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On a characterization of the cosine

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The problem of a functional characterization of the trigonometric functions has often been dealt with in the mathematical literature (cf. the excellent bibliography in [1]). In most cases the addition and subtraction formulae of the sine and cosine were used for this purpose. Equations of this type involve both the sine and cosine, and thus they contain two unknown functions. It should be noticed that a single equation can characterize more than one function. E.g. H. E. Vaughan proved [7] that if $\varphi(x)$ and $\psi(x)$ satisfy

$$\varphi(x-y) = \varphi(x)\varphi(y)+\psi(x)\psi(y) \tag{1}$$

and

$$\lim_{h\to 0}\frac{\psi(h)}{h} = 1\,, \tag{2}$$

then $\varphi(x) = \cos x$ and $\psi(x) = \sin x$. (1) and (2) present a nice system of conditions for a characterization of the trigonometric functions. If, however, one wants to define the cosine alone, one may use the equation

$$\varphi(x+y) = \varphi(x)\varphi(y)-\sqrt{1-\varphi^2(x)}\sqrt{1-\varphi^2(y)}\,, \tag{3}$$

whose general continuous solution is $\varphi(x) = \cos px$ (cf. [4], [1], p. 74).

Although equations (1) or (3) are quite sufficient for a characterization of the cosine, it would be perhaps desirable (and at least interesting) to characterize the latter by an equation in a single variable. Attempts have already been made to do this, but since the problem is more difficult, the results are less satisfactory. Equations in a single variable are essentially weaker (contain less information) than those in two variables (like (1) or (3)) and therefore much stronger additional conditions are needed to determine a single function among all their solutions.

A functional relation which has been used for a characterization of the cosine is the duplication formula

$$\varphi(2x) = 2\varphi^2(x)-1\,, \tag{4}$$

which may be obtained from (3) on setting $y = x$. H. G. Forder [3] proved that $\varphi(x) = \cos x$ is the only even solution of (4) which is twice differentiable at $x = 0$ and such that $\varphi(0) = 1$, $\varphi''(0) = -1$. This system of conditions is somewhat inconvenient for didactic purposes. The condition of twice differentiability of φ at 0 requires a knowledge of the calculus. It would be nicer to be able to introduce the cosine before introducing derivatives, with no stronger means than e.g. continuity. Unfortunately, as R. Cooper showed in [2], the condition that φ be twice differentiable cannot be weakened, even under the additional assumption of convexity.

In this note we discuss the possibility of a characterization of the cosine by an equation in a single variable and some more elementary conditions. As we shall see, the equation

$$\varphi\left(\frac{x}{2}\right) = s(x)\sqrt{\tfrac{1}{2}[1+\varphi(x)]}\,, \tag{5}$$

where the function $s(x)$ is defined by

$$s(x) = \begin{cases} +1 & \text{for} \quad x \in \langle(4k-1)\pi, (4k+1)\pi), \\ -1 & \text{for} \quad x \in \langle(4k+1)\pi, (4k+3)\pi), \end{cases} \quad k = 0, \pm 1, \pm 2, \dots, \tag{6}$$

can better serve this aim than equation (4). We shall prove the following

THEOREM 1. *$\varphi(x) = \cos x$ is the only function defined for all x, continuous in a neighbourhood of $x = 0$, satisfying equation* (5) *and periodic with the period 2π.*

Proof. Let $\varphi(x)$ be a function fulfilling the conditions of the theorem. Putting $x = 0$ in (5) we obtain the equation for $\varphi(0)$:

$$2\varphi^2(0) - \varphi(0) - 1 = 0\,,$$

which has the roots $+1$ and $-\frac{1}{2}$. According to (6) we have $\varphi(0) \geqslant 0$ and consequently $\varphi(0) = 1$. Since $\varphi(x)$ is periodic, we have

$$\varphi(2k\pi) = 1 \quad \text{for integral } k. \tag{7}$$

By (7) and (5) $\varphi(x)$ is determined on the set of the numbers of the form $k\pi/2^n$. This set is dense in $(-\infty, \infty)$. Since $\varphi(x)$ is continuous in a neighbourhood R of $x = 0$, it is uniquely determined in the whole of R. Being a solution of (5), $\varphi(x)$ satisfies also equation (4). The latter allows us to extend $\varphi(x)$ uniquely from R onto the whole of $(-\infty, \infty)$. On the other hand, $\varphi(x) = \cos x$ is known to fulfil all the conditions of the theorem.

The above theorem yields a characterization of the cosine. It is, however, far from being satisfactory. The main objection is that equation (5) is rather clumsy; in particular, it contains the function $s(x)$, which is somewhat disagreeble. So the natural question arises, whether the cosine cannot be defined with the aid of equation (4), periodicity and continuity. This problem turns out to be very interesting in itself and presents some unexpected difficulties.

DEFINITION. For a given number $h > 0$ let $D(h)$ denote the set of those $x \in (-\infty, \infty)$ for which the sequence $u_n = \frac{2^n x}{h} - \left[\frac{2^n x}{h}\right]$ (where $[a]$ denotes the greatest integer not exceeding a) is dense in the interval $(0,1)$.

As has been proved in [5] (cf. also [6], ch. I), for every $h > 0$ the set $D(h)$ is not empty.

Now we shall prove the following

THEOREM 2. *For given numbers $h > 0$, $x_0 \in D(h)$ and y_0, there exists at most one function $\varphi(x)$ defined for all x and fulfilling the following conditions:*

(8) $\varphi(x)$ *satisfies equation* (4),

(9) $\varphi(x)$ *is continuous in a neighbourhood of* $x = 0$,

(10) $\varphi(x)$ *is periodic with the period* h,

(11) $\varphi(x_0) = y_0$.

Proof. Let $\varphi(x)$ be a function fulfilling conditions (8) through (11). According to (9), (10) and (11) $\varphi(x)$ is uniquely determined at all the points of the sequence $2^n x_0 - kh$ (n, k — positive integers). Since $x_0 \in D(h)$, the sequence $2^n x_0 - kh$ is dense in $(0, h)$. In view of condition (9) the function $\varphi(x)$ is uniquely determined in a right neighbourhood of $x = 0$. Equation (4) yields a unique extension of $\varphi(x)$ onto $(0, \infty)$ and condition (10) allows us to extend $\varphi(x)$ to the whole real axis.

Unfortunately, Theorem 2 cannot be used to characterize the cosine. Firstly, we are not able to indicate an $x_0 \in D(2\pi)$. We only know that one does exist. Without knowledge of an x_0 for which the sequence $2^n x_0 - 2k\pi$ is dense in $(0, 2\pi)$ we cannot choose the cosine out of all the functions fulfilling (8), (9) and

$$\varphi(x + 2\pi) = \varphi(x). \tag{12}$$

Further, if we even knew an $x_0 \in D(2\pi)$, the condition

$$\varphi(x_0) = \cos x_0 \tag{13}$$

would be useless from the practical point of view. Either x_0, or $\cos x_0$ would have no simple, known value, definable without the notion of cos and arccos.

We see that the problem of a characterization of the cosine by a functional equation in a single variable remains open. But we would also like to call the reader's attention to the following interesting and somewhat curious fact. We may ask for what initial values $y_0 \epsilon \langle -1, 1\rangle$ there exists a function $\varphi(x)$ fulfilling conditions (8), (9), (12) and (11). Evidently $\varphi(x) = \cos px$ is the solution for $y_0 = \cos px_0$ if p is an integer. But if p is not an integer, then $\varphi(x) = \cos px$ does not fulfil condition (12). We are not able to say whether for $y_0 = \cos px_0$ with a non-integral p there exists a $\varphi(x)$ satisfying (8), (9), (12) and (11), but we conjecture that there does not. In the favour of this conjecture we shall prove the following

THEOREM 3. *If l, m are positive integers, l odd, and $x_0 \epsilon D(2\pi)$, then there does not exist a function $\varphi(x)$ defined for all x and fulfilling conditions* (8), (9), (12) *and*

$$\varphi(x_0) = \cos \frac{l}{2^m} x_0 \,. \tag{14}$$

Proof. Suppose that a function $\varphi(x)$ satisfies (8), (9), (12) and (14). Then $\varphi(x)$ fulfils also (8), (9), (14) and

$$\varphi(x + 2^{m+1}\pi) = \varphi(x) \,.$$

$x_0 \epsilon D(2\pi)$ implies $x_0 \epsilon D(2^{m+1}\pi)$. Consequently, by Theorem 2,

$$\varphi(x) = \cos \frac{l}{2^m} x \,. \tag{15}$$

But this is impossible, since, in view of the fact that l is odd, function (15) does not satisfy (12). Thus the system of conditions (8), (9), (12), (14) has no solution at all.

So we have the following strange result:

THEOREM 4. *There exist two sets A, B, dense in $\langle -1, 1\rangle$ and such that for $y_0 \epsilon A$ there exists a (unique) function $\varphi(x)$ defined for all x and fulfilling conditions* (8), (9), (12) *and* (14) (*where $x_0 \epsilon D(2\pi)$*), *whereas for $y_0 \epsilon B$ there does not.*

Proof. The set A contains all the numbers $y_0 = \cos px_0$ with integral p, and by Theorem 3 the set B contains all the numbers $y_0 = \cos \frac{l}{2^m} x_0$, where l, m are positive integers, l odd. So both these sets are dense in $\langle -1, 1\rangle$.

Finally let us notice that a similar difficulty did not occur in the case of equation (5). The reason that the latter is stronger than (4) lies in this "clumsy, disagreeable" function $s(x)$.

References

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