FINITE SUBGROUPS OF LOCALLY COMPACT GROUPS

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The aim of this note* is to solve a problem of J. Mycielski ([4], P 316, p. 137) and another one put forward by S. Hartman and J. Mycielski ([2], P 214, p. 169).

Mycielski conjectured in [4] that the infinite alternating group (i.e. the group of all even permutations of a countable infinite set) cannot be a subgroup (closed or not) of any locally compact connected group. This is so, by the following

THEOREM. Every simple and periodic (not necessarily closed) subgroup $K$ of a locally compact connected group $G$ is finite.

Proof. If $G$ is a Lie group, the lemma is a consequence of the known fact that

(*) every infinite, periodic subgroup of $GL(n, R)$ contains an infinite abelian normal subgroup ([1], 36. 17, p. 260, and 36. 11, p. 257).

Indeed, let $K$ be simple, periodic and contained in $G$, and let $a: G \rightarrow GL(n, R)$ be the adjoint representation of $G$ in its Lie algebra, where $n = \dim G$. Since $\ker a \subseteq$ Centre $G$, it follows that $K \cap \ker a = \{1\}$ and thus the composite $K \subseteq G \rightarrow GL(n, R)$ is a monomorphism. Thus, by (*), $K$ cannot be infinite.

In the general case, when $G$ is any locally compact connected group, we apply Yamabe's theorem ([5], Th. 5', p. 364) by which $G$ has arbitrarily small normal subgroups $N$ such that $G/N$ is a Lie group. Since $K \cap N$ is either $\{1\}$ or $K$, there must be some $N$ for which $K \cap N = \{1\}$. For such $N$, the composite $K \subseteq G \rightarrow G/N$ is a monomorphism, moreover, $G/N$ being a Lie group, the image of $K$ in $G/N$ is finite, by the above. Thus $K$ is finite.

The problem put forward by J. Mycielski and S. Hartman in [2] was to prove the following

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Theorem. Every periodic discrete subgroup of a connected locally compact group $G$ is finite.

Proof. Note first of all that, by the quoted theorem of Yamabe, the general case reduces to the one where $G$ is a Lie group. In that latter case the lemma is a consequence of ($\ast$) and another known fact that

($\ast\ast$) every abelian discrete subgroup of a connected Lie group is finitely generated ([3], Th. 1', p. 250).

Indeed, let $\Gamma$ be a discrete, periodic subgroup of the locally compact, connected $G$. Denote the centre of $G$ by $Z$. Then $\Gamma \cap Z$ is finite, by ($\ast\ast$). Let $G_1 = G/(\Gamma \cap Z)$ and let $\Gamma_1$ be the image of $\Gamma$ in $G_1$. Then $\Gamma_1$ is still discrete in $G_1$. Moreover, the adjoint representation of $G_1$ on its Lie algebra is faithful on $\Gamma_1$. Hence $\Gamma_1$ is isomorphic to a subgroup of $GL(n, R)$, where $n = \dim G_1$. If $\Gamma_1$ were infinite, then it would contain an infinite abelian subgroup, by ($\ast$). This group would then be a subgroup of $G_1$ which is discrete, periodic, infinite and abelian. But such a group cannot exist, by ($\ast\ast$). Thus $\Gamma_1$ is finite and from $\Gamma_1 \cong \Gamma/(\Gamma \cap Z)$ we infer that $\Gamma$ is finite.

References


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