$H^p$ spaces on bounded symmetric domains

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Abstract. Let $D$ be a bounded symmetric domain in $C^N \,(N > 1)$ with Bergman–
Šilov boundary $b$ and $H^p \,(p > 0)$ the Hardy space of functions on $D$. If $f \in H^p \,(p > 1)$,
a Fourier series expansion is obtained for $f$ which gives Cauchy and Poisson integral
formulas for $f$. $H^p \,(p > 1)$ can be identified with the class $\tilde{f} \in L^p(b)$ whose
Cauchy and Poisson integrals are the equal. $H^2$ can be identified with the class $\tilde{f} \in L^2(b)$ whose
Fourier coefficients $a_{kp} = 0$ for $k < 0$. Several properties of weak convergence in $H^p$
are proved. In particular, if a bounded sequence converges pointwise on $D$, then it
converges weakly. D. J. Newman's result on pseudo-uniform convexity of $H^1$
for the disc is extended to $D$.

1. Introduction. Let $D$ be a bounded symmetric domain in the
complex vector space $C^N \,(N > 1)$, $0 \in D$, with Bergman–Šilov boundary $b$, $I'$ the group of holomorphic automorphisms of $D$ and $I'_0$ its isotropy
group. It is known that $D$ is circular and star-shaped with respect to
0 and that $b$ is circular. The group $I'_0$ is transitive on $b$ and $b$ has a unique normalized $I'_0$-invariant measure $V^{-1} ds_b$, $V$ the euclidean volume of $b$ and $ds_t$ euclidean volume element at $t \in b$. See [11], [17].

The Hardy space $H^p = H^p(D)$, $0 < p < \infty$, is the set of holomorphic
functions on $D$ with

$$
\|f\|_p = \sup_{0 < r < 1} \left\{ \frac{1}{V} \int_b |f(rt)|^p ds_t \right\}^{1/p} < \infty.
$$

For $p \geq 1$ $H^p$ is a Banach space and for $0 < p < 1$ a complete linear
Hausdorff space [6].

In Section 2 we derive a Fourier series representation for any holomorphic function in $D$. If $f \in H^p \,(p > 1)$ a better representation is obtained which gives Cauchy and Poisson integral formulas for $f$. The space $H^p$
can be identified with the class of functions $\tilde{f} \in L^p(b)$ whose Cauchy and
Poisson integrals are the equal (Theorem 2). Theorem 3 gives another
characterization of $H^2$. Theorem 4 proves that if a bounded sequence
in $H^p \,(p > 1)$ converges pointwise on $D$, then it converges weakly; thus

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Remarks.

1. The Bergman–Šilov boundary $b$ is the smallest closed set in $\overline{D}$ on which functions holomorphic on $\overline{D}$ take their maximum [5], p. 215. The real dimension of $b$ is $\geq N$. If $N = 2$ any bounded symmetric domain is biholomorphically equivalent to the bidisc $\{|x_1| < 1, |x_2| < 1\}$ or to the ball $\{|x_1|^2 + |x_2|^2 < 1\}$. Their Bergman–Šilov boundaries are $\{|x_1| = 1, |x_2| = 1\}$ of real dimension 2 and $\{|x_1|^2 + |x_2|^2 = 1\}$ of real dimension 3, respectively. If $N = 3$ an example of a bounded symmetric domain besides the polydisc and ball is $\{|x_1|^2 + |x_2|^2 < 1, |x_3| < 1\}$ with Bergman–Šilov boundary $\{|x_1|^2 + |x_2|^2 = 1, |x_3| = 1\}$ of real dimension 4 [5], p. 313.

2. Bergman and Weil have generalized Cauchy's integral formula for functions holomorphic on closed analytic polyhedra [2], [15].

2. Cauchy and Poisson formulas for functions of class $H^p$ ($p \geq 1$). Let $Z_{k\nu}$ denote the monomial $x_1^{k_1} \ldots x_N^{k_N} (k = k_1 + \ldots + k_N, k_0 = 0, 1, 2, \ldots, \nu = 1, \ldots, m_k = \binom{N + k - 1}{k})$. From the set $\{Z_{k\nu}\}$ Hua constructed by group representation theory a system $\Phi_0 = \{\varphi_{k\nu}\}$ of homogeneous polynomials, complete and orthogonal on $D$ and orthonormal on $b$ [9]. For each $k$ the sets $\{\varphi_{k\nu}\}$ and $\{Z_{k\nu}\}$ are in 1-1 correspondence so that to a set of constants $\{a_{k\nu}\}$ corresponds a set $\{A_{k\nu}\}$ with $\sum_{\nu=1}^{m_k} a_{k\nu} \varphi_{k\nu} = \sum_{\nu=1}^{m_k} A_{k\nu} Z_{k\nu}$ and conversely. Let $f$ be holomorphic on $D$. Then $f_q$, defined by $f_q(z) = f(qz)$, $0 < q < 1$, is holomorphic on $\overline{D}$ and has the series expansions

$$f_q(z) = \sum_{k\nu} a_{k\nu}(f_q) \varphi_{k\nu}(z) = \sum_{k\nu} A_{k\nu}(f_q) Z_{k\nu}$$

($\sum_{k\nu} = \sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k}$), both series converging uniformly to $f_q$ on compact subsets of $D$. Term by term differentiation in (1) gives

$$A_{k\nu}(f_q) = \frac{\partial^k f}{\partial z_1^{k_1} \ldots \partial z_N^{k_N}} \bigg|_{z_1 = \ldots = z_N = 0} (0) = \varphi_{k\nu}(f)$$

so that

$$f_q(z) = \sum_{k\nu} \varphi_{k\nu}(f) \varphi_{k\nu}(z).$$

To find $a_{k\nu}(f)$ set $z = e^t \bar{b} (t \in b, 0 < e^t < 1)$ in (2), multiply by $\varphi_{j\mu}(t)$ and integrate over $b$. This gives

$$a_{j\mu}(f) q^t = \int_{b} f(qt) \varphi_{j\mu}(t) ds_t = \langle f_q, \varphi_{j\mu} \rangle$$
(r = \varrho^s). Replacing \varrho^s by z in (2) gives

\[ f(z) = \sum_{k,r} a_{k,r}(f) \varpi_{k,r}(z), \quad a_{k,r}(f) = \lim_{r \to 1} (f_r, \varpi_{k,r}), \]

which converges uniformly on compact subsets of D.

**Theorem 1.** Let \( f \in H^p \) (\( p \geq 1 \)) with boundary values \( f^* \) on b. Then \( f \) has a Cauchy integral representation

\[ f(z) = \int_b S(z, t) f^*(t) ds_t \equiv (f^*, S_z) \quad (z \in D) \]

and a Poisson integral representation

\[ f(z) = \int_b P(z, t) f^*(t) ds_t \equiv (f^*, P_z) \quad (z \in D). \]

**Proof.** By a theorem of Bochner on circular sets [3] if \( f \in H^p \) (\( p > 0 \)) there exists \( f^* \in L^p(b) \) such that \( \lim_{r \to 1} \| f_r - f^* \|_p = 0 \). Since \( \varpi_{k,r} \) is bounded independently of \( r \) on b, Hölder's inequality for \( p > 1 \) and (4) give

\[ a_{k,r}(f) = \lim_{r \to 1} (f_r, \varpi_{k,r}) = (f^*, \varpi_{k,r}). \]

This also holds for \( p = 1 \). (5) follows from (4) and (7) and the fact that the series \( \sum_{k,r} \varpi_{k,r}(z) \varpi_{k,r}(t) \) for \( S(z, t) \) converges uniformly for z on compact subsets of D and \( t \in b \) [9]. (6) follows by writing (5) for the function \( g \in H^p \) defined by \( g(z) = f(\zeta) S(\zeta, \bar{z}) S(z, \bar{z}), \zeta \in D. \)

By (6) if \( f^* \) is real, then \( f \) is real on D, and a real holomorphic function is a constant. Thus

**Corollary.** If \( f^* \) is real on b and \( f \in H^p \) (\( p \geq 1 \)), then \( f \) is constant on D.

3. **Characterization of** \( H^p(D) \) (\( p \geq 1 \)). Set

\[ S^p(b) = \{ \tilde{f} \in L^p(b); (\tilde{f}, S_z) = (\tilde{f}, P_z) \}. \]

Then

**Theorem 2.** For \( p \geq 1 \) \( S^p(b) \) is a closed subspace of \( L^p(b) \) which is isometrically isomorphic to \( H^p(D) \) under the correspondence \( f \to \tilde{f} \) given by \( f(z) = (\tilde{f}, P_z), \tilde{f} \in S^p(b). \) Also if \( f^* \) is the boundary value of \( f \), then \( f^* = \tilde{f} \) a.e. on b.

**Proof.** From Theorem 1, \( f \in H^p(D) \) implies \( f^* \in S^p(b). \) Conversely let \( \tilde{f} \in S^p(b) \) and set \( f(z) = (\tilde{f}, S_z). \) Then \( f \) is holomorphic on D. Since \( P(z, t) \geq 0 \) and \( \int_b P(z, t) ds_t = 1, \) from \( f(z) = (\tilde{f}, P_z) \) follows by Hölder's inequality for \( p > 1 \)

\[ |f(z)| = |(\tilde{f}, P_z)| \leq |(\tilde{f})|^p, P_z |^{1/p}, \]

where
where \(|\tilde{f}|^p \in L^p(b)\). (1) also holds for \(p = 1\). Now \(h(z) = (|\tilde{f}|^p, P_z) \in \mathcal{S}(D)\) and \(f \in H^p(D)\) ([6], p. 521) and by Theorem 3 of [6] \(f \in H^p(D)\). Thus \(S^p(b)\) and \(H^p(D)\) are in 1-1 correspondence. For \(\tilde{f} \in S^p(b)\) set \(f(z) = (P_z, \tilde{f})\).

By [10], Proposition 2.5; \(\|f_r - \tilde{f}\|_p \to 0\) as \(r \to 1\) if \(p \geq 1\). By [3] there exists \(f^* \in L^p(b)\) such that \(\|f_r - f^*\|_p \to 0\) as \(r \to 1\). Hence \(\tilde{f} = f^*\) a.e. on \(b\).

Clearly \(\|\tilde{f}\|_p = \|f^*\|_p = \|f\|_p\) so that \(S^p(b)\) is isometrically isomorphic to \(H^p(D)\). Since \(H^p(D)\) is complete, \(S^p(b)\) is a closed subspace of \(L^p(b)\).

A second characterization of \(H^2(D)\) follows easily from Hilbert space theory. By Weyl [16] the orthonormal system \(\Phi_0\) can be extended to a complete orthonormal system of continuous functions on \(b\): \(\Phi = \{\varphi_{kr}, k = 0, \pm 1, \pm 2, \ldots; 1 \leq \nu \leq m_k\) if \(k \geq 0\), \(\nu = 0\) if \(k < 0\}, where the additional terms have been indexed by negative indices.

Set

\[
T^a(b) = \{f \in L^2(b); \quad a_{kr}(f) = (f, \varphi_{kr}) = 0 \text{ for } k < 0\}.
\]

Then

**Theorem 3.** \(T^a(b)\) is a closed subspace of \(L^2(b)\) which is isometrically isomorphic to \(H^2(D)\). If \(f^*\) is the boundary value of \(f\), then \(f^* = \tilde{f}\) a.e. on \(b\).

**Proof.** If \(f \in H^2(D)\), then \(f \to f \in T^a(b)\) by (2.4) and (2.7). Conversely let \(\tilde{f} \in T^a(b)\) and set

\[
f(z) = \sum_{k, r} a_{kr}(\tilde{f}) \varphi_{kr}(z) \quad (k \geq 0).
\]

From the Schwarz inequality and Bessel’s inequality follow that the series in (2) converges absolutely and uniformly on compact subsets of \(D\). Hence \(f\) is holomorphic on \(D\). By a calculation

\[
\|f_r\|^2 = \sum_{k, r} |a_{kr}(\tilde{f})|^2 \leq \sum_{k, r} |a_{kr}(\tilde{f})|^2
\]

so that \(f \in H^2(D)\). Also \(\|f\|_2 = \|f\|_2\). The rest of Theorem 3 follows as in the proof of Theorem 2.

Schmid obtained an analogous characterization of \(H^2(D)\) when \(D\) is a non-compact hermitian symmetric space by using Lie group theory [14].

4. **Convergence in** \(H^p\). The following properties of weak convergence are known or are easy to prove:

If \(f_n \to \gamma f\) in \(H^p\), then \(f_n \to f\) uniformly on compact subsets of \(D\) for every \(p > 0\) ([6], Theorem 9). If \(f_n \to f\) strongly in \(H^p (p > 0)\), then \(f_n \to \gamma f\) in \(H^p\).

This follows from the inequality \(|\gamma(f_n) - \gamma(f)| \leq \|\gamma\| \|f_n - f\|_p (\gamma \in (H^p^*)^*\).

Since \(H^p (p > 1)\) is a Banach space, the norms of the elements of a weakly convergent sequence are bounded. Let \(\{f_n\}\) be a bounded sequence in \(H^p (p > 0)\). Then \(f_n \to f\) pointwise on \(D\) if and only if \(f_n \to f\) uniformly on compact subsets of \(D\); also \(\gamma f \in H^p\).
Proof. By Lemma 3 of [6], boundedness of \( \{f_n\} \) in \( H^p \) implies that \( \{f_n\} \) is uniformly bounded in compact subsets of \( D \). Then by Lemma 4 of [6] \( f_n \to f \) pointwise on \( D \) implies uniform convergence of \( \{f_n\} \) to \( f \) on compact subsets of \( D \). The converse is trivial. Since \( \{f_n\} \) is bounded in \( H^p \) and \( f_n \to f \) uniformly on \( b_r = \{ rt : t \in b \} \), \( 0 < r < 1 \), \( \|f_n\|_p \) is bounded independently of \( r \) so that \( f \in H^p \).

The next theorem generalizes to bounded symmetric domains a result of Rudin [13] for the disc.

**Theorem 4.** Let \( \{f_n\} \) be a bounded sequence in \( H^p \) \( (p \geq 1) \). If \( f_n \to f \) pointwise in \( D \), then \( f_n \to f \) in \( H^p \) for \( p > 1 \) but not for \( p = 1 \).

**Proof.** See [13] for a counter-example when \( p = 1 \). Assume that \( \|f_n\|_p < 1 \) for all \( n \). By Lemma 3 of [6] the boundedness of \( \{\|f_n\|_p\} \) implies that \( \{f_n(z)\} \) is bounded independently of \( n \) and \( z \) on \( D_r(0 < r < 1) \). Hence by Vitali's theorem [6] \( f_n \to f \) uniformly on compact subsets of \( D \). Thus \( f \in H^p \) and we may assume that \( f = 0 \). Show that \( f_n \to 0 \) in \( H^p \).

\( f_n \in H^p \) has the series representation (2.4) with Fourier coefficients \( \varphi_{b_\nu}(f_n) \) given by (2.3). Since \( \{f_{r,n}\} \) converges uniformly to 0 on the compact set \( b, \) (2.3) gives \( \lim \varphi_{b_\nu}(f_n) = 0 \) for all \( k \geq 0 \) and \( n \). Hence by (2.7) \( \lim \varphi_{b_\nu}(f_n) = 0 \) for all \( k \geq 0 \). In (2.4) with \( f = f_n \) set \( z = rt \), multiply by \( \varphi_{b_\nu}(t) \) \((k < 0)\) and integrate over \( b \). By orthogonality of \( \Phi \) \((f_{r,n}, \varphi_{b_\nu}) = 0 \) for all \( k < 0 \) and \( n \). Since \( \varphi_{b_\nu} \in C(b) \) as in (2.7) \( \varphi_{b_\nu}(f_n) = \lim_{r \to 1} (f_{r,n}, \varphi_{b_\nu}) = 0 \) for \( k < 0 \). Hence

\[
\lim_{n \to \infty} \varphi_{b_\nu}(f_n, P(\Phi)) = 0,
\]

where \( P(\Phi) \) is any linear combination of the \( \varphi_{b_\nu} \).

Let \( \gamma \in (H^p)^* \). Since \( H^p \) is a closed subspace of \( L^p(b) \) by the Hahn–Banach theorem every bounded linear functional on \( H^p \) can be extended to \( L^p(b) \). Then by a well-known representation theorem for \( p > 1 \) [7] there exists a function \( g \in L^p(b), 1/p + 1/q = 1 \), such that \( \gamma(F) = (F, g) \) for all \( F \in L^p(b) \). In particular \( \gamma(f_n) = (f_n^*, g) \). Now approximate \( g \) in \( L^p(b) \) by a continuous function \( h \). By [16] \( h \) can be approximated on \( b \) in the norm by a linear combination \( P(\Phi) \) of \( \varphi_{b_\nu} \)'s. These approximations along with Hölder's inequality and the equality \( \|f_n\|_p = \|f_n^*\|_p \) give \( \lim \gamma(f_n) = 0 \), which proves the theorem.

**5. Pseudo-uniform convexity of \( H^1(D) \).**

**Theorem 5.** Let \( f_n \to f \) uniformly on compact subsets of \( D \) and \( \|f_n\|_p \to \|f\|_p \) as \( n \to \infty \), where \( f_n, f \in H^p(D) \) \((p \geq 1)\). Then \( \|f_n - f\|_p \to 0 \) as \( n \to \infty \).

**Proof.** For \( p > 1 \) the result follows from Theorem 4 and the local uniform convexity of \( H^p \) ([7], p. 233). \( H^1(D) \) is not locally uniformly convex but a proof due to L. D. Hoffman [8] in case \( D \) is the unit polydisc or ball in \( C^N \) can be extended to all bounded symmetric domains in \( C^N \).
If \( f \in H^1(D) \), then the function \( f_t \), defined on \( D^1 = \{ z : |z| < 1 \} \) by \( f_t(z) = f(tz) \) for any \( t \in b \), belongs to \( H^1(D^1) \) for almost all \( t \in b \) and

\[
\|f\|_1 = \frac{1}{V} \int_b \|f_t\|_{1,t} ds_t,
\]

where \( \| \cdot \|_{1,t} \) is the \( H^1 \) norm on \( D^1 \).

Proof. Since \( f \in H^1(D) \) and the rotation \( t = t'e^\theta \) preserves \( b \) and the measure \( V^{-1}ds_t \)

\[
(1) \quad \|f\|_1 \geq \|f_t\|_1 = (2\pi)^{-1} \int_0^{2\pi} d\theta \|f_r\|_1 = V^{-1} \int_b ds_t I_{r,t}
\]

by Fubini, where \( I_{r,t} = \|f_r\|_{1,1} \) so that

\[
(2) \quad \sup_{0 < r < 1} \frac{1}{V} \int_b I_{r,t} ds_t \leq \|f\|_1.
\]

(2) implies that \( I_{r,t} \) is bounded independently of \( r \) for \( 0 \leq r < 1 \) and almost all \( t \in b \). Since also \( f_t \) is holomorphic on \( D^1 \), \( f_t \in H^1(D^1) \) for almost all \( t \in b \). Thus \( I_{r,t} \) is monotone in \( r \). Interchanging sup and \( \int \) on the left-hand side of (2) gives \( V^{-1} \int_b \|f\|_{1,1} ds_t \). By the transformation in (1) the left-hand side of (2) equals \( \|f\|_1 \). Similarly \( f_{t,n} \) has these properties.

It follows as in Hoffman's paper by means of his lemma in integration theory and D. J. Newman's theorem [12] for the case \( N = 1 \) that \( \|f_n - f\|_1 \to 0 \) as \( n \to \infty \).

Professor Charles Chui independently obtained the same proof of Theorem 5.

References


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