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On approximate solutions of a functional equation in the class of differentiable functions

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Abstract. In the paper, we have proved an existence theorem for the inequality $\|\varphi(x)-h(x, \varphi[f(x)])\| \leqslant \varepsilon$ in the class C^m of functions fulfilling the system of conditions $\|\varphi^{(r)}(x)-\psi^{(r)}(x)\| \leqslant \varepsilon$ for $r \in \{1, \ldots, m\}$, where f, h are given functions, φ is an unknown function and $\psi(x) = h(x, \varphi[f(x)])$.

Consider the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]) \tag{1}$$

with φ as an unknown function defined on a normed space X and taking values in a normed space Y, and the following problem: under what conditions, on the given functions $f\colon X \to X$ and $h\colon X \times Y \to Y$, for every $\varepsilon > 0$, does there exist a continuous function $\varphi\colon X \to Y$ such that the inequality

$$\|\varphi(x)-h(x, \varphi[f(x)])\| \leqslant \varepsilon$$

holds for every $x \in X$? This problem was considered first by Buck in [3] in connection with the problem of approximation of continuous functions of two variables by those of the form $A(x)+B(x')$. Paper [2] by Baron and Jarczyk brings a general theorem which answers the above question. The aim of this note is to obtain a similar result for functions of the class C^m, where m is a fixed positive integer. As in [2], we base on two extension theorems. The first one is Lemma 2 being a corollary from a result of Sablik proved in [5] as Lemma 0.4 (Lemma 1). The second one concerns extending solutions of functional equations from a neighbourhood of a distinguished point (Lemma 3).

Our considerations concern in fact Banach spaces satisfying the following condition:

(C) There exist a functional $q\colon X \to R$ of the class C^∞, and a positive integer N such that

$$\bigvee_{c\in[1,+\infty)} \bigwedge_{x\in X} (\|x\|^N \leqslant q(x) \leqslant c\|x\|^N),$$

$$\bigvee_{d\in(0,+\infty)} \bigwedge_{k\in\{1,\ldots,N\}} \bigwedge_{x\in X} (\|q^{(k)}(x)\| \leqslant d\|x\|^{N-k})$$

and $q^{(k)} = 0$ for every positive integer $k > N$.

Condition (C) was found by M. Sablik to get an extension lemma quoted as Lemma 1. It has been also pointed out by M. Sablik that (cf. [5], § 0, Examples): (1) every real Hilbert space (with the square of the norm as q) satisfies condition (C); (2) every space L^{2p} of real functions integrable with $2p$-th power satisfies condition (C) (with $q = \|\ \|^{2p}$).

LEMMA 1 (M. Sablik). *Assume that X and Y are Banach spaces and let condition* (C) *be fulfilled.*

If the function F, defined on a neighbourhood of the origin of the space X and taking values in the space Y, is of the class C^m and $F(0) = 0$, then, for every $\varepsilon > 0$, there exist $\delta > 0$, $\theta \in (0, 1)$ and $L > 0$ such that for every $\varrho \in (0, \delta)$ there exists a function $G\colon X \to Y$ of the class C^m and such that

$$G(x) = F(x) \qquad \text{for } \|x\| < \theta\varrho,$$

$$G(x) = F'(0)x \qquad \text{for } \|x\| > \varrho,$$

$$\|G'(x)\| \leqslant \|F'(0)\| + \varepsilon \qquad \text{for } x \in X,$$

$$\|(G - F'(0))^{(r)}(x)\| \leqslant L \sum_{j=0}^{r} \varrho^{j-r} \|(F - F'(0))^{(j)}(x)\|$$

for $r \in \{1, \ldots, m\}$ and $\|x\| < \varrho$; moreover, if $F'(0)$ is a bijection and ε is small enough, then G is a bijection and $\|(G^{-1})'(x)\| \leqslant 1/(\|F'(0)^{-1}\|^{-1} - \varepsilon)$ for every $x \in X$.

In fact, Lemma 1 is not exactly the one stated in paper [5] as Lemma 0.4 but, in fact, the present stronger result was proved there.

From this theorem we shall derive the following corollary:

LEMMA 2. *Assume that X and Y are Banach spaces and let condition* (C) *be fulfilled.*

If the function F, defined on a neighbourhood of a point $\xi \in X$ and taking values in the space Y, is of the class C^m and if

(2) $$F^{(r)}(\xi) = 0 \qquad \text{for } r = 0, 1, \ldots, m,$$

then, for every $\varepsilon > 0$, there exists a function $G\colon X \to Y$ of the class C^m such that $G(x) = F(x)$ on a neighbourhood of the point ξ and

(3) $$\|G^{(r)}(x)\| \leqslant \varepsilon \qquad \text{for } x \in X,\ r = 0, 1, \ldots, m.$$

Proof. Fix an $\varepsilon > 0$, define the function F_0 on a neighbourhood of the origin putting

$$F_0(x) = F(x+\xi)$$

and, making use of Lemma 1, fix numbers $\delta > 0$, $\theta \in (0, 1)$ and $L > 0$ such that for every $\varrho \in (0, \delta)$ there exists a function G_0: $X \to Y$ of the class C^m and such that

$$G_0(x) = F_0(x) \quad \text{for } \|x\| < \theta\varrho, \tag{4}$$

$$G_0(x) = 0 \quad \text{for } \|x\| > \varrho \tag{5}$$

and

$$\|G_0^{(r)}(x)\| \leqslant L \sum_{j=0}^{r} \varrho^{j-r} \|F_0^{(j)}(x)\| \quad \text{for } r = 1, \ldots, m, \ \|x\| < \varrho. \tag{6}$$

Now, let us choose a positive number $\varrho < \min\{\delta, 1\}$ such that

$$\|F_0^{(m)}(x)\| < \varepsilon/(m+1)L \quad \text{for } \|x\| < \varrho$$

and let G_0: $X \to Y$ be a function of class C^m and such that conditions (4), (5) and (6) are fulfilled. Putting

$$G(x) = G_0(x-\xi) \quad \text{for } x \in X,$$

we see that

$$G(x) = F(x)$$

whenever $\|x-\xi\| < \theta\varrho$ and

$$G(x) = 0 \quad \text{whenever} \quad \|x-\xi\| > \varrho.$$

We shall show that

$$\|G^{(r)}(x)\| \leqslant \varepsilon \quad \text{for } \|x-\xi\| < \varrho, \ r = 0, 1, \ldots, m.$$

To this end let us observe that if $\|x\| < \varrho$, then

$$\begin{aligned} \|F_0^{(m-1)}(x)\| &= \|F_0^{(m-1)}(x) - F_0^{(m-1)}(0)\| \\ &\leqslant \|x\| \sup\{\|F_0^{(m)}(x')\|: \|x'\| < \varrho\} < \varrho \cdot \varepsilon/(m+1)L \end{aligned}$$

and, by induction,

$$\|F_0^{(j)}(x)\| < (\varepsilon/(m+1)l)\varrho^{m-j} \quad \text{for } \|x\| < \varrho, \ j = 0, 1, \ldots, m.$$

Hence and from property (6) of the function G_0 we infer that

$$\|G_0^{(r)}(x)\| \leqslant L \sum_{j=0}^{r} \varrho^{j-r} \frac{\varepsilon}{(m+1)l} \varrho^{m-j} = \frac{\varepsilon}{m+1} \sum_{j=0}^{r} \varrho^{m-r} \leqslant \varepsilon$$

whenever $\|x\| < \varrho$ and $r \in \{1, \ldots, m\}$, i.e., $\|G^{(r)}(x)\| \leqslant \varepsilon$ whenever $\|x-\xi\| < \varrho$, then

$$\|G(x)\| = \|G(x)-G(\xi)\| \leqslant \|x-\xi\| \sup \{\|G'(x')\colon \|x'-\xi\| < \varrho\} < \varrho\varepsilon < \varepsilon.$$

Lemma 2 is proved.

Passing to equation (1), we assume the following:

(i) The Banach space X fulfils condition (C), U is an open subset of the space X and ξ is a given (and fixed) point of the set U.

(ii) The function $f\colon U \to U$ is of the class C^m, each neighbourhood of the point ξ contains a neighbourhood $W \subset U$ such that $f(W) \subset W$ and $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in U$.

(iii) Y is a Banach space and the function $h\colon U \times Y \to Y$ is of the class C^m.

(iv) The (fixed) function φ_0, defined on a neighbourhood of the point ξ and taking values in the space Y, is of the class C^m and $\psi_0^{(r)}(\xi) = \varphi_0^{(r)}(\xi)$ for every $r \in \{0, 1, \ldots, m\}$, where the function ψ_0 is defined in a neighbourhood of the point ξ by the formula

$$\psi_0(x) = h\big(x, \varphi_0[f(x)]\big).$$

The function φ_0 may be considered as a formal solution of equation (1) (cf. [5], Definition 1.1 and Remark 1.1).

The following lemma is well known (cf. [1], Chapter 3; also [4], Theorem 4.2; in the case $X = Y = \mathbf{R}$).

LEMMA 3. *Let hypotheses* (i)–(iii) *be fulfilled and let $W \subset U$ be an open neighbourhood of the point ξ such that $f(W) \subset W$.*

If $\varphi\colon W \to Y$ is a solution of equation (1) *of the class C^m, then there exists exactly one solution $\Phi\colon U \to Y$ of this equation which is an extension of the function φ. The solution Φ is of the class C^m.*

THEOREM. *Assume* (i)–(iv).

Then, for every positive real number ε, there exists a function $\varphi\colon U \to Y$ of the class C^m which coincides with φ_0 in a neighbourhood of the point ξ and is such that

$$\|\varphi^{(r)}(x) - \psi^{(r)}(x)\| \leqslant \varepsilon$$

for every $x \in U$ and $r \in \{0, 1, \ldots, m\}$, where the function $\psi\colon U \to Y$ is defined by the formula

$$\psi(x) = h\big(x, \varphi[f(x)]\big). \tag{7}$$

Proof. Fix an $\varepsilon > 0$ and put

$$F(x) := \varphi_0(x) - h\big(x, \varphi_0[f(x)]\big)$$

for x from a neighbourhood of the point ξ. The function F is of the class C^m

and condition (2) is fulfilled. Thus, by Lemma 2, there exist a neighbourhood W of the point ξ and a function $G: X \to Y$ of the class C^m such that $G(x) = F(x)$ for every $x \in W$ and condition (3) holds. We may assume that $f(W) \subset W$. Put

$$h_0(x, y) := G(x) + h(x, y)$$

for every $(x, y) \in U \times Y$. Then the function h_0 is of the class C^m and for $x \in W$ we have

$$h_0(x, \varphi_0[f(x)]) = \varphi_0(x),$$

i.e., $\varphi_0|_W$ is a solution of the equation

$$\varphi(x) = h_0(x, \varphi[f(x)]).$$

Making use of Lemma 3 we may extend uniquely the solution $\varphi_0|_W$ to a solution $\varphi: U \to Y$ of this equation, and this solution φ is of the class C^m. Now, defining the function $\psi: U \to Y$ by formula (7), we see that

$$\varphi(x) - \psi(x) = h_0(x, \varphi[f(x)]) - h(x, \varphi[f(x)]) = G(x)$$

for every $x \in U$, and it follows from condition (3) that the function φ has all the required properties.

At the end let us observe that the above presented approach is a little bit different from that presented by Baron and Jarczyk in [2] (where, roughly speaking, a restriction of the function of two variables $\varphi_0(x) - h(x, y)$ was extended rather than a restriction of the function of one variable $\varphi_0(x) - h(x, \varphi_0[f(x)])$). By this approach our theorem may be stated also for equations of the form

$$\varphi(x) = h(x, \varphi \circ f(\cdot, x))$$

without any restrictions on their order. On the other hand, this approach does not allow us to obtain a full counterpart of Theorem 1 from paper [2], although we have the following corollary from the above-proved theorem.

COROLLARY. *Assume* (i)–(iv).

Then, for every positive real number ε, there exists a function $H: U \times Y \to Y$ of the class C^m such that the equation

$$\varphi(x) = H(x, \varphi[f(x)])$$

has a solution $\varphi: U \to Y$ of the class C^m which coincides with φ_0 in a neighbourhood of the point ξ and

$$\|h^{(r)}(x, y) - H^{(r)}(x, y)\| \leqslant \varepsilon$$

for every $(x, y) \in U \times Y$ and $r \in \{0, 1, \ldots, m\}$.

Proof. If $\varphi\colon U \to Y$ is a function obtained via the theorem, then it is enough to put

$$H(x, y) = h(x, y) + \varphi(x) - h(x, \varphi[f(x)]), \quad (x, y) \in U \times Y.$$

References

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