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Control of quasi-differential equations

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1. Fully controllable linear differential systems. Consider a linear control system

$$(\mathscr{L}) \qquad \dot{x} = A(t)x + B(t)u$$

(note: $\dot{x}(t) = dx/dt$) with state vector x and control vector u:

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \cdot \\ \cdot \\ \cdot \\ x^n \end{bmatrix} \in \boldsymbol{R}^n, \quad u = \begin{bmatrix} u^1 \\ u^2 \\ \cdot \\ \cdot \\ \cdot \\ u^m \end{bmatrix} \in \boldsymbol{R}^m,$$

at each time $t \in \mathscr{J}$, where $\mathscr{J}$ is a prescribed open interval in $\boldsymbol{R}$. Here the coefficient matrices $A(t) \in L^1_{\mathrm{loc}}(\mathscr{J}, \boldsymbol{R}^{n\times n})$ and $B(t) \in L^1_{\mathrm{loc}}(\mathscr{J}, \boldsymbol{R}^{n\times m})$, that is, the corresponding entries

$$a_{ij}(t) \quad \text{for } 1 \leqslant i, j \leqslant n, \quad \text{and} \quad b_{ij}(t) \quad \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$$

are each integrable on every compact subinterval of $\mathscr{J}$.

For each initial state $x_0 \in \boldsymbol{R}^n$, and controller $u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{J}_1, \boldsymbol{R}^{m\times 1})$ for some subinterval $\mathscr{J}_1 = [t_0, t_1] \subset \mathscr{J}$, there exists a unique (absolutely continuous) solution or response $x(t)$ on $\mathscr{J}_1$:

$$x(t) = X(t)x_0 + X(t)\int_{t_0}^{t} X(s)^{-1} B(s)u(s)\,ds.$$

Here, $X(t)$ is the usual fundamental solution matrix defined by

$$\dot{X}(t) = A(t)X(t), \qquad X(t_0) = I.$$

DEFINITION. The linear control system $(\mathscr{L})$ for state vector $x \in \boldsymbol{R}^n$ and controllers $u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{I}_1, \boldsymbol{R}^{m \times 1})$ is called *fully controllable in* $\boldsymbol{R}^n$ *starting at instant* $t_0 \in \mathscr{I}$ in case:

> for each pair of initial state x_0 and target state $x_T \in \boldsymbol{R}^n$ and for each positive duration $\mathscr{I}_1 = [t_0, T] \subset \mathscr{I}$, there exists a controller $u(t)$ on $t_0 \leqslant t \leqslant T$ such that the corresponding response $x(t)$ from $x(t_0) = x_0$ is controlled or steered to $x(T) = x_T$.

If $(\mathscr{L})$ is fully controllable in $\boldsymbol{R}^n$, starting at each initial instant $t_0 \in \mathscr{I}$, then $(\mathscr{L})$ is called *fully controllable on* $\mathscr{I}$.

We recall that the set of attainability from x_0, on the duration $[t_0, T] \subset \mathscr{I}$ is the subset of $\boldsymbol{R}^n$

$$K_{x_0}(t_0, T) = \{X(T)x_0 + \int_{t_0}^{T} X(T)X(s)^{-1}B(s)u(s)ds \mid \text{all controllers } u(s) \text{ on } t_0 \leqslant s \leqslant T\}.$$

The corresponding set of attainability from the origin $x_0 = 0$, say starting at initial time $t_0 = 0$ in $\mathscr{I}$, is denoted $K_0(T)$. In this notation the linear system $(\mathscr{L})$ is fully controllable in $\boldsymbol{R}^n$, starting at the instant $t_0 = 0$, in case

$$K_0(T) = \boldsymbol{R}^n \quad \text{for each } T > t_0 \text{ in } \mathscr{I}.$$

In the autonomous case, where $A(t) = A$ and $B(t) = B$ are constant, and where $\mathscr{I} = \boldsymbol{R}$, the classical necessary and sufficient condition [5] for $(\mathscr{L})$ to be fully controllable on $\mathscr{I}$ is

$$\operatorname{rank}[B, AB, A^2B, \ldots, A^{n-1}B] = n.$$

However, in the case of time-dependent coefficient matrices $A(t)$ and $B(t)$, no similar criterion is available – unless these coefficient matrices are assumed to be highly differentiable [4], [5].

In this note, we present an appropriate sufficiency test for the full controllability of $(\mathscr{L})$, under the hypothesis that $(A(t), B(t))$ have the format of a Zettl–Everitt pair [2], [3], and hence $(\mathscr{L})$ is equivalent to a scalar quasi-differential equation (Q), as will be described in the next Section 2. An extensive treatment of the general control theory of quasi-differential equations can be found in [1], but our treatment seems more direct and elementary. In Section 3 we present new results on bounded controllers and on the bang-bang principle for minimal-time optimal controllers.

2. Controllability of quasi-differential equations. One of the first problems of a first course in control theory concerns the controllability of the real scalar linear differential equation

$$(S) \qquad \chi^{(n)} - a_{n-1}(t)\chi^{(n-1)} - a_{n-2}(t)\chi^{(n-2)} - \ \ldots\ - a_0(t)\chi = b(t)u,$$

with coefficients $a_i(t)$, $b(t)$ in $L^1_{\mathrm{loc}}(\mathscr{I})$ on a prescribed open interval $\mathscr{I} \subset \boldsymbol{R}$, and with controllers $u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{I}_1)$ for various subintervals $\mathscr{I}_1 \subset \mathscr{I}$. It is elementary to verify by direct arguments that (S) is fully controllable in the state space $\boldsymbol{R}^n$ (where the state is specified by $x^1 = \chi$, $x^2 = \chi^{(1)}, \ldots, x^n = \chi^{(n-1)}$, as usual), provided $b(t) \neq 0$ a.e. on $\mathscr{I}$. This is easily seen because we merely interpolate a smooth function $\chi(t)$ between the prescribed initial and final states in $\boldsymbol{R}^n$; and use this response $\chi(t)$ to compute $b(t)u(t)$ and hence the desired controller $u(t)$ (then adjust the controller to be bounded by means of routine techniques involving linearity and approximation).

In this paper we shall provide the control theory for the corresponding linear system with time-varying coefficients; in fact for a more extensive and generalized type of linear system where the coefficient matrices $(A(t), B(t))$ constitute a Zettl–Everitt pair, as defined next. For simplicity of exposition *we henceforth assume that* $t_0 = 0$ *lies in* $\mathscr{I}$, *and that only scalar controllers* $u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{I}, \boldsymbol{R})$ *are involved, that is* $m = 1$.

DEFINITION. The pair of matrices, defined on the open interval $\mathscr{I} \subset \boldsymbol{R}$,

$$A(t) \in L^1_{\mathrm{loc}}(\mathscr{I}, \boldsymbol{R}^{n \times n}) \quad \text{and} \quad B(t) \in L^1_{\mathrm{loc}}(\mathscr{I}, \boldsymbol{R}^{n \times 1})$$

constitute a Zettl–Everitt pair in case they display the format

$$A(t) = \begin{bmatrix} * & a_{12}(t) & 0 & 0\ 0 \ldots & 0 \\ * & * & a_{23}(t) & 0\ 0\ \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & & & a_{n-1,n}(t) \\ * & * & & & * \end{bmatrix}, \qquad B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}.$$

In greater detail the conditions are:

$$a_{i,i+1}(t) \neq 0 \quad \text{a.e. on } \mathscr{I} \text{ for } 1 \leqslant i \leqslant n-1$$

and

$$a_{ij}(t) \equiv 0 \quad \text{a.e. on } \mathscr{I} \text{ for } 1 \leqslant i \leqslant n-2,\ j \geqslant i+2.$$

Remarks.

1. We say that $A(t)$ is a Zettl matrix, $A(t) \in Z_n(\mathscr{I})$, and note that this format provides a generalization of the companion-form matrix arising from the scalar differential equation (S).

2. In the notation of the quasi-derivatives $\chi^{[s]}_A(t)$ for $0 \leqslant s \leqslant n$, for a suitably differentiable real scalar function $\chi(t)$, we can describe such a Zettl–Everitt control system $(\mathscr{L})$ by a scalar quasi-differential control equation

$$(Q) \qquad \chi^{[n]}_A = b(t)u.$$

Here the quasi-derivatives of $\chi(t)$, constructed relative to a prescribed matrix $A(t) = (a_{ij}(t)) \in Z_n(\mathscr{I})$, are defined as follows:

$$\chi^{[0]}_A(t) = \chi(t)$$

which we also denote by $\chi_A^{[0]}(t) = x^1(t)$,

$$\chi_A^{[1]}(t) = a_{12}(t)^{-1}\{\dot{\chi}_A^{[0]}(t) - a_{11}(t)\chi_A^{[0]}(t)\}$$

which we also denote by $\chi_A^{[1]}(t) = x^2(t)$, in accord with the first row of the differential system $(\mathscr{L})$ $\dot{x} = A(t)x + b(t)u$ with Zettl–Everitt pair $(A(t), B(t))$, namely,

$$\dot{x}^1 = a_{11}x^1 + a_{12}x^2 \quad \text{or} \quad x^2 = a_{12}^{-1}\{\dot{x}^1 - a_{11}x^1\}.$$

Continue to define

$$\chi_A^{[2]}(t) = a_{23}^{-1}\{\dot{\chi}_A^{[1]} - (a_{21}\chi_A^{[0]} + a_{22}\chi_A^{[1]})\}$$

which we also denote by $\chi_A^{[2]}(t) = x^3(t)$, in accord with

$$\dot{x}^2 = a_{21}x^1 + a_{22}x^2 + a_{23}x^3,$$

and so forth

$$\vdots$$

$$\chi_A^{[n-1]}(t) = x^n(t).$$

We observe that if $A(t)$ is the companion matrix corresponding to the scalar differential equation (S), then these quasi-derivatives are merely the classical derivatives,

$$\chi_A^{[s]}(t) = \chi^{(s)}(t) \quad \text{for } 0 \leqslant s \leqslant n-1.$$

However, we define $\chi_A^{[n]}(t)$, as based on a general matrix $A(t) \in Z_n(\mathscr{I})$, to include the information of the last row of the linear system $(\mathscr{L})$. Thus we define

$$\chi_A^{[n]}(t) = \dot{\chi}_A^{[n-1]}(t) - \sum_{s=1}^{n} a_{ns}(t)\chi_A^{[s-1]}(t)$$

or equally well

$$\chi_A^{[n]}(t) = \dot{x}^n - a_{n1}(t)x^1 - a_{n2}(t)x^2 - \ldots - a_{nn}(t)x^n.$$

In this notation the first $(n-1)$-rows of the linear system $(\mathscr{L})$ merely define the quasi-derivatives $\chi_A^{[s]}(t)$ for $0 \leqslant s \leqslant n-1$, and the last row can be written as the scalar quasi-differential control equation

$$(Q) \qquad \chi_A^{[n]} = b(t)u.$$

In this sense the two differential control equations $(\mathscr{L})$ and (Q) are merely different notations for the same control system in $\boldsymbol{R}^n$ (see [1] for further details). In this paper we shall usually employ the more familiar notation of the matrix differential system in $\boldsymbol{R}^n$, $(\mathscr{L})$ with the Zettl–Everitt pair $(A(t), B(t))$.

THEOREM 1. *Consider the linear control system $(\mathscr{L})$ in $\boldsymbol{R}^n$, on the open interval $\mathscr{I} \subset \boldsymbol{R}$, where*

$$A(t) \in Z_n(\mathscr{I}), \qquad B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix} \in L^1_{\mathrm{loc}}(\mathscr{I}),$$

so $(A(t), B(t))$ constitute a Zettl–Everitt pair, and the controllers $u(t) \in L^{\infty}_{\mathrm{loc}}(\mathscr{I}_1)$ for intervals $\mathscr{I}_1 \subset \mathscr{I}$.

If $b(t) \neq 0$ a.e. on $\mathscr{I}$, then $(\mathscr{L})$ is fully controllable in the state space $\boldsymbol{R}^n$, starting at any $t_0 \in \mathscr{I}$.

Proof. Take $x_0 = 0$ and fix $t_0 = 0$ in $\mathscr{I}$ for simplicity. The set of attainability, for each $T > 0$ in $\mathscr{I}$, is then

$$K_0(T) = \left\{\int_0^T X(T)X(s)^{-1}B(s)u(s)ds \,\middle|\, \text{all controllers } u(s) \text{ on } [0, T]\right\}.$$

Clearly $K_0(T)$ is a linear subspace of $\boldsymbol{R}^n$. We must prove that $K_0(T) = \boldsymbol{R}^n$.

Suppose to the contrary that $\dim K_0(T) < n$ for some $T > 0$. In this case there exists a constant row vector $\eta_0 \neq 0$ so that $\eta_0 K_0(T) = 0$. Now define $\eta_T \neq 0$ by $\eta_0 = \eta_T X(T)^{-1}$, and then define the row vector function $\eta(t) = \eta_T X(t)^{-1}$, which is a nontrivial solution of the adjoint equation

$$\dot{\eta}(t) = -\eta(t)A(t) \qquad \text{from } \eta(0) = \eta_T \neq 0.$$

But then, for all controllers $u(t)$ on $0 \leqslant t \leqslant T$,

$$\eta_T X(T)^{-1} X(T) \int_0^T X(s)^{-1} B(s)u(s)ds = 0,$$

and hence the integrand must be zero:

$$\eta(s)B(s) \equiv 0 \qquad \text{a.e. on } 0 \leqslant s \leqslant T.$$

Recall that $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$, $B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}$ so we conclude that

$$\eta_n(t)b(t) \equiv 0 \qquad \text{a.e. on } 0 \leqslant t \leqslant T.$$

This implies that $\eta_n(t) \equiv 0$ a.e. on $0 \leqslant t \leqslant T$.

Next examine carefully the structure of the adjoint differential equation:

$$\begin{aligned}
\dot{\eta}_1 &= -\eta_1 a_{11} - \eta_2 a_{21} - \cdots && - \eta_n a_{n1} \\
\dot{\eta}_2 &= -\eta_1 a_{12} - \eta_2 a_{22} - \cdots && - \eta_n a_{n2} \\
\dot{\eta}_3 &= \phantom{-\eta_1 a_{12}} - \eta_2 a_{23} - \cdots && - \eta_n a_{n3} \\
&\vdots \\
\dot{\eta}_{n-1} &= -\eta_{n-2} a_{n-2,n-1} - \eta_{n-1} a_{n-1,n-1} && - \eta_n a_{n,n-1} \\
\dot{\eta}_n &= -\eta_{n-1} a_{n-1,n} && - \eta_n a_{nn}.
\end{aligned}$$

We note that $\eta_n(t) \equiv 0$ a.e., so its derivative is zero, $\dot{\eta}_n(t) \equiv 0$ a.e. on $[0, T]$. But then

$$-\eta_{n-1}(t)a_{n-1,n}(t) \equiv 0 \quad \text{a.e., so } \eta_{n-1}(t) \equiv 0 \text{ a.e. on } [0, T].$$

Since $\eta_{n-1}(t) \equiv 0$ a.e., so its derivative is zero, $\dot{\eta}_{n-1}(t) \equiv 0$ a.e. on $[0, T]$.

From this result, and the $(n-1)$-row of the adjoint differential equation, we deduce $\eta_{n-2}(t) \equiv 0$ a.e.

Continue in this way to show that

$$\eta_n(t) \equiv 0,\ \eta_{n-1}(t) \equiv 0, \ldots, \eta_2(t) \equiv 0 \quad \text{a.e. on } [0, T].$$

From these results, and the second row of the adjoint differential equation, we deduce $\eta_1(t) \equiv 0$ a.e. But this implies that the vector $\eta(t) \equiv 0$ a.e. on $[0, T]$, which contradicts our earlier assertion that $\eta(t)$ is a nontrivial solution of the homogeneous differential equation $\dot{\eta} = -\eta A(t)$ on $0 \leqslant t \leqslant T$.

Hence we conclude that $(\mathscr{L})$ is fully controllable in $\boldsymbol{R}^n$, starting at any instant $t_0 \in \mathscr{I}$, as required. ∎

In the notation of quasi-differential equations the conclusion of Theorem 1 takes the following form.

COROLLARY. *The scalar quasi-differential equation* (Q) *on* $\mathscr{I} \subset \boldsymbol{R}$ *with* $A(t) \in Z_n(\mathscr{I})$ *and* $b(t) \neq 0$ *a.e. on* $\mathscr{I}$, *is fully controllable in the state space* $\boldsymbol{R}^n$, *starting at any initial instant* $t_0 \in \mathscr{I}$.

3. Minimal-time optimal control of quasi-differential equations. In this final section we turn to the problem of minimal-time optimal control of an initial state $x_0 \neq 0$ to the origin in $\boldsymbol{R}^n$, with bounded controllers $u(t)$ in the linear system $(\mathscr{L})$.

Again we assume that the coefficient matrices

$$A(t) \in Z_n(\mathscr{I}), \qquad B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix} \in L^1_{\mathrm{loc}}(\mathscr{I}) \qquad \big(b(t) \neq 0 \text{ a.e. on } \mathscr{I}\big)$$

constitute a Zettl–Everitt pair on the open interval $\mathscr{I} \subset \boldsymbol{R}$. Also for simplicity we assume $t_0 = 0$ lies in $\mathscr{I}$, and the control restraint is defined by

$$|u(t)| \leqslant 1 \qquad \big(\text{so } u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{I})\big).$$

Under these circumstances it is known [5] that the set of attainability $K_0(T)$, from the origin in duration $[0, T]$, is a compact convex subset of $\boldsymbol{R}^n$, and furthermore that $K_0(T)$ varies continuously with $T > 0$ (say, in terms of the Hausdorff metric). In addition the controllability asserted in Theorem 1 guarantees that $K_0(T)$ has non-empty interior, so $K_0(T)$ is a compact convex-body in $\boldsymbol{R}^n$.

For the given linear control system $(\mathscr{L})$ fix an initial state $x_0 \in \boldsymbol{R}^n$ and consider the corresponding set of attainability $K_{x_0}(T)$, from x_0 on $0 \leqslant t \leqslant T$, using the controllers restrained by $|u(t)| \leqslant 1$. Then

$$K_{x_0}(T) = X(T)x_0 + K_0(T) \quad \text{(vector sum)},$$

so $K_{x_0}(T)$ is merely a translate of $K_0(T)$ in $\boldsymbol{R}^n$.

If a controller $u(t)$ on $0 \leqslant t \leqslant T$, with $|u(t)| \leqslant 1$, steers x_0 to a point $x(T)$ on the boundary $\partial K_{x_0}(T)$, then $u(t)$ is called an *extremal controller*, and the corresponding response $x(t)$ on $0 \leqslant t \leqslant T$, is an extremal solution of the control system $(\mathscr{L})$, see [5]. A necessary and sufficient condition that $u(t)$ on $0 \leqslant t \leqslant T$ is an extremal controller is that the Pontryagin Maximal Principle holds – namely:

there exists a nontrivial solution $\eta(t)$ of

$$\dot{\eta} = -\eta A(t)$$

for which

$$\eta(t)B(t)u(t) = \max_{|u| \leqslant 1} \eta(t)B(t)u \qquad \text{a.e. on } [0, T].$$

But since $b(t) \neq 0$ a.e. on $\mathscr{I}$, we note that

$$\eta(t)B(t) = \eta_n(t)b(t) \neq 0 \qquad \text{a.e. on } [0, T],$$

for otherwise, by the same argument as before, the vector $\eta(t) \equiv 0$ on some set of positive measure.

THEOREM 2 (Bang-bang). *Consider the linear control system* $(\mathscr{L})$ in $\boldsymbol{R}^n$, *on the open interval* $\mathscr{I} \subset \boldsymbol{R}$, *where*

$$A(t) \in Z_n(\mathscr{I}), \qquad B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix} \in L^1_{\mathrm{loc}}(\mathscr{I}) \qquad (b(t) \neq 0 \ \textit{a.e. on}\ \mathscr{I}),$$

so $(A(t), B(t))$ *forms a Zettl–Everitt pair. We use controllers* $u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{I}_1)$ *for intervals* $\mathscr{I}_1 \subset \mathscr{I}$, *subject to the restraint* $|u(t)| \leqslant 1$.

If a controller $u(t)$ *on* $0 \leqslant t \leqslant T$ *steers an initial state* $x_0 \in \boldsymbol{R}^n$ *to the boundary* $\partial K_{x_0}(T)$, *then there exists a nontrivial solution* $\eta(t)$ *of*

$$\dot{\eta} = -\eta A(t)$$

so that the extremal controller $u(t)$ *satisfies*

$$u(t) = \operatorname{sgn}[\eta_n(t)b(t)] \qquad \textit{a.e. on}\ 0 \leqslant t \leqslant T.$$

Thus $|u(t)| = 1$ *a.e., which means that* $u(t)$ *is of bang-bang type.*

Proof. The proof of Theorem 2 follows immediately from the Pontryagin Maximal Principle. Since

$$u(t) = \operatorname{sgn}[\eta_n(t)b(t)] \quad \text{a.e. on } 0 \leqslant t \leqslant T,$$

and since $\eta_n(t)b(t) \neq 0$ a.e, we conclude that

$$|u(t)| = 1 \quad \text{a.e. on } [0, T].$$

This asserts that an extremal controller $u(t)$ must be of bang-bang type. ∎

Theorem 2 asserts that an extremal controller $u(t)$ of system $(\mathscr{L})$ assumes only the extreme values ± 1 (a.e.), and moreover, $u(t)$ is thus determined by $\eta(t)$. It is known [5] that under such circumstances the attainable set $K_{x_0}(T)$ is a strictly convex body — that is, each supporting hyperplane meets $K_{x_0}(T)$ in precisely one point. Furthermore each point on the boundary $\partial K_{x_0}(T)$ is attained by means of a unique (a.e.) extremal controller.

For an extremal controller $u(t)$ on $0 \leqslant t \leqslant T$, with

$$u(t) = \operatorname{sgn}[\eta_n(t)b(t)]$$

there can be an infinite number of switches for $u(t)$ between the values $+1$ and -1. But if, in addition, the coefficient matrices $A(t)$, $B(t)$ are real analytic on $\mathscr{I}$, then there are only a finite number of switches for the extremal controller $u(t)$ on $0 \leqslant t \leqslant T$.

We next apply these considerations to the minimal-time optimal control problem, where we seek to steer $x_0 \neq 0$ to the origin in $\boldsymbol{R}^n$, in the minimal possible time $T^* > 0$ by an admissible controller $u(t)$ on $0 \leqslant t \leqslant T^*$, subject to the restraint $|u(t)| \leqslant 1$. This minimal time T^* will correspond to the first instant when the compact set of attainability $K_{x_0}(T)$ hits the origin. Thus $T^* = \inf\{T \mid 0 \in K_{x_0}(T)\}$ is attained at a minimal-time by an extremal controller $u^*(t)$ on $0 \leqslant t \leqslant T^*$, with the extremal response $x^*(t)$ steered from $x^*(0) = x_0$ to $x^*(T^*) = 0$ (provided the control of x_0 to the origin is possible in a finite duration). Since the first time $T^* > 0$ when $K_{x_0}(T)$ hits the origin must occur when $0 \in \partial K_{x_0}(T)$, we conclude that a minimal-time optimal controller $u^*(t)$ on $0 \leqslant t \leqslant T^*$ must be an extremal controller. We summarize these conclusions in Theorem 3.

THEOREM 3. *Consider the linear control system* $(\mathscr{L})$ *in* $\boldsymbol{R}^n$, *on the open interval* $\mathscr{I} \subset \boldsymbol{R}$, *where* $(A(t), B(t))$ *form a Zettl–Everitt pair with* $b(t) \neq 0$ *a.e. on* $\mathscr{I}$, *as before. The controllers* $u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{I}_1)$ *for intervals* $\mathscr{I}_1 \subset \mathscr{I}$, *satisfy the restraint*

$$|u(t)| \leqslant 1.$$

If $x_0 \neq 0$ *can be steered to the origin of* $\boldsymbol{R}^n$ *by some admissible controller, then there exists a minimal-time optimal controller* $u^*(t)$ *on* $0 \leqslant t \leqslant T^*$ *for the minimal-time* $T^* > 0$. *Moreover,* $u^*(t)$ *is unique (a.e.), and* $u^*(t)$ *is necessarily of bang-bang type.*

The next two corollaries give the interpretation of Theorem 3 for scalar differential, and quasi-differential control systems — which are merely notational reformulations of system $(\mathscr{L})$.

COROLLARY 1. *Consider the scalar control system (S) with coefficients* $a_i(t)$, $b(t)$ *in* $L^1_{\mathrm{loc}}(\mathscr{I})$ *and* $b(t) \neq 0$ *a.e. on the open interval* $\mathscr{I} \subset \boldsymbol{R}$. *Controllers* $u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{I}_1)$, *on various intervals* $\mathscr{I}_1 \subset \mathscr{I}$, *satisfy the restraint* $|u(t)| \leqslant 1$.

If an initial state $(x_0^1 = \chi(0), \ldots, x_0^n = \chi^{(n-1)}(0))$ *can be steered to the zero state in* $\boldsymbol{R}^n$, *then there exists a unique (a.e.) minimal-time optimal controller* $u^*(t)$ *on* $0 \leqslant t \leqslant T^*$, *and* $|u^*(t)| = 1$ *a.e., so* $u^*(t)$ *is of bang-bang type.*

COROLLARY 2. *Consider the scalar quasi-differential equation (Q), where* $A(t) \in Z_n(\mathscr{I})$ *and* $b(t) \neq 0$ *a.e. is in* $L^1_{\mathrm{loc}}(\mathscr{I})$, *on the open interval* $\mathscr{I} \subset \boldsymbol{R}$. *Controllers* $u(t) \in L^\infty_{\mathrm{loc}}(\mathscr{I}_1)$, *on various intervals* $\mathscr{I}_1 \subset \mathscr{I}$, *satisfy the restraint* $|u(t)| \leqslant 1$.

If an initial state $(\chi_A^{[0]}(0), \ldots, \chi_A^{[n-1]}(0))$ *can be steered to the zero state in* $\boldsymbol{R}^n$, *then there exists a unique (a.e.) minimal-time optimal controller* $u^*(t)$ *on* $0 \leqslant t \leqslant T^*$, *and* $|u^*(t)| = 1$, *so* $u^*(t)$ *is of bang-bang type.*

As interesting applications of Theorem 3 we relate our theories of controllability to some particular scalar differential equations. In these examples special problems arise in demonstrating the global controllability of an initial state $x_0 \in \boldsymbol{R}^n$ to the zero state, with controllers subject to the restraint $|u(t)| \leqslant 1$.

EXAMPLE 1. Consider the linear control system in $\boldsymbol{R}^n$

$$(\mathscr{L}_p) \qquad \dot{x} = A(t)x + B(t)u \quad \text{on } \mathscr{I} = \boldsymbol{R},$$

where $(A(t), B(t))$ form a Zettl–Everitt pair, with $b(t) \neq 0$ a.e. Use controllers $u(t) \in L^\infty_{\mathrm{loc}}(\boldsymbol{R})$ satisfying the restraint $|u(t)| \leqslant 1$. In this example assume the periodicity conditions,

$$A(t) \equiv A(t+1), \qquad B(t) \equiv B(t+1)$$

and that the corresponding Floquet (Lyapunov) characteristic roots of the system $\dot{X} = A(t)X$ all lie in the left-half complex plane, so that the free $(u(t) \equiv 0)$ homogeneous differential equation is asymptotically stable towards the origin.

Under these circumstances each initial state $x_0 \neq 0$ can be steered to the origin in $\boldsymbol{R}^n$ in some finite duration. To show this null controllability of $(\mathscr{L}_p)$ consider the set of attainability $K_{x_0}(T)$, which is merely a translate of $K_0(T)$. Now note that $K_0(T)$ covers a closed ball $B_0(\varrho)$, centered at the origin in $\boldsymbol{R}^n$ and with some radius $\varrho > 0$, uniformly on $1 \leqslant T \leqslant 2$. Start from initial state $x_0 \neq 0$ at $t_0 = 0$ and observe the free trajectory $x(t) = X(t)x_0$ (corresponding to $u = 0$) for a suitably long duration $0 \leqslant t \leqslant N$. Take the integer N so large that $X(t)x_0$ lies within the ball $B_0(\varrho/2)$ for all $t \geqslant N$ — which is possible since the free motion is asymptotically stable towards the origin.

Next consider the set of attainability $K_{x(N)}(T)$ for $1 \leqslant T \leqslant 2$. Because of the periodicity of the coefficients of $(\mathscr{L}_p)$, $K_{x(N)}(T) \subset K_{x_0}(N+T)$. Thus $K_{x_0}(N+T)$ contains the ball $B_0(\varrho/2)$, and hence x_0 can be steered to the origin in some minimal-time $0 \leqslant t \leqslant T^*$, where $T^* \leqslant N+2$.

EXAMPLE 2. Consider Hill's differential equation

$$\ddot{x} + p(t)x = u$$

for $p(t) = p(t+1) \in L^1_{\mathrm{loc}}(\boldsymbol{R})$, and for controllers $u(t) \in L^\infty_{\mathrm{loc}}(\boldsymbol{R})$ satisfying the restraint $|u(t)| \leqslant 1$. We assume that the Floquet (Lyapunov) characteristic roots are purely imaginary (stable case), so that each solution of the free Hill's equation (with $u = 0$) is almost periodic in $\boldsymbol{R}^2$, see [6]. In this case we shall show that each initial state $x_0 = (x(0), \dot{x}(0))$ can be steered to the origin of $\boldsymbol{R}^2$ in a finite time duration.

As before, note that $K_0(T)$ covers a closed ball $B_0(\varrho)$ about the origin in $\boldsymbol{R}^2$, uniformly on $1 \leqslant T \leqslant 2$. Suppose first that the initial state x_0, at the initial instant $t_0 = 0$, lies within the ball $B_0(\varrho/4)$. Consider the almost periodic solution $x(t) = X(t)x_0$ of the homogeneous ($u = 0$) Hill's differential equation. Give $\varepsilon = \varrho/4$ and then there is a very long ε-almost period L_ε so that $X(L_\varepsilon)x_0$ again lies in $B_0(\varrho/2)$. Then take a positive integer N so that $1 \leqslant L_\varepsilon - N \leqslant 2$, and start to apply the controller $u(t)$ at the instant $t = N$. Then $K_0(T) \subset K_0(N+T)$ for $T = L_\varepsilon - N$ so the set of attainability for x_0

$$K_{x_0}(N+T) = X(L_\varepsilon)x_0 + K_0(N+T)$$

must contain the ball $B_0(\varrho/2)$. Therefore x_0 can be steered to the origin in a finite time $T^* \leqslant N + T = L_\varepsilon$.

Finally take any $x_0 \neq 0$ in $\boldsymbol{R}^2$. By the same type of argument, after some long ε-almost period L_ε, the attainable set $K_{x_0}(L_\varepsilon)$ contains a state γx_0 for some positive constant $\gamma < 1$ which depends only on $K_0(L_\varepsilon)$. Repeat this process several times (note, $\gamma < 1$ is unchanged when $|x_0|$ decreases) until the initial state x_0 is steered into the ball $B_0(\varrho/4)$. Then use the first argument to steer the subsequent trajectory precisely to the origin in a finite time duration.

Therefore each initial state $x_0 \in \boldsymbol{R}^2$ can be steered to the origin in a finite duration of time. Hence there exists a minimal-time optimal controller $u^*(t)$ on $0 \leqslant t \leqslant T^*$, steering x_0 to the origin of $\boldsymbol{R}^2$. As asserted in Theorem 3, $u^*(t)$ is unique (a.e.) and satisfies the bang-bang principle.

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