COLLOQUIUM MATHEMATICUM

XLII DÉDIÉ À LA MÉMOIRE D'EDWARD MARCZEWSKI 1979

P R O B L È M E S

P 734, R 1. The problem has been commented by the author (1).

XXIII.1, p. 177.

(1) S. Hartman, *On harmonic separation*, ce volume, p. 209-222.

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P 1075 - P 1079. Formulés dans la communication *Sets of uniqueness on noncommutative locally compact groups. II.*

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P 1149 - P 1154. Formulés dans la communication *On problems related to characteristic vertices of graphs.*

Ce volume, p. 369, 371, 372 et 375.

P. ERDÖS (BUDAPEST)

P 1155. Let $x_1, x_2, \ldots$ be an infinite sequence of real numbers. Prove that there is a set E of reals of positive measure which contains no subset similar (in the Euclidean sense) to E. We can of course assume that $x_n \to 0$.

P 1156. Let $1 \leqslant u_1 < \ldots < u_l \leqslant n$ be a sequence of reals. Assume that all the products $u_i u_j$, $1 \leqslant i \leqslant j \leqslant l$, differ by at least one, i.e.

(1) $$|u_i u_j - u_r u_p| \geqslant 1 \quad \text{if } \{i, j\} \neq \{r, p\}.$$

Determine or estimate $\max l$. Prove that

$$\frac{\max l}{n} \to 0 \quad \text{as } n \to \infty.$$

If the u's are integers, (1) simply means that the products $u_i u_j$ are all different. In this case we proved (²) that for $0 < c_1 < c_2 < \infty$

(2) $$\pi(n) + c_1 n^{3/4}/(\log n)^{3/2} < \max l < \pi(n) + c_2 n^{3/4}/(\log n)^{3/2}.$$

It seems true that for some absolute constant C

(3) $$\max l = \pi(n) + \big(C + o(1)\big) n^{3/4}/(\log n)^{3/2}.$$

We have not been able to prove (3).

P 1157. Let E be a measurable subset of $(0, 1)$ and $m(E)$ its measure. Put

$$f_n(E, a) = \sum_{1 \leqslant k \leqslant n} 1,$$

where the summation is extended over those k for which $(ka) = ka - [ka]$ is in E.

Khintchine conjectured that, for every E, if one neglects a set of α's of measure 0 (depending on E), then

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n} f_n(E, \alpha) = m(E).$$

It was a great surprise when Marstrand [3] disproved (1). Perhaps Khintchine's conjecture can be saved if in (1) density is replaced by logarithmic density. In other words, for almost all α

$$(2) \qquad \lim_{n\to\infty} \frac{1}{\log n} \sum{}' \frac{1}{k} = a,$$

where the dash indicates that the summation is extended over those k, $1 \leqslant k \leqslant n$, for which $(k\alpha)$ is in E. It seems that Marstrand's proof does not disprove (2) — at least not without significant modifications.

P 1158. W. Schmidt posed the following problem:

Does there exist a set S of real numbers of infinite measure such that the quotient of two elements of S is never an integer?

Haight [4] and Szemerédi [5] proved that the answer is affirmative. One can now ask:

Denote by $m(S(x))$ the measure of the intersection of S with $(0, x)$. Assume that the quotient of two elements of S is never an integer. How fast can $m(S(x))$ tend to infinity? (It is easy to see that $m(S(x)) = o(x)$.)

More precisely: Try to characterize those functions $F(x)$ and $f(x)$ for which

(a) there is a set S_1 with $m(S_1(x)) > f(x)$ for all large x and the quotient of two elements of S_1 is never an integer,

(b) there are a set S_2 and a sequence $x_n \to \infty$ such that $m(S_2(x_n)) > F(x_n)$ and the quotient of two elements of S_2 is never an integer.

In particular, can we take $f(x) = x^a$ and $F(x) = x^a$, $a < 1$?

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(2) P. Erdös, *On some applications of graph theory to number theoretic problems*, Publications of the Ramanujan Institute 1 (1969), p. 131-136.

(3) I. M. Marstrand, *On Khintchine's conjecture about strong uniform distribution*, Proceedings of the London Mathematical Society 21 (1970), p. 540-556.

(4) J. A. Haight, *A linear set of infinite measure with no two points having integral ratio*, Mathematika 17 (1970), p. 133-138.

(5) E. Szemerédi, *On a problem of W. Schmidt*, Studia Scientiarum Mathematicarum Hungarica 6 (1971), p. 287-288.

P. ERDÖS AND A. HAJNAL (BUDAPEST)

P 1159. Let S be an infinite set of elements $x_1, x_2, \ldots$ To every edge (x_i, x_j) make correspond a subset $E_{i,j}$ of $(0, 1)$ of measure greater than $\varepsilon > 0$ (ε does not depend on i and j). Is there an infinite path

$x_{i_1}, x_{i_2}, x_{i_3}, \ldots$ such that the intersection of all sets $E_{i_r, i_{r+1}}$, $r = 1, 2, \ldots$, is non-empty?

Further problem: Let $|S| = \aleph_1$. To each triple $(x_{a_1}, x_{a_2}, x_{a_3})$ of S make correspond a measurable subset E_{a_1, a_2, a_3} of measure greater than ε. Is it true that there must be four points $x_{a_1}, x_{a_2}, x_{a_3}, x_{a_4}$ such that the intersection of the four sets $E_{a_{i_1}, a_{i_2}, a_{i_3}}$ ($a_{i_1}, a_{i_2}, a_{i_3}$ run over the four triples from a_1, a_2, a_3, a_4) is non-empty?

For further problems of this type see (6).

Letter of P. Erdös, November 1977.

(6) P. Erdös and A. Hajnal, *Some remarks on set theory, IX. Combinatorial problems in measure theory and set theory*, Michigan Mathematical Journal 11 (1964), p. 107-127.

P. ERDÖS (BUDAPEST) AND D. PREISS (PRAGUE)

P 1160. Let $E_{\aleph_1}$ be the unit sphere of the $\aleph_1$-dimensional Hilbert space. Join two points of $E_{\aleph_1}$ if their distance is greater than $\sqrt{2}$. Is the chromatic number of this graph $\aleph_1$ or $\aleph_0$?

We proved (7) that if $\sqrt{2}$ is replaced by $\sqrt{2}+\varepsilon$, then the chromatic number is $\aleph_0$. If $> \sqrt{2}$ is replaced by $\geqslant \sqrt{2}$, then the chromatic number is $\aleph_1$, since our graph contains a $K_{\aleph_1}$.

Letter of P. Erdös, November 1977.

(7) P. Erdös and D. Preiss, *Decomposition of spheres in Hilbert spaces*, Commentationes Mathematicae Universitatis Carolinae 17 (1976), p. 791-795.

P. ERDÖS (BUDAPEST) AND D. SILVERMAN (LOS ANGELES, CALIFORNIA)

P 1161. Define an infinite graph whose vertices are the integers defined as follows. Join i and j if $i+j$ is a square. Prove (or disprove) that the chromatic number of this graph is infinite.

P 1162. Let $1 \leqslant u_1 < \ldots < u_l$ be a sequence of integers such that none of the sums $u_i + u_j$ is a square. Determine or estimate $\max l$. The u's can be chosen $\equiv 1 \pmod 3$, i.e., $\max l \geqslant n/3$. Is $\max l > (1+\varepsilon)n/3$ possible? Clearly, the squares can be replaced by other sets of numbers, e.g., cubes, etc. If $u_i + u_j$ is replaced by $u_i - u_j$, then L. Lovász conjectured some time ago that $l = o(n)$. This was proved by Sárközy (8) and quite independently by H. Fürstenberg.

Letter of P. Erdös, November 1977.

(8) A. Sárközy, *On difference sets of sequence of integers*, I, II and III, I and III in Acta Mathematica Academiae Scientiarum Hungaricae (to appear), II in Annales Universitatis Scientiarum Budapestinensis (to appear).

JERZY ŁOŚ (WARSZAWA)

P 1163. The characteristic set of an n-tuple of equivalence relations $R_1, \dots, R_n$ in a set X is defined as

$$C = \{(\chi_{R_1}(x, y), \dots, \chi_{R_n}(x, y)) \mid (x, y) \in X^2\} \subset \{0, 1\}^n,$$

where χ_{R_i} denote the characteristic functions of R_i in X^2. Obviously, not every set $C \subset \{0, 1\}^n$ can serve as a characteristic set of an n-tuple of equivalence relations, since $\mathit{1} = (1, \dots, 1)$ (the sequence of only ones) must belong to it. This is not, however, the only condition on C.

If R and Q are two equivalence relations on X, then for all $a, b, c, d \in X$ the following is true:

if $(a, b) \in R \cap \bar{Q}$ and $(c, d) \in \bar{R} \cap Q$, then at least one pair (a, c), (a, d), (b, c) or (b, d) belongs to $\bar{R} \cap \bar{Q}$.

Here $\bar{R}$ and $\bar{Q}$ denote the complements of R and Q, respectively, in X^2.

It follows that, with $n = 2$, the set consisting of three pairs $(1, 1)$ $(1, 0)$ and $(0, 1)$ is not a characteristic set. Actually, something more can be said about a characteristic set. It fulfils the following

CONDITION. For every $p, q \in C$ there exist $r, s, t, u \in C$ such that for every i, j $(i, j = 1, 2, \dots, n)$ if $p_i = 1 = q_j$ and $p_j = 0 = q_i$, then $r_i = r_j = 0$ or $s_i = s_j = 0$ or $t_i = t_j = 0$ or $u_i = u_j = 0$.

PROBLEM. Is the condition above sufficient in order to represent a set $C \subset \{0, 1\}^n$, to which $\mathit{1}$ belongs, as a characteristic set of an n-tuple of equivalence relations?

P 1163, R 1. A partial answer can be given. If $\mathit{0} = (0, \dots, 0)$ (the sequence of only zeroes) belongs to C, then the condition is obviously satisfied. Such a set to which $\mathit{1}$ and $\mathit{0}$ belong can always be represented by a very peculiar kind of equivalence relations, what is to be seen from the following construction.

To every $t \in C$ we make correspond a pair of distinct elements $\{a_t, b_t\}$. The set X of all a_t's and b_t's which contains twice the number of elements in C is already partitioned into pairs $\{a_t, b_t\}$. We define each relation R_i by a partition of X, being a subpartition of that partition. Namely, to define the relation R_i, we keep the pair $\{a_t, b_t\}$ together if $t_i = 1$ and we split it apart replacing by $\{a_t\}$, $\{b_t\}$ if $t_i = 0$. It is easy to check that this construction provides us with an n-tuple of equivalence relations having the characteristic set $C \cup \{\mathit{0}, \mathit{1}\}$; thus, if $\mathit{0}, \mathit{1} \in C$, it is C. The problem is therefore reduced to sets C with $\mathit{1} \in C$ and $\mathit{0} \notin C$.

TABLE DES MATIÈRES DU VOLUME XLII

C O M M U N I C A T I O N S

PROBLÈMES

P 1, **P 2**, ... désignent les problèmes posés; **R 1**, **R 2**, ... désignent les réponses et remarques concernant le problème en tête de ligne. Les autres numéros indiquent les pages. Celles des communications déjà citées dans les remarques sont omises.

AUTEURS