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ON RIEMANNIAN CONFORMALLY SYMMETRIC SPACES ADMITTING PROJECTIVE COLLINEATIONS

BY

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1. According to Chaki and Gupta [1] an n-dimensional ($n > 3$) Riemannian space is called *conformally symmetric* or, briefly, a CS_n*-space* if its Weyl's conformal tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(R_{hk}g_{ij} - R_{hj}g_{ik} + R_{ij}g_{hk} - R_{ik}g_{hj}) + \\ + \frac{1}{(n-1)(n-2)} R(g_{hk}g_{ij} - g_{hj}g_{ik}) \tag{1}$$

satisfies the condition

$$C_{hijk,l} = 0, \tag{2}$$

where the comma indicates covariant differentiation with respect to the metric of the space.

In the first part of this paper we shall show that a CS_n-space with the definite metric form (positive or negative) and a constant scalar curvature, whose curvature tensor satisfies a special algebraic condition, reduces to a symmetric space (in the sense of E. Cartan). In the second part we shall investigate projective collineations in CS_n-spaces. Notation is the same as in [5].

2. We put for brevity

$$M = A_s A^s, \tag{3}$$

$$P = \frac{1}{M(1-n)}\left(R^{sr}R_{sr} - \frac{1}{n}R^2\right), \tag{4}$$

$$T_{pijk} = -MR_{pijk} - \frac{1}{n(n-1)(n-2)} RA_i(A_k g_{jp} - A_j g_{kp}) + \\ + \frac{1}{n(n-2)} MR(g_{ik}g_{jp} - g_{ij}g_{kp}) + \frac{1}{(n-1)(n-2)} A_i(A_k R_{jp} - A_j R_{kp}) + \\ + \frac{1}{n-1} M(R_{ij}g_{kp} - R_{ik}g_{jp}) - \frac{1}{n-2} M(R_{jp}g_{ik} - R_{kp}g_{ij}), \tag{5}$$

where $A_j \neq 0$ is a gradient vector field.

we have, by transvection with $R^s{}_{ijk}$, in view of (6) and (13),

$$(15)\quad R^m{}_p R_{sm} R^s{}_{ijk} = -\frac{1}{n(n-1)(n-2)} RPA_i(A_k g_{pj} - A_j g_{pk}) + \\ + \frac{1}{n(n-2)} RA_p P(A_k g_{ij} - A_j g_{ik}) + \frac{1}{n(n-2)} RMP(g_{ik} g_{jp} - g_{ij} g_{pk}) + \\ + \frac{1}{n(n-1)} R(R_{ij} R_{kp} - R_{ik} R_{jp}) + \frac{1}{n-1} PA_p(R_{ij} A_k - R_{ik} A_j) - MPR_{pijk}.$$

By virtue of (5), the comparison of (14) and (15) yields the relation $PT_{pijk} = 0$ which completes the proof.

Lemma 2. *If the curvature tensor of a CS_n-space with the constant scalar curvature and definite metric form satisfies equations* (6) *and* $T_{pijk} = 0$, *then the space is necessarily symmetric in the sense of E. Cartan.*

Proof. Since, by virtue of (5), $T_{pijk} = 0$ and $MR_{pijk} - MR_{jkpi} = 0$, we have

$$\frac{1}{n} R(A_k A_p g_{ij} - A_i A_j g_{pk}) + (A_i A_j R_{kp} - A_k A_p R_{ij}) - M(g_{ij} R_{kp} - R_{ij} g_{kp}) = 0.$$

This, by contraction with g^{pk}, gives $R_{ij} = Rg_{ij}/n$.

Now, since one can easily verify that a conformally symmetric Einstein space is necessarily symmetric in the sense of E. Cartan, we have our lemma.

Lemma 3. *If the curvature tensor of a CS_n-space with the constant scalar curvature and definite metric form satisfies equations* (6) *and* $P = 0$, *then the space is symmetric in the sense of E. Cartan.*

Proof. Since $P = 0$, relation (12) yields

$$(16)\qquad R^s{}_m R_{sj} = \frac{1}{n} RR_{mj},$$

which, by covariant differentiation, gives

$$(17)\qquad R^s{}_{m,p} R_{sj} + R^s{}_m R_{sj,p} = \frac{1}{n} RR_{mj,p}.$$

On the other hand, contracting (2) with g^{hl} and using (1), we have the equation

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)} (g_{ij} R_{,k} - g_{ik} R_{,j})$$

which leads immediately to

$$(18)\qquad R_{ij,k} = R_{ik,j}.$$

Interchanging now in (17) the indices m and p and substituting (18), we find that

$$R^s{}_m R_{sj,p} = R^s{}_p R_{sj,m}. \tag{19}$$

Because of (18) and (19), the last relation can be written in the form

$$R^s{}_m R_{sj,p} = R^s{}_p R_{sj,m} = R^s{}_p R_{sm,j} = R^s{}_j R_{sm,p}.$$

This, together with (17), gives

$$R^s{}_m R_{sj,p} = \frac{1}{2n} R R_{mj,p}. \tag{20}$$

Transvecting now (20) with A^m and using (7), we obtain $RA^s R_{sj,p} = 0$, whence we see that either of the two following cases holds: $R = 0$ or $A^s R_{sj,p} = 0$.

It follows easily from (16) that the assumption $R = 0$ gives $R^{sr} R_{sr} = 0$ which, because of (1) and (2), leads immediately to $R_{hijk,l} = 0$.

In the second case, since $P = 0$, (13) can be written as

$$R_{sm} R^s{}_{ijk} = \frac{1}{n-1} (R_{ij} R_{km} - R_{ik} R_{jm}),$$

which, by covariant differentiation, gives

$$R_{sm,p} R^s{}_{ijk} + R_{sm} R^s{}_{ijk,p} = \frac{1}{n-1} (R_{ij,p} R_{km} - R_{ik,p} R_{jm}) + \\ + \frac{1}{n-1} (R_{ij} R_{km,p} - R_{ik} R_{jm,p}). \tag{21}$$

Making now use of (1) and (2), we find that

$$R_{sm} R^s{}_{ijk,p} = \frac{1}{2n(n-2)} R (R_{mk,p} g_{ij} - R_{mj,p} g_{ik}) + \frac{1}{n-2} (R_{ij,p} R_{km} - R_{ik,p} R_{jm}),$$

which, together with (20) and (21), yields

$$R_{sm,p} R^s{}_{ijk} = -\frac{1}{2n(n-2)} R (R_{mk,p} g_{ij} - R_{mj,p} g_{ik}) - \\ - \frac{1}{(n-1)(n-2)} (R_{ij,p} R_{km} - R_{ik,p} R_{jm}) + \frac{1}{n-1} (R_{ij} R_{mk,p} - R_{ik} R_{jm,p}).$$

Transvecting the last equation with $A^m A^k$, and using (7) and $A^s R_{sj,p} = 0$, we obtain $MRR_{ij,p} = 0$, whence $R = 0$ or $R_{ij,p} = 0$. But the first relation leads to our first case and, consequently, to $R_{hijk,l} = 0$. The same holds for $R_{ij,p} = 0$.

Combining lemmas 1, 2 and 3, we have the following

THEOREM 1. *If the curvature tensor of a CS_n-space with the constant scalar curvature and definite metric form satisfies* (6), *then the space is necessarily symmetric.*

3. Let a Riemannian space V_n admit a projective collineation with respect to the vector field v^i. Denote by L the Lie derivative with respect to this field. Then we have [4]

$$LΓ^i_{jk} = \delta^i_j A_k + \delta^i_k A_j, \qquad LR^i{}_{jkl} = \delta^i_k A_{j,l} - \delta^i_l A_{j,k}, \tag{22}$$

$$LR_{jk} = (1-n)A_{j,k}, \tag{23}$$

where Γ^i_{jk} are the Christoffel symbols and $A_j \neq 0$ is a gradient vector.

Applying to Ricci's tensor the well-known formula

$$(LT_{jk})_{,m} - L(T_{jk,m}) = -T_{js}L\Gamma^s_{km} - T_{sk}L\Gamma^s_{jm}$$

and making use of (22) and (23), we get

$$LR_{jk,m} = (1-n)A_{j,k,m} - 2R_{jk}A_m - R_{jm}A_k - R_{mk}A_j,$$

whence

$$L(R_{jk,m} - R_{jm,k}) = (n-1)A_s R^s{}_{jkm} - R_{jk}A_m + R_{\ m}A_k. \tag{24}$$

Suppose now that the considered space is a CS$_n$-space with the constant scalar curvature and definite metric form. Then, because of (24) and (18), we find that

$$(n-1)A_s R^s{}_{jkm} = R_{jk}A_m - R_{jm}A_k.$$

Since every symmetric Riemannian n-space admitting a projective collineation is a space of constant curvature [3], we have, by (25) and Theorem 1,

THEOREM 2. *If a CS_n-space with the constant scalar curvature and definite metric form admits a projective collineation, then it is of constant curvature.*

Since in a CS$_n$-space with the definite metric, which is not conformally flat, the scalar curvature is constant [2], there do not exist CS$_n$-spaces with the definite metric, which admit projective collineations and are not conformally flat.

REFERENCES

[1] M. C. Chaki and B. Gupta, *On conformally symmetric spaces*, Indian Journal of Mathematics, 5 (1963), p. 113-122.

[2] E. Głodek, *Some remarks on conformally symmetric Riemannian spaces*, Colloquium Mathematicum 23 (1971), p. 121-123.

[3] T. Sumitomo, *Projective and conformal transformations in compact Riemannian manifolds*, Tensor, New Series, 9 (1959), p. 113-135.

[4] K. Yano, *The theory of Lie derivatives and its applications*, Amsterdam-Groningen 1955.

[5] — and S. Bochner, *Curvature and Betti-numbers*, Princeton 1953.

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