.$$  

Thus

$$k'(s+q) < m.$$  \hfill (6)

Now we can prove that $v'$ is non-$\Gamma$-regular. Assume that $v'$ is $\Gamma$-regular. We have $\varrho(v') = k'$ and $v_i \in \Gamma(v')$. Consequently, since $G$ is $m$-$\Gamma$-regular, we get $\varrho_i(v') = k'(s + q) = m$, which contradicts (6).

**Corollary.** If a graph $G = (V; X)$ is $m$-$\Gamma$-regular for $m \geq 0$, then any vertex $v \in V$ is $\Gamma$-regular.

**Proof.** For $m = 0$ the proof is obvious. If $m > 0$, then by Lemma 1 all vertices of $G$ have to be $\Gamma$-regular. Otherwise, using Lemma 1 we obtain an infinite sequence $v, v', v'', \ldots$ of non-$\Gamma$-regular vertices such that $\varrho(v) > \varrho(v') > \varrho(v'') > \ldots$, which is impossible. Let us recall that if $\varrho_i(u) = 1$ or $\varrho_i(u) = 0$, then $u$ is $\Gamma$-regular.

**Lemma 2.** If in a connected graph $G = (V; X)$ any vertex is $\Gamma$-regular and for some vertex $v$ we have $\varrho_i(v) = m \geq 0$, where $m$ is an integer, then
\( G \) is either \( k \)-regular, where \( k = \sqrt{m} \), or \( G \) is a bipartite \((k, s)\)-regular graph, where \( ks = m \).

Proof. If \( m = 0 \), then \( V = \{v\}, X = \emptyset \), and \( G \) is 0-regular. Assume \( m > 0 \). Put \( q(v) = k \). So for \( w \in \Gamma(v) \) we have \( q(w) = m/k = s \). Let \( v' \) be a vertex of \( G \) different from \( v \). Since \( G \) is connected, there exists a simple chain from \( v \) to \( v' \). Let \( v = v_1, v_2, \ldots, v_p = v' \) be such a chain. Since \( v_2 \) is \( \Gamma \)-regular and \( v_1, v_2 \in \Gamma(v_2) \), we obtain \( q(v_2) = q(v_1) = q(v) = k \). Since \( v_3 \) is \( \Gamma \)-regular, we have \( q(v_3) = q(v_4) = s \). In general, \( q(v_{2r}) = s, q(v_{2r-1}) = k \) \((r = 1, \ldots, [p + 1])\). If \( k = s \), then \( q(v') = k \) and \( G \) is \( k \)-regular, where \( k^2 = m \). Let \( k \neq s \). If \( p \) is odd, then \( q(v') = k \), and since \( v' \leftrightarrow v_{p-1} \) and \( q(v_{p-1}) = s \), so by \( \Gamma \)-regularity of \( v' \) we have \( q(w) = s \) for each \( w \in \Gamma(v') \). Thus \( v' \) is adjacent only to \( k \) vertices having degrees equal to \( s \). Analogously, if \( p \) is even, then \( v' \) is adjacent only to \( s \) vertices having degrees equal to \( k \). Now it is enough to put \( V_1 = \{u : q(u) = k\} \) and \( V_2 = \{w : q(w) = s\} \) to see that \( G \) is a bipartite \((k, s)\)-regular graph and \( ks = m \). Thus the proof is complete.

Let \( m \) be a non-negative integer.

Theorem. A graph \( G \) is \( m-\Gamma \)-regular if and only if each of the components of \( G \) is either \( k \)-regular subgraph of \( G \), where \( k^2 = m \), or a \((k, s)\)-regular bipartite subgraph of \( G \), where \( ks = m \).

Proof. The proof of the sufficiency is obvious. The necessity follows from the Corollary and Lemma 2.

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