

COLLOQUIUM MATHEMATICUM

XLII

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P 734, R 1. The problem has been commented by the author ⁽¹⁾.
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(¹) S. Hartman, *On harmonic separation*, ce volume, p. 209-222.

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P. ERDÖS (BUDAPEST)

P 1155. Let x_1, x_2, \dots be an infinite sequence of real numbers. Prove that there is a set E of reals of positive measure which contains no subset similar (in the Euclidean sense) to E . We can of course assume that $x_n \rightarrow 0$.

P 1156. Let $1 \leq u_1 < \dots < u_l \leq n$ be a sequence of reals. Assume that all the products $u_i u_j$, $1 \leq i \leq j \leq l$, differ by at least one, i.e.

$$(1) \quad |u_i u_j - u_r u_p| \geq 1 \quad \text{if } \{i, j\} \neq \{r, p\}.$$

Determine or estimate $\max l$. Prove that

$$\frac{\max l}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

If the u 's are integers, (1) simply means that the products $u_i u_j$ are all different. In this case we proved (2) that for $0 < c_1 < c_2 < \infty$

$$(2) \quad \pi(n) + c_1 n^{3/4} / (\log n)^{3/2} < \max l < \pi(n) + c_2 n^{3/4} / (\log n)^{3/2}.$$

It seems true that for some absolute constant C

$$(3) \quad \max l = \pi(n) + (C + o(1)) n^{3/4} / (\log n)^{3/2}.$$

We have not been able to prove (3).

P 1157. Let E be a measurable subset of $(0, 1)$ and $m(E)$ its measure. Put

$$f_n(E, a) = \sum_{1 \leq k \leq n} 1,$$

where the summation is extended over those k for which $(ka) = ka - [ka]$ is in E .

Khinchine conjectured that, for every E , if one neglects a set of α 's of measure 0 (depending on E), then

$$(1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} f_n(E, \alpha) = m(E).$$

It was a great surprise when Marstrand ⁽³⁾ disproved (1). Perhaps Khinchine's conjecture can be saved if in (1) density is replaced by logarithmic density. In other words, for almost all α

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{\log n} \sum' \frac{1}{k} = \alpha,$$

where the dash indicates that the summation is extended over those k , $1 \leq k \leq n$, for which $(k\alpha)$ is in E . It seems that Marstrand's proof does not disprove (2) — at least not without significant modifications.

P 1158. W. Schmidt posed the following problem:

Does there exist a set S of real numbers of infinite measure such that the quotient of two elements of S is never an integer?

Haight ⁽⁴⁾ and Szemerédi ⁽⁵⁾ proved that the answer is affirmative. One can now ask:

Denote by $m(S(x))$ the measure of the intersection of S with $(0, x)$. Assume that the quotient of two elements of S is never an integer. How fast can $m(S(x))$ tend to infinity? (It is easy to see that $m(S(x)) = o(x)$.)

More precisely: Try to characterize those functions $F(x)$ and $f(x)$ for which

(a) there is a set S_1 with $m(S_1(x)) > f(x)$ for all large x and the quotient of two elements of S_1 is never an integer,

(b) there are a set S_2 and a sequence $x_n \rightarrow \infty$ such that $m(S_2(x_n)) > F(x_n)$ and the quotient of two elements of S_2 is never an integer.

In particular, can we take $f(x) = x^\alpha$ and $F(x) = x^\alpha$, $\alpha < 1$?

⁽²⁾ P. Erdős, *On some applications of graph theory to number theoretic problems*, Publications of the Ramanujan Institute 1 (1969), p. 131-136.

⁽³⁾ I. M. Marstrand, *On Khinchine's conjecture about strong uniform distribution*, Proceedings of the London Mathematical Society 21 (1970), p. 540-556.

⁽⁴⁾ J. A. Haight, *A linear set of infinite measure with no two points having integral ratio*, Mathematika 17 (1970), p. 133-138.

⁽⁵⁾ E. Szemerédi, *On a problem of W. Schmidt*, Studia Scientiarum Mathematicarum Hungarica 6 (1971), p. 287-288.

P. ERDÖS AND A. HAJNAL (BUDAPEST)

P 1159. Let S be an infinite set of elements x_1, x_2, \dots . To every edge (x_i, x_j) make correspond a subset $E_{i,j}$ of $(0, 1)$ of measure greater than $\varepsilon > 0$ (ε does not depend on i and j). Is there an infinite path

$x_{i_1}, x_{i_2}, x_{i_3}, \dots$ such that the intersection of all sets $E_{i_r, i_{r+1}}, r = 1, 2, \dots$, is non-empty?

Further problem: Let $|S| = \aleph_1$. To each triple $(x_{a_1}, x_{a_2}, x_{a_3})$ of S make correspond a measurable subset E_{a_1, a_2, a_3} of measure greater than ε . Is it true that there must be four points $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}$ such that the intersection of the four sets $E_{a_{i_1}, a_{i_2}, a_{i_3}}$ ($a_{i_1}, a_{i_2}, a_{i_3}$ run over the four triples from a_1, a_2, a_3, a_4) is non-empty?

For further problems of this type see (⁶).

Letter of P. Erdős, November 1977.

(⁶) P. Erdős and A. Hajnal, *Some remarks on set theory, IX. Combinatorial problems in measure theory and set theory*, Michigan Mathematical Journal 11 (1964), p. 107-127.

P. ERDÖS (BUDAPEST) AND D. PREISS (PRAGUE)

P 1160. Let E_{\aleph_1} be the unit sphere of the \aleph_1 -dimensional Hilbert space. Join two points of E_{\aleph_1} if their distance is greater than $\sqrt{2}$. Is the chromatic number of this graph \aleph_1 or \aleph_0 ?

We proved (⁷) that if $\sqrt{2}$ is replaced by $\sqrt{2} + \varepsilon$, then the chromatic number is \aleph_0 . If $> \sqrt{2}$ is replaced by $\geq \sqrt{2}$, then the chromatic number is \aleph_1 , since our graph contains a K_{\aleph_1} .

Letter of P. Erdős, November 1977.

(⁷) P. Erdős and D. Preiss, *Decomposition of spheres in Hilbert spaces*, Commentationes Mathematicae Universitatis Carolinae 17 (1976), p. 791-795.

P. ERDÖS (BUDAPEST) AND D. SILVERMAN (LOS ANGELES, CALIFORNIA)

P 1161. Define an infinite graph whose vertices are the integers defined as follows. Join i and j if $i+j$ is a square. Prove (or disprove) that the chromatic number of this graph is infinite.

P 1162. Let $1 \leq u_1 < \dots < u_l$ be a sequence of integers such that none of the sums $u_i + u_j$ is a square. Determine or estimate $\max l$. The u 's can be chosen $\equiv 1 \pmod{3}$, i.e., $\max l \geq n/3$. Is $\max l > (1 + \varepsilon)n/3$ possible? Clearly, the squares can be replaced by other sets of numbers, e.g., cubes, etc. If $u_i + u_j$ is replaced by $u_i - u_j$, then L. Lovász conjectured some time ago that $l = o(n)$. This was proved by Sárközy (⁸) and quite independently by H. Fürstenberg.

Letter of P. Erdős, November 1977.

(⁸) A. Sárközy, *On difference sets of sequence of integers*, I, II and III, I and III in Acta Mathematica Academiae Scientiarum Hungaricae (to appear), II in Annales Universitatis Scientiarum Budapestinensis (to appear).

JERZY ŁOŚ (WARSAWA)

P 1163. The characteristic set of an n -tuple of equivalence relations R_1, \dots, R_n in a set X is defined as

$$C = \{(\chi_{R_1}(x, y), \dots, \chi_{R_n}(x, y)) \mid (x, y) \in X^2\} \subset \{0, 1\}^n,$$

where χ_{R_i} denote the characteristic functions of R_i in X^2 . Obviously, not every set $C \subset \{0, 1\}^n$ can serve as a characteristic set of an n -tuple of equivalence relations, since $\mathbf{1} = (1, \dots, 1)$ (the sequence of only ones) must belong to it. This is not, however, the only condition on C .

If R and Q are two equivalence relations on X , then for all $a, b, c, d \in X$ the following is true:

if $(a, b) \in R \cap \bar{Q}$ and $(c, d) \in \bar{R} \cap Q$, then at least one pair (a, c) , (a, d) , (b, c) or (b, d) belongs to $\bar{R} \cap \bar{Q}$.

Here \bar{R} and \bar{Q} denote the complements of R and Q , respectively, in X^2 .

It follows that, with $n = 2$, the set consisting of three pairs $(1, 1)$, $(1, 0)$ and $(0, 1)$ is not a characteristic set. Actually, something more can be said about a characteristic set. It fulfils the following

CONDITION. For every $p, q \in C$ there exist $r, s, t, u \in C$ such that for every i, j ($i, j = 1, 2, \dots, n$) if $p_i = 1 = q_j$ and $p_j = 0 = q_i$, then $r_i = r_j = 0$ or $s_i = s_j = 0$ or $t_i = t_j = 0$ or $u_i = u_j = 0$.

PROBLEM. Is the condition above sufficient in order to represent a set $C \subset \{0, 1\}^n$, to which $\mathbf{1}$ belongs, as a characteristic set of an n -tuple of equivalence relations?

P 1163, R 1. A partial answer can be given. If $\mathbf{0} = (0, \dots, 0)$ (the sequence of only zeroes) belongs to C , then the condition is obviously satisfied. Such a set to which $\mathbf{1}$ and $\mathbf{0}$ belong can always be represented by a very peculiar kind of equivalence relations, what is to be seen from the following construction.

To every $t \in C$ we make correspond a pair of distinct elements $\{a_i, b_i\}$. The set X of all a_i 's and b_i 's which contains twice the number of elements in C is already partitioned into pairs $\{a_i, b_i\}$. We define each relation R_i by a partition of X , being a subpartition of that partition. Namely, to define the relation R_i , we keep the pair $\{a_i, b_i\}$ together if $t_i = 1$ and we split it apart replacing by $\{a_i\}, \{b_i\}$ if $t_i = 0$. It is easy to check that this construction provides us with an n -tuple of equivalence relations having the characteristic set $C \cup \{\mathbf{0}, \mathbf{1}\}$; thus, if $\mathbf{0}, \mathbf{1} \in C$, it is C . The problem is therefore reduced to sets C with $\mathbf{1} \in C$ and $\mathbf{0} \notin C$.

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