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Mean growth and Fourier coefficients of some classes of holomorphic functions on bounded symmetric domains

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Abstract. Let D be a bounded symmetric domain in C^n with Bergman–Silov boundary B and $0 \in D$. Let $N(D)$ denote the Nevanlinna class of holomorphic functions f on D for which $\sup_{0<r<1} \int_B \log^+ |f(rt)|\, d\lambda(t) < \infty$, and let $N_*(D)$ denote the subspace of $N(D)$ for which the family $\{\log^+ |f(rt)|: 0 < r < 1\}$ is uniformly integrable on D. If $D = \bigtimes_{i=1}^{k} D_i$, where each D_i is irreducible of dimension n_i, let

$$M_\infty(f;(r)) = \sup \{|f(r_1 t_1, \ldots, r_k t_k)|: (t) \in \bigtimes_{i=1}^{k} B_i\}.$$

Let $F_*(D)$ denote the space of holomorphic functions f on D for which

$$\|f\|_c = \int_{I^k} \exp[-c \prod_{i=1}^{n} (1-r_i)^{-n_i}] M_\infty(f;(r))\, dr_1 \ldots dr_k < \infty$$

for all $c > 0$. In the paper it is shown that $F_*(D)$ is a countably normed Fréchet space containing $N_*(D)$ as a dense subspace. Furthermore, if D is irreducible of dimension n and $f(z) = \sum_{k=0}^{\infty} \sum_{v=1}^{m_k} a_{kv}(f) \varphi_{kv}(z)$, then $f \in F_*(D)$ if and only if $|a_{kv}(f)| \leqslant \exp[\lambda_k k^{n/(n+1)}]$ for some sequence λ_k decreasing to zero. This result is then used to characterize the continuous linear functionals on $F_*(D)$. The paper also contains results concerning the rate of growth of the means M_∞ of Poisson integrals of measures and of functions in the space $N(D)$ and $N_*(D)$.

1. Introduction. For the unit disc $U = \{z: |z| < 1\}$ in C, the Nevanlinna class N is the algebra of holomorphic functions f on U for which the characteristic function

$$T(f, r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{it})|\, dt$$

is bounded for $0 \leqslant r < 1$. The Smirnov class N_* is the set of functions $f \in N$ for which

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{it})| dt = \int_0^{2\pi} \log^+ |f^*(e^{it})|\, dt,$$

where $f^*(e^{it}) = \lim_{r\to 1} f(re^{it})$ a.e. If one defines $d(f, g) = \|f-g\|$, where for $f \in N$,

$$\|f\| = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \log(1+|f(re^{it})|)\, dt,$$

then d is a complete, translation invariant metric on N which induces a topology stronger than that of uniform convergence on compact subsets of U. The space $N(U)$, especially some of its unusual topological properties, has been investigated by J. H. Shapiro and A. L. Shields in [9]. The space $N_*(U)$ and its containing Fréchet space F^+ has been thoroughly investigated by N. Yanagihara in [14]–[16] among others.

For domains in C^n, the spaces N and N_* have been considered by K. T. Hahn for star-shaped circular domains in [1] and the space N_* has also been considered by the author for bounded symmetric domains in [10], [11]. Analogues of some of the results of [9] have also been looked at in [7] for the polydisc in C^n.

The purpose of the present paper is to extend a number of the one variable results for the space N and N_* to bounded symmetric domains in C^n. In Sections 2 and 3 we introduce the necessary notation and include some preliminary results about functions in N and N_*. In Section 4 we consider the rate of growth of the means $M_\infty(f; r)$ for $f \in N$ and N_*. A classical result due to S. N. Mergeljan (see [8], p. 106) states that if $f \in N(U)$, then

$$\lim_{r\to 1} \sup (1-r) \log^+ M_\infty(f, r) < \infty.$$

For functions in N_*, N. Yanagihara has shown in [16] that

$$\lim_{r\to 1} (1-r) \log^+ M_\infty(f, r) = 0. \tag{1.1}$$

In Section 4 we obtain analogues of these results for products of irreducible bounded symmetric domains by investigating the rate of growth of Poisson integrals of measures.

In Sections 5 and 6 we consider the space F_*, first for irreducible domains in Section 5 and for arbitrary domains in Section 6. In the unit disc, the space F_* (F^+ in the notation of [15]) consists of all holomorphic functions f on D for which (1.1) holds. This is equivalent to

$$\|f\|_c = \int_0^1 \exp\left[\frac{-c}{(1-r)}\right] M_\infty(f, r)\, dr < \infty$$

for all $c > 0$. If D is an irreducible bounded symmetric domain of dimension

n, then for functions $f \in N_*(D)$, the analogue of (1.1) becomes

$$\lim_{r \to 1} (1-r)^n \log^+ M_\infty(f, r) = 0 \tag{1.2}$$

and thus F_* is defined as the space of holomorphic functions f on D for which (1.2) holds. In Section 5, we show that this is equivalent to the following integrals being finite for every $c > 0$:

$$\int_0^1 \exp[-c(1-r)^{-n}] M_\infty(f, r)\, dr.$$

We also show that if $f(z)$ has the series expansion $f(z) = \sum_{k,v} a_{kv} \varphi_{kv}(z)$, where the $\{\varphi_{kv}\}$ are a complete orthonormal system on D, then $f \in F_*$ if and only if

$$|a_{kv}| \leqslant \exp[\lambda_k k^{n/(n+1)}]$$

for some sequence λ_k decreasing to zero. For the case $n = 1$, this was proved in [15]. This section also gives a characterization of the continuous linear functionals on F_*. Finally in Section 6 we indicate what modifications need to be made in the definition of F_* for products of irreducible domains.

2. Preliminaries. Let D be a bounded symmetric domain in C^n with Bergman–Silov boundary B and $0 \in D$. The domain D is circular and starlike, that is $tz \in D$ whenever $z \in D$, $t \in C$, $|t| \leqslant 1$ [5]. Let G denote the connected component of the identity of the group of holomorphic automorphisms of D, and K the isotropy subgroup of G at the origin. The group G is transitive on D and extends continuously to the topological boundary of D. The Bergman–Silov boundary B is circular and invariant under G. The group K acts transitively on B and B admits a unique K-invariant normalized measure which we denote by λ. The domain D is irreducible if it cannot be written as a product of domains D_i of lower dimension.

The Poisson kernel P on $D \times B$ is given by

$$P(z, t) = \frac{|S(z, t)|^2}{S(z, z)}, \quad (z, t) \in D \times B,$$

where S is the Szegö kernel (reproducing kernel of the Hardy space $H^2(D)$) on D. By Proposition 2 of [4],

$$\left(\frac{1-r}{1+r}\right)^n \leqslant P(ru, t) \leqslant \left(\frac{1+r}{1-r}\right)^n \tag{2.1}$$

for all u, $t \in B$ and

$$P(rt, t) = \left(\frac{1+r}{1-r}\right)^n. \tag{2.2}$$

For the classical Cartan domains of type I–IV, inequality (2.1) has also been proved in [13].

If μ is a finite signed Borel measure on B, we denote by $P_z[d\mu]$ the Poisson integral of μ, that is,

$$P_z[d\mu] = \int_B P(z, t)\,d\mu(t), \qquad z \in D. \tag{2.3}$$

For an integrable function f on B, we set $P_z[f] = P_z[f d\lambda]$. If μ is a finite signed Borel measure on B, or if $f \in L^1(B)$, then the function $F(z) = P_z[d\mu]$ ($P_z[f]$) is harmonic on D in the sense that $\Delta F = 0$ for all G-invariant differential operators Δ on D which annihilate constants. Such functions are often referred to as *strongly harmonic*.

Finally, for a real valued function f on D, we let $f^+ = \max\{f, 0\}$, and for $0 < r < 1$, $z \in \bar{D}$, $f_r(z) = f(rz)$. For a signed measure μ on D, $|\mu|$, μ^+, and μ^- denote the total variation, positive variation, and negative variation of μ respectively.

3. Functions of bounded characteristic. In analogy with the one variable case, as in [1], we define the Nevanlinna class $N(D)$ as the algebra of holomorphic functions f on D for which

$$\sup_{0<r<1} \int_B \log^+ |f(rt)|\,d\lambda(t) < \infty.$$

The subspace $N_* = N_*(D)$ consists of those $f \in N(D)$ for which the family $\{\log^+ |f_r|: 0 < r < 1\}$ is uniformly integrable on B.

For completeness, we state the following result which was proved in [10] as a lemma.

LEMMA 3.1. (a) *If $f \in N(D)$, $f \not\equiv 0$, then there exists a minimal Borel measure μ_f on B such that*

$$\log|f(z)| \leqslant P_z[d\mu_f], \qquad z \in D, \tag{3.1}$$

with $d\mu_f = \log|f^*|\,d\lambda + d\sigma_f$, where $f^*(t) = \lim_{r\to 1} f(rt)$ a.e. on B, $\log|f^*| \in L^1(B)$, and σ_f is singular with respect to λ on B.

(b) *If $f \in N_*(D)$, then $\sigma_f \leqslant 0$ and*

$$\log|f(z)| \leqslant P_z[\log|f^*|], \qquad z \in D. \tag{3.2}$$

Furthermore,

$$\lim_{r\to 1} \int_B \log^+ |f_r|\,d\lambda = \int_B \log^+ |f^*|\,d\lambda. \tag{3.3}$$

Note. Inequalities (3.1) and (3.2) remain valid if log is replaced by $\log^+$ and μ_f by μ_f^+. The measure μ_f is minimal in the sense that if ν is any other measure for which inequality (3.1) holds, then $\int \varphi\,d\mu_f \leqslant \int \varphi\,d\nu$ for all non-negative continuous functions φ on B.

For $f \in N$, we will refer to the measure μ_f as the *boundary measure* of

$\log|f|$. The measure μ_f is the weak star limit of $\log|f_r|$, that is

$$\lim_{r\to 1} \int_B \varphi(t) \log|f(rt)|\, d\lambda(t) = \int_B \varphi(t)\, d\mu_f(t)$$

for all φ continuous on B. Similarly, μ_f^+ is the weak star limit of $\log^+|f_r|$.

As in the one variable case, for $f \in N(D)$ we set

$$\|f\| = \lim_{r\to 1} \int_B \log(1+|f_r|)\, d\lambda.$$

Since

$$\log^+ x \leqslant \log(1+x) \leqslant \log 2 + \log^+ x \quad (x > 0), \tag{3.4}$$

$f \in N$ if and only if $\|f\| < \infty$. By Theorem 2 of [11], if $f \in N_*$, then

$$\|f\| = \int_B \log(1+|f^*|)\, d\lambda. \tag{3.5}$$

For $f \in N$, we have the following

PROPOSITION 3.2. *Let $f \in N$ with boundary measure μ_f. Then*

$$\|f\| = \int_B \log(1+|f^*|)\, d\lambda + \sigma_f^+(B), \tag{3.6}$$

where σ_f is the singular part of μ_f. Furthermore

$$\lim_{a\to 0} \|af\| = \sigma_f^+(B). \tag{3.7}$$

Proof. Since $\sup_{0<r<1} \int_B \log(1+|f_r|)\, d\lambda < \infty$, by Theorem 1 of [10], there exists a nonnegative Borel measure ν on B, with $d\nu = \log(1+|f^*|)\, d\lambda + d\beta$ where β is singular, which is the weak star limit of $\log(1+|f_r|)$. If φ is continuous and nonnegative on B, $0 < r < 1$, then by (3.4)

$$\int_B \varphi \log^+|f_r|\, d\lambda \leqslant \int_B \varphi \log(1+|f_r|)\, d\lambda \leqslant \int_B \varphi(\log 2 + \log^+|f_r|)\, d\lambda.$$

Letting $r \to 1$ we obtain

$$\int_B \varphi \log^+|f^*|\, d\lambda + \int_B \varphi d\sigma_f^+$$
$$\leqslant \int_B \varphi \log(1+|f^*|)\, d\lambda + \int_B \varphi d\beta \leqslant \int_B \varphi(\log 2 + \log^+|f^*|)\, d\lambda + \int_B \varphi d\sigma_f^+.$$

Since this inequality holds for all nonnegative continuous functions φ on B, the same inequality must hold among the respective measures, and therefore among their singular parts. Hence $\beta = \sigma_f^+$ and thus $d\nu = \log(1+|f^*|)\, d\lambda + {} + d\sigma_f^+$ which proves (3.6).

To prove (3.7), we first show that for $a \neq 0$, the singular part of the boundary measure μ_{af} of $\log|af|$ is precisely σ_f. As above, if φ is continuous on B,

$$\int_B \varphi d\mu_{af} = \lim_{r\to 1} \int_B \varphi(t)\log|af(rt)|\,d\lambda(t) = \int_B (\log|af^*|)\,\varphi(t)\,d\lambda + \int_B \varphi(t)\,d\sigma_f.$$

Since this holds for all φ continuous on B,

$$d\mu_{af} = \log|af^*|\,d\lambda + d\sigma_f.$$

Thus by (3.6)

$$\|af\| = \int_B \log(1+|af^*|)\,d\lambda + \sigma_f^+(B).$$

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{a\to 0} \int_B \log(1+|af^*|)\,d\lambda = 0.$$

Hence $\lim_{a\to 0}\|af\| = \sigma_f^+(B)$, which completes the proof.

For functions f in N_*, $\sigma_f^+ = 0$ and thus (3.6) reduces to (3.5). Another immediate consequence of (3.7) is that scalar multiplication is not continuous in $N(D)$. Consequently $N(D)$ is not a topological vector space. However, as in the one variable case, we do have the following

PROPOSITION 3.3. *$N(D)$ with the translation invariant metric $d(f, g) = \|f-g\|$ is a complete metric space whose topology is stronger than that of uniform convergence on compact subsets of D.*

The proof is identical to the one variable proof and follows immediately from the inequality.

$$\log(1+|f(rz)|) \leqslant \left(\frac{1+r}{1-r}\right)^n \|f\|$$

for all r, $0 < r < 1$, $z\in D$. The above follows from the fact that $\log(1+|f_r(z)|)$ is plurisubharmonic on $\bar{D}$.

As was shown in [11], the space $N_*(D)$ is a topological vector space with a complete translation invariant metric in which multiplication is also continuous. For the unit ball in C^n and the polydisc it is known that $N_* \subsetneqq N$. Whether this is true for all bounded symmetric domains is still not known.

4. Rate of growth of Poisson integrals. Throughout this section we will assume that D is the product of k irreducible bounded symmetric domains D_i, $i = 1, \ldots, k$. Then the Bergman–Silov boundary B of D is given by $B = \bigtimes_{i=1}^{k} B_i$, where B_i is the Bergman–Silov boundary of D_i, and the Poisson kernel P on $D \times B$ is given by

$$P(z, t) = P\big((z_1, \ldots, z_k), (t_1, \ldots, t_k)\big) = \prod_{i=1}^{k} P_i(z_i, t_i),$$

where P_i is the Poisson kernel on $D_i \times B_i$.

For $0 \leqslant r < 1$, $i = 1, \ldots, k$, we will denote by (r) the vector $(r_1, \ldots, r_k)$, and for $u = (u_1, \ldots, u_k) \in B$ with $u_i \in B_i$, we denote $(r_1 u_1, \ldots, r_k u_k)$ by $(r)u$. Then by (2.1), for all t, $u \in B$

$$\prod_{i=1}^{k} \left(\frac{1-r_i}{1+r_i}\right)^{n_i} \leqslant P\big((r)u, t\big) \leqslant \prod_{i=1}^{k} \left(\frac{1+r_i}{1-r_i}\right)^{n_i} \tag{4.1}$$

and

$$P\big((r)t, t\big) = \prod_{i=1}^{k} \left(\frac{1+r_i}{1-r_i}\right)^{n_i}, \tag{4.2}$$

where $n_i = \dim D_i$. Also, if f is a continuous function on D, we set

$$M_\infty\big(f;(r)\big) = \sup_{(t_1, \ldots, t_k) \in B} |f(r_1 t_1, \ldots, r_k t_k)|. \tag{4.3}$$

For $r_i = r$ for all i, this becomes

$$M_\infty(f; r) = \sup_{t \in B} |f(rt)|. \tag{4.4}$$

For the proof of Theorem 4.2, we need the following lemma.

LEMMA 4.1. *For each bounded symmetric domain D, there exists a constant p, depending only on D, such that:*

(a) *For each $u \in B$ and each neighborhood $\mathscr{U}$ of u, there exists a constant M such that for each $t \in B \setminus \mathscr{U}$*

$$\limsup_{r \to 1} (1-r)^p P(rt, u) \leqslant M;$$

(b) *For each $q < p$, there exists $t \neq u$ such that*

$$\limsup_{r \to 1} (1-r)^q P(rt, u) = +\infty.$$

If D is irreducible of dimension n and rank l (as a symmetric space), then $p - n - 2n/l$.

For the classical Cartan domains of type I–IV, this result was first proved in [12]. For arbitrary bounded symmetric domains the result has recently been proved by M. Lassalle in [6].

THEOREM 4.2. *Let $D = \bigtimes_{i=1}^{k} D_i$ be the product of irreducible bounded symmetric domains D_i, each of (complex) dimension n_i.*

(a) *If* $F(z) = P_z[f]$, $f \in L^1(B)$, *then*

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) M_\infty(F, (r)) = 0. \tag{4.5}$$

(b) *If* μ *is a complex or finite Borel measure on* B *and* $F(z) = P_z[d\mu]$, *then for all* $t \in B$,

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) F((r)t) = 2^n \sigma(\{t\}), \tag{4.6}$$

where σ *is the singular part of* μ *(with respect to* λ*) and* $n = n_1 + \ldots + n_k$.

(c) *If in addition* μ *is continuous on* B, *that is,* $\mu(\{t\}) = 0$ *for each* $t \in B$, *then*

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) M_\infty(F; (r)) = 0. \tag{4.7}$$

Proof. (a) Suppose $F(z) = P_z[f]$ with $f \in L^1(B)$. Let $\varepsilon > 0$ be given. Choose g bounded on B with $0 \leqslant g \leqslant |f|$ and $\int_B (|f| - g)\,d\lambda < \varepsilon$. Then for $u \in B$, $r = (r_1, \ldots, r_k)$,

$$|F((r)u)| \leqslant \int_B P((r)u, t)(|f(t)| - g(t))\,d\lambda(t) + \int_B P((r)u, t) g(t)\,d\lambda(t),$$

and by (4.1)

$$|F((r)u)| \leqslant \varepsilon \prod_{i=1}^{k} \left(\frac{1+r_i}{1-r_i}\right)^{n_i} + \|g\|_\infty \int_B P((r)u, t)\,d\lambda(t).$$

Since $\int_B P(z, t)\,d\lambda(t) = 1$ for all $z \in D$.

$$M_\infty(F, (r)) \leqslant \varepsilon \prod_{i=1}^{k} \left(\frac{1+r_i}{1-r_i}\right)^{n_i} + \|g\|_\infty,$$

from which (4.5) now follows.

(b) Suppose μ is a complex Borel measure on B. By linearity it suffices to assume that μ is a nonnegative finite Borel measure. Let σ denote the singular part of μ in the Lebesgue decomposition of μ with respect to λ, i.e. $d\mu = f d\lambda + d\sigma$, $f \in L^1(B)$. By (a), it suffices to show that

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) F_\sigma((r)u) = 2^n \sigma(\{u\}),$$

for every $u \in B$, where $F_\sigma = P[d\sigma]$.

Fix $u \in B$. We first consider the case where $\sigma(\{u\}) = 0$. Let $\varepsilon > 0$ be given. Then there exists a neighborhood $\mathscr{U}$ of u such that $\sigma(\mathscr{U}) < \varepsilon$. Let r

$= (r_1, \ldots, r_k)$ with $0 \leqslant r_j < 1$. By (4.1)

$$F_\sigma\big((r)u\big) \leqslant \sigma(\mathscr{U}) \prod_{i=1}^{k} \left(\frac{1+r_i}{1-r_i}\right)^{n_i} + \int_{B-\mathscr{U}} P\big((r)u, t\big)\, d\sigma(t). \tag{4.8}$$

Let P_i denote the Poisson kernel on $D_i \times B_i$. By (4.1), for each i,

$$(1-r_i)^{n_i} P_i(r_i u_i, t_i) \leqslant 2^{n_i}$$

for all u_i, $t_i \in B_i$, and by Lemma 4.1, for $t_i \neq u_i$,

$$\lim_{r_i \to 1} (1-r_i)^{n_i} P_i(r_i u_i, t_i) = 0.$$

Hence for all $t \neq u$,

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) P\big((r)u, t\big) = \lim_{(r)\to(1)} \prod_{i=1}^{k} (1-r_i)^{n_i} P_i(r_i u_i, t_i) = 0.$$

Therefore $\lim\limits_{(r)\to(1)} \prod\limits_{i=1}^{k} (1-r_i)^{n_i} \int\limits_{B-\mathscr{U}} P\big((r)u, t\big)\, d\sigma(t) = 0$, and hence by (4.8),

$$\limsup_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) F\big((r)u\big) \leqslant 2^n \varepsilon,$$

where $n = n_1 + \ldots + n_k$. Since $\varepsilon > 0$ was arbitrary,

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) F\big((r)u\big) = 0.$$

Now suppose $\sigma(\{u\}) = a > 0$. Let ν be the measure defined by $\nu = \sigma - a\delta_u$, where δ_u is point mass at u. Then ν is a nonnegative Borel measure with $\nu(\{u\}) = 0$. Hence if $H(z) = P_z[d\nu]$,

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) H\big((r)u\big) = 0.$$

But $H\big((r)u\big) = F_\sigma\big((r)u\big) - aP\big((r)u, u\big)$, and by (4.2),

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) P\big((r)u, u\big) = 2^n,$$

from which the result now follows.

(c) Suppose $\mu(\{t\}) = 0$ for each $t \in B$. By considering the total variation measure $|\mu|$, which is also continuous, we may assume $\mu \geqslant 0$. Let $F(z) = P_z[d\mu]$.

Let A denote the limit superior in (4.7), which is clearly finite. Then for each i, $1 \leqslant i \leqslant k$, there exists a sequence $\{r_{i(n)}\}_{n=1}^{\infty}$ with $r_{i(n)} \to 1$ such that

$$\Big(\prod_{i=1}^{k} (1-r_{i(n)})^{n_i}\Big) M_\infty\big(F; (r_{i(n)})\big) \to A.$$

By continuity of F and compactness of B_i, for each $i(n)$, there exists $u_{i(n)} \in B_i$ such that

$$M_\infty(F, (r_{i(n)})) = F(r_{1(n)} u_{1(n)}, \ldots, r_{k(n)} u_{k(n)}).$$

Let $u^{(n)} = (u_{1(n)}, \ldots, u_{k(n)})$. Since B is compact, by choosing subsequences if necessary, we may assume that $u^{(n)} \to u \in B$. Since μ is continuous at u, the proof of part (b) shows that

$$\lim_{n\to\infty} \int_B \prod_{i=1}^{k} (1-r_{i(n)})^{n_i} P_i(r_{i(n)} u_{i(n)}, t_i)\, d\mu(t) = 0.$$

Thus $A = 0$, which proves (4.6).

We now apply the results of Theorem 4.2 to functions in $N(D)$ and $N_*(D)$.

THEOREM 4.3. *Suppose $D = \bigtimes_{i=1}^{k} D_i$, where each D_i is irreducible of dimension n_i.*

(a) *Let $f \in N(D)$ with boundary measure μ_f. Then for every $t \in B$,*

$$\limsup_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) \log^+ |f((r)t)| \leqslant 2^n \sigma_f^+(\{t\}), \tag{4.9}$$

and

$$\limsup_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) \log^+ M_\infty(f;\, (r)) \leqslant 2^n \mu_f^+(B). \tag{4.10}$$

(b) *If $f \in N_*(D)$, then*

$$\lim_{(r)\to(1)} \Big(\prod_{i=1}^{k} (1-r_i)^{n_i}\Big) \log^+ M_\infty(f;\, (r)) = 0. \tag{4.11}$$

The proofs of (4.9) and (4.11) is an immediate consequence of Theorem 3. Inequality (4.10) follows immediately from (3.1) with $\log^+$ and μ_f^+ replacing log and μ_f and (4.1).

Remark. If $D = U^n$ is the unit polydisc in C^n, then by Proposition 4.2 of [7], equality holds in (4.9). For arbitrary bounded symmetric domains, even the unit ball, the question of whether equality holds in (4.9) is still unresolved. It certainly holds if $f(z)$ is nonzero in D.

5. The space F_* for irreducible domains. In this section we consider the space F_* which contains N_* as a dense subspace. We first consider the case where D is irreducible of dimension n ($n \geqslant 1$). By Theorem 4.3, if $f \in N_*(D)$, then

$$\lim_{r\to 1} (1-r)^n \log^+ M_\infty(f;\, r) = 0, \tag{5.1}$$

where here $M_\infty(f; r)$ is given by (4.4). Using the results of [15] as motivation, we define $F_*(D)$ as the class of holomorphic function f on D for which (5.1) holds.

If f is holomorphic on D, then as in [2], [3], f has a Fourier series expansion

$$f(z) = \sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k} a_{k\nu}(f)\, \varphi_{k\nu}(z), \tag{5.2}$$

where $\{\varphi_{k\nu}(z)\}$, $k = 0, 1, 2, \ldots$, $\nu = 1, \ldots$, $m_k = \binom{n+k-1}{k}$, is a complete orthogonal system of homogeneous polynomials on D which are normalized with respect to the measure λ on B, and

$$a_{k\nu}(f) = \lim_{r \to 1} \int_B f(rt)\, \overline{\varphi_{k\nu}(t)}\, d\lambda(t). \tag{5.3}$$

The series (5.1) converges absolutely and uniformly on compact subsets of D.

The following theorem characterizes F_* both in terms of an integral condition and also in terms of the Fourier coefficients $a_{k\nu}(f)$.

THEOREM 5.1. *Let D be irreducible of dimension n. Then for f holomorphic on D, the following are equivalent.*

(a) $f \in F_*(D)$.

(b) *For all* $c > 0$,

$$\|f\|_c = \int_0^1 \exp[-c(1-r)^{-n}]\, M_\infty(f; r)\, dr < \infty. \tag{5.4}$$

(c) *If $f(z)$ is given by the series* (5.2), *then*

$$|a_{k\nu}(f)| \leqslant \exp[\lambda_k k^{n/(n+1)}], \qquad \nu = 1, \ldots, m_k, \tag{5.5}$$

for some sequence λ_k decreasing to zero.

(d) *For any* $c > 0$

$$\sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k} |a_{k\nu}(f)| \exp[-ck^{n/(n+1)}] < \infty. \tag{5.6}$$

Proof. (i) Suppose $f \in F_*(D)$. Let $c > 0$ be arbitrary. Then by (5.1), there exists R, $0 < R < 1$, such that

$$M_\infty(f; r) \leqslant \exp\left[\frac{c}{(1-r)^n}\right]$$

for all $r \geqslant R$. For $r < R$, $M_\infty(f; r) \leqslant M_\infty(f; R)$. Therefore

$$\|f\|_{2c} \leqslant M_\infty(f; R) \int_0^R \exp[-2c(1-r)^{-n}]\, dr + \int_R^1 \exp[-c(1-r)^{-n}]\, dr$$

which is finite for all $c > 0$. Thus $\|f\|_c < \infty$ for all $c > 0$.

(ii) We now show that (b)$\Rightarrow$(a). For $R<1$ and $c>0$, set

$$I(R;c)=\int_R^1 \exp[-c(1-r)^{-n}]\,M_\infty(f;r)\,dr.$$

Since $M_\infty(f;r)$ is nondecreasing as a function of r,

$$I(R;c)\geqslant M_\infty(f;R)\int_R^1 \exp[-c(1-r)^{-n}]\,dr. \tag{5.7}$$

By the change of variable $t=(1-r)^{-n}$,

$$\int_R^1 \exp[-c(1-r)^{-n}]\,dr=\frac{1}{n}\int_T^\infty \exp(-ct)\,t^{-\gamma}\,dt,$$

where $T=(1-R)^{-n}$ and $\gamma=(n+1)/n$. The function $t^\gamma \exp[-ct]$, $t\in(0,\infty)$, has a maximum at $t=\gamma/c$. Thus

$$t^\gamma \exp(-ct)\leqslant(\gamma/ce)^\gamma,$$

and hence

$$t^{-\gamma}\exp(-ct)\geqslant(ce/\gamma)^\gamma\exp[-2ct].$$

Therefore

$$\frac{1}{n}\int_T^\infty \exp(-ct)\,t^{-\gamma}\,dt\geqslant\frac{1}{2cn}\left(\frac{ce}{\gamma}\right)^\gamma\exp[-2cT].$$

Combining this with (5.7) gives

$$M_\infty(f;R)\leqslant 2cn(\gamma/ce)^\gamma\exp[2c/(1-R)^n]\,I(R;c). \tag{5.8}$$

Thus

$$\limsup_{R\to1}(1-R)^n\log^+ M_\infty(f;R)\leqslant 2c$$

for all $c>0$. Thus $f\in F_*(D)$.

(iii) (a)$\Rightarrow$(c). Since $M_2(f;r)\leqslant M_\infty(f;r)$, $\lim_{r\to1}(1-r)^n\log^+ M_2(f;r)=0$. Hence there exists $\omega(r)$ decreasing to zero as $r\to1$ such that

$$M_\infty(f;R)\leqslant 2cn(\gamma/ce)^\gamma\exp[2c/(1-R)^n]\,I(R;c). \tag{5.8}$$

Since each $\varphi_{k\nu}$ is homogeneous of degree k, for $0<r<1$,

$$f_r(z)=f(rz)=\sum_{k,\nu}a_{k\nu}(f)\,r^k\varphi_{k\nu}(z).$$

Thus $a_{k\nu}(f_r) = a_{k\nu}(f)r^k$. Since f_r is continuous on $\bar{D}$,

$$a_{k\nu}(f_r) = \int_B f(rt)\overline{\varphi_{k\nu}(t)}\,d\lambda(t),$$

and hence by Hölder's inequality,

$$|a_{k\nu}(f_r)| \leqslant M_2(f;\, r)\,\|\varphi_{k\nu}\|_2.$$

Since $\|\varphi_{k\nu}\|_2 = 1$, from (5.9) and the above,

$$|a_{k\nu}(f)| \leqslant \frac{1}{r^k}\exp[\omega(r)(1-r)^{-n}].$$

Let $\varepsilon_j \downarrow 0$ (with $\varepsilon_1 < 2^{-(n+1)}$) and choose $\varrho_j \uparrow 1$ such that $\omega(r) \leqslant \varepsilon_j$ for $r \geqslant \varrho_j$. Choose a sequence of integers k_j with $k_{j+1} > k_j$ such that

$$1-(\varepsilon_j/k_j)^{1/(n+1} \geqslant \varrho_j.$$

For $k_j \leqslant k < k_{j+1}$, set

$$r_k = 1-(\varepsilon_k/k)^{1/(n+1)}.$$

Then $r_k \geqslant \varrho_j$ and

$$\exp[\omega(r_k)(1-r_k)^{-n}] \leqslant \exp[\varepsilon_k^{1/(n+1)}k^{n/(n+1)}].$$

Also, by the inequality $1-x > e^{-2x}$, which is valid for x sufficiently small $(x \leqslant \frac{1}{2})$,

$$(r_k)^k = \left(1-(\varepsilon_k/k)^{1/(n+1)}\right)^k \geqslant \exp[-2\varepsilon_k^{1/(n+1)}k^{n/(n+1)}].$$

Therefore

$$|a_{k\nu}(f)| \leqslant \exp[3\varepsilon_k^{1/(n+1)}k^{n/(n+1)}],$$

from which (5.5) follows with $\lambda_k = 3\varepsilon_k^{1/(n+1)}$, $k_j \leqslant k < k_{j+1}$.

(iv) (c) $\Rightarrow$ (d). Suppose (5.5) holds for some positive seguence λ_k decreasing to zero. Let $c > 0$ be arbitrary. Choose K such that $\lambda_k < c/2$ for all $k \geqslant K$. Then

$$\sum_{k=K}^{\infty}\sum_{\nu=1}^{m_k}|a_{k\nu}(f)|\exp[-ck^{n/(n+1)}] \leqslant \sum_{k=K}^{\infty}\binom{n+k-1}{k}\exp[-\tfrac{1}{2}ck^{n/(n+1)}]$$

which converges.

(v) (d) $\Rightarrow$ (a). Suppose f is given by (5.2). Then

$$M_1(f;\, r) = \int_B |f(rt)|\,d\lambda(t) \leqslant \sum_{k,\nu}|a_{k\nu}|\,r^k\int_B|\varphi_{k\nu}(t)|\,d\lambda(t).$$

But $\|\varphi_{k\nu}\|_1 \leqslant 1$. Consider $I(c) \equiv \int_0^1 \exp[-c(1-r)^{-n}] M_1(f; r)\,dr$. Then

$$I(c) \leqslant \sum_{k,\nu} |a_{k\nu}| \int_0^1 r^k \exp[-c(1-r)^{-n}]\,dr.$$

Define the sequence $\{r_k'\}$ and $\{r_k''\}$ by

$$r_k' = 1-(c/k)^{1/(n+1)}, \quad r_k'' = 1-2(c/k)^{1/(n+1)}. \tag{5.10}$$

Then for $r_k'' \leqslant r \leqslant r_k'$

$$\begin{aligned} r^k \exp[-c(1-r)^{-n}] &\leqslant \exp[-(1+2^{-n})c^{1/(n+1)} k^{n/(n+1)}] \\ &\leqslant \exp[-c^{1/(n+1)} k^{n/(n+1)}]. \end{aligned}$$

For $r \leqslant r_k''$,

$$\begin{aligned} r^k \exp[-c(1-r)^{-n}] &\leqslant r^k \leqslant \left(1-2(c/k)^{1/(n+1)}\right)^k \\ &\leqslant \exp[-2c^{1/(n+1)} k^{n/(n+1)}] \leqslant \exp[-c^{1/(n+1)} k^{n/(n+1}], \end{aligned}$$

and for $r \geqslant r_k'$,

$$r^k \exp[-c(1-r)^{-n}] \leqslant \exp[-c^{1/(n+1)} k^{n/(n+1)}].$$

Therefore for all r, $0 < r < 1$,

$$r^k \exp[-c(1-r)^{-n}] \leqslant \exp[-c^{1/(n+1)} k^{n/(n+1)}],$$

and hence by assumption $I(c) < \infty$.

As in the proof of (5.8)

$$M_1(f; r) \leqslant A \exp[2c/(1-r)^n]\, I(c),$$

where A is a constant depending only on c and n. Let $0 < r \leqslant \varrho < 1$. Since $f_\varrho(z)$ is holomorphic in D and continuous in $\bar{D}$,

$$f_\varrho(z) = \int_B P(z, t) f_\varrho(t)\,d\lambda(t).$$

Thus by (2.1)

$$M_\infty(f_\varrho, r) \leqslant \frac{2^n}{(1-r)^n} M_1(f; \varrho).$$

By the above, taking $\varrho = r$ gives

$$M_\infty(f; r^2) \leqslant 2^n A (1-r)^{-n} \exp[2c/(1-r)^n]\, I(c),$$

from which it follows that

$$\limsup_{r \to 1} (1-r)^n \log^+ M_\infty(f, r) \leqslant 2^n c.$$

Thus $f \in F_*$.

The family of (semi) norms $\{\|\ \|_c\}_{c>0}$ given by (5.4) defines a locally convex topology on F_*. Furthermore, since $\|f\|_{c_2} \leqslant \|f\|_{c_1}$ whenever $0 < c_1 < c_2$, the topology on F_* can be defined by a countable family of seminorms.

THEOREM 5.2.

(a) *F_* is a countably normed Fréchet space containing N_* as a dense subspace. The topology of F_* restricted to N_* is weaker than the topology in N_* defined by the distance function* (3.5).

(b) *For $f \in F_*$, $f_r \to f$ in F_* as $r \to 1$.*

Proof. Suppose $\{f_j\}$ is a Cauchy sequence in F_*. Then by inequality (5.8), for $0 < r < 1$

$$M_\infty(f_i - f_j, r) \leqslant A \exp[2c(1-r)^{-n}]\,\|f_i - f_j\|_c,$$

from which it follows that $\{f_j\}$ converges uniformly on compact subsets of D to a holomorphic function f. The proof that $f_j \to f$ in F^* is straightforward.

Suppose $\{f_j\} \subset N_*$ and $f_j \to f \in N_*$ with respect to the metric $d(f, g) = \|f - g\|$ given by (3.5). Since $\log^+ x \leqslant \log(1+x)$, by (2.1) and (3.2), for $u \in B$,

$$\log^+ |f_j(ru) - f(ru)| \leqslant \int_B P(ru, t) \log^+ |f_j^*(t) - f^*(t)|\, d\lambda(t)$$

$$\leqslant \frac{2^n}{(1-r)^n} \int_B \log(1 + |f_j^* - f^*|)\, d\lambda.$$

Therefore,

$$M_\infty(f_j - f, r) \leqslant \exp[2^n \|f_j - f\| (1-r)^{-n}].$$

Let $\varepsilon > 0$ be given and let $c > 0$ be arbitrary. Choose R, $0 < R < 1$ such that

$$\int_R^1 \exp[-\tfrac{1}{2}c(1-r)^{-n}]\, dr < \tfrac{1}{2}\varepsilon.$$

Also, choose J, such that $2^n \|f_j - f\| < \frac{1}{2}c$ for all $j \geqslant J$, and

$$\int_0^R \exp[-c(1-r)^{-n}]\, M_\infty(f_j - f; r)\, dr < \tfrac{1}{2}\varepsilon.$$

The last follows since $f_j \to f$ uniformly on compact subsets of D. Then for all $j \geqslant J$,

$$\|f_j - f\|_c \leqslant \int_0^R \exp[-c(1-r)^{-n}]\, M_\infty(f_j - f; r)\, dr + \int_R^1 \exp[-\tfrac{1}{2}c(1-r)^{-n}]\, dr < \varepsilon.$$

Therefore $\|f_j - f\|_c \to 0$ as $j \to \infty$. Since this holds for all $c > 0$, $f_j \to f$ in the topology of F_*. Therefore the topology of F_* restricted to N_* is weaker than the topology on N_* given by the metric (3.5).

Let $f \in F_*$. Then for all r, $0 < r < 1$, $f_r \in N_*$. Thus to show N_* is dense in F_* we only need to show that $f_r \to f$ in F_*. But this is an immediate consequence of the fact that f_r converges uniformly to f on compact subsets of D and that for $0 < \varrho < 1$,

$$M_\infty(f_r - f; \varrho) \leqslant M_\infty(f, r\varrho) + M_\infty(f, \varrho) \leqslant 2M_\infty(f, \varrho).$$

We conclude this section by giving a characterization of the continuous linear functionals on F_*.

THEOREM 5.3. *If Γ is a continuous linear functional on F_*, then there exists a sequence $\{b_{k,v}\}$, $k = 0, 1, 2, \ldots$; $v = 1, \ldots, m_k$ of complex numbers with*

$$b_{k,v} = O(\exp[-ck^{n/(n+1)}]) \tag{5.11}$$

for some $c > 0$, such that

$$\Gamma(f) = \sum_{k=0}^{\infty} \sum_{v=1}^{m_k} a_{kv}(f) b_{kv}, \tag{5.12}$$

where $f \in F_*$ is given by (5.2), *with convergence being absolute. Conversely if $\{b_{kv}\}$ satisfies* (5.11), *then* (5.12) *defines a continuous linear functional on F_*.*

Proof. Suppose Γ is a continuous linear functional on F_*. Put

$$b_{kv} = \Gamma(\varphi_{kv}).$$

For $f(z) = \sum_{k,v} a_{kv} \varphi_{kv} \in F_*$, let

$$f_r^j(z) = \sum_{k=0}^{j} \sum_{v=1}^{m_k} a_{kv} r^k \varphi_{kv}(z).$$

Since $f_r^j \to f_r$ in F_* as $j \to \infty$,

$$\Gamma(f_r) = \lim_{j \to \infty} \Gamma(f_r^j) = \sum_{k,v} a_{kv} b_{kv} r^k.$$

Since $f_r \to f$ in F_*,

$$\Gamma(f) = \sum_{k,v} a_{kv} b_{kv}. \tag{5.13}$$

By Theorem 5.1 (c), the above series must converge for any sequence $\{a_{kv}\}$ satisfying (5.5). Hence the argument of a_{kv} can be chosen arbitrarily. Therefore the series in (5.13) converges absolutely. Also, the sequence $\{b_{kv}\}$ must satisfy

$$b_{kv} = O(\exp[-\lambda_k k^{n/(n+1)}])$$

for any sequence $\{\lambda_k\}$ with $\lambda_k \to 0$. Therefore as in Lemma 1 of [14]

$$b_{kv} = O(\exp[-ck^{n/(n+1)}])$$

for some $c > 0$.

Conversely, if $\{b_{k\nu}\}$ is a sequence of complex numbers satisfying (5.11), by (5.6), Γ defined by (5.12) is a linear functional on F_*. To show Γ is continuous, we will show that $|\Gamma(f)| \leqslant A\|f\|_{c'}$ for some constant A and some c' (depending on c).

For $t \in B$, let

$$g(t) = \int_0^1 \exp[-c(1-r)^{-n}] f(rt)\, dr.$$

Then $|g(t)| \leqslant \|f\|_c$, and

$$g(t)\overline{g(t)} = \int_0^1\int_0^1 \exp[-c(1-r)^{-n}]\exp[-c(1-\varrho)^{-n}] f(rt)\overline{f(\varrho t)}\, dr\, d\varrho.$$

Using the orthonormality of the $\{\varphi_{k\nu}\}$ on B,

$$\int_B |g(t)|^2\, d\lambda(t) = \sum_{k,\nu} |a_{k\nu}|^2 \int_0^1\int_0^1 \exp\left[\frac{-c}{(1-r)^n}\right]\exp\left[\frac{-c}{(1-\varrho)^n}\right] r^k \varrho^k\, dr\, d\varrho$$

$$= \sum_{k,\nu} |a_{k\nu}|^2 \left(\int_0^1 \exp\left[\frac{-c}{(1-r)^n}\right] r^k\, dr\right)^2.$$

With r_k' and r_k'' defined as in (5.10), for $r_k'' \leqslant r \leqslant r_k'$,

$$r^k \exp\left[\frac{-c}{(1-r)^n}\right] \geqslant \exp[-5c^{1/(n+1)} k^{n/(n+1)}].$$

Therefore

$$\int_0^1 r^k \exp[-c(1-r)^{-n}]\, dr \geqslant (c/k)^{1/(n+1)} \exp[-5c^{1/(n+1)} k^{n/(n+1)}]$$

$$\geqslant \exp[-6c^{1/(n+1)} k^{n/(n+1)}]$$

for sufficiently large k. Therefore

$$\|f\|_c^2 \geqslant \sum_{k,\nu} |a_{k\nu}|^2 \exp[-12c^{1/(n+1)} k^{n/(n+1)}].$$

For any $c' > 0$,

$$\Big(\sum_{k,\nu} |a_{k\nu}| \exp[-c' k^{n/(n+1)}]\Big)^2$$

$$\leqslant \Big(\sum_{k,\nu} |a_{k\nu}|^2 \exp[-c' k^{n/(n+1)}]\Big)\Big(\sum_{k,\nu} \exp[-c' k^{n/(n+1)}]\Big).$$

Therefore, if $b_{k\nu} = O(\exp[-ck^{n/(n+1)}])$,

$$|\Gamma(f)|^2 \leqslant A(c)\|f\|_{c'}^2$$

where $c' = (c/12)^{n+1}$.

Remark. As in [15], one can use (5.6) to define an alternate family of seminorms on F_*. For $f(z) = \sum a_{k\nu}\varphi_{k\nu}(z) \in F_*$, set

$$\|\|f\|\|_c \coloneqq \sum_{k,\nu} |a_{k\nu}| \exp[-ck^{n/(n+1)}]. \tag{5.14}$$

In the proof of Theorem (5.3) we showed that

$$\|\|f\|\|_c \leqslant A(c)\|f\|_{c'}$$

where $c' = (c/12)^{n+1}$ and $A(c)$ is a constant depending only on c. A similar method of proof can be used to show that

$$\|f\|_c \leqslant \|\|f\|\|_{c'},$$

where $c' = c^{1/(n+1)}$. Thus the topologies on F_* given by the (semi)norms. (5.4) and (5.14) are equivalent.

6. The space F_* for general domains. In this section we assume that D is the product of k irreducible bounded symmetric domains D_i, $i = 1, \ldots, k$, each of dimension n_i. Let $I^k = [0, 1)^k$. As in (5.4) we define $F_*(D)$ as the class of holomorphic functions f on D for which

$$\|f\|_c = \int_{I^k} \exp[-c\prod_{i=1}^{k}(1-r_i)^{-n_i}]\, M_\infty(f; (r))\, dr < \infty, \tag{6.1}$$

where $dr = dr_1 \ldots dr_k$.

By Theorem 4.3 (d), if $f \in N_*(D)$, then

$$\lim_{(r)\to(1)} \left(\prod_{i=1}^{k}(1-r_i)^{n_i}\right)\log^+ M_\infty(f; (r)) = 0. \tag{6.2}$$

We note, however, that (6.2) in itself is not sufficient for a holomorphic function f to belong to $F_*(D)$. As an example, consider $D = U \times U$, where $U = \{z \in \mathbf{C}, |z| < 1\}$ and set

$$f(z_1, z_2) = \exp\left[\frac{1}{\sqrt{1-z_1}}\frac{1}{(1-z_2)}\right].$$

Then

$$M_\infty(f; r_1, r_2) = \exp\left[\frac{1}{\sqrt{1-r_1}}\frac{1}{(1-r_2)}\right].$$

Thus

$$(1-r_1)(1-r_2)\log^+ M_\infty(f; r_1, r_2) = \sqrt{1-r_1}$$

which tends to zero as $(r)\to(1)$. However, $\|f\|_c = +\infty$ for all c sufficiently small. To see this we first note that if $0 < c < \frac{1}{2}$, then there exists an R_1 such that

$$\frac{-c}{(1-r_1)} + \frac{1}{\sqrt{1-r_1}} \geqslant c$$

for all r_1, $0 \leqslant r_1 \leqslant R_1$. Thus

$$\int_{I^2} \exp\left[\frac{-c}{(1-r_1)(1-r_2)}\right] M_\infty(f; r_1, r_2)\, dr_1\, dr_2$$

$$\geqslant \int_0^1 \int_0^{R_1} \exp\left[\frac{1}{(1-r_2)}\left(\frac{-c}{(1-r_1)} + \frac{1}{\sqrt{1-r_1}}\right)\right] dr_1\, dr_2$$

$$\geqslant R_1 \int_0^1 \exp\left[\frac{c}{(1-r_2)}\right] dr_2 = +\infty.$$

The following theorem gives a characterization of F_* in terms of limits of $\log^+ M_\infty(f; (r))$.

THEOREM 6.1. *Let f be holomorphic on $D = \bigtimes_{i=1}^{k} D_i$, where each D_i is irreducible of dimension n_i. Then $f \in F_*(D)$ if and only if for every nonempty subset S of $\{1, \ldots, k\}$,*

$$\lim_{r_S \to (1)} \left(\prod_{i \in S} (1-r_i)^{n_i}\right) \log^+ M_\infty(f; (r)) = 0, \tag{6.3}$$

where $r_S = (r_i)$, $i \in S$, and r_j is fixed with $0 < r_j < 1$ for all $j \notin S$.

For convenience we will give the proof only for the case $k = 2$. For the proof of the theorem we will need the following lemma.

LEMMA 6.2. *If F is holomorphic on $D_1 \times D_2$, then for all $r_1 \leqslant \varrho_1$, $r_2 \leqslant \varrho_2$,*

$$M_\infty(f; r_1, r_2) \leqslant M_\infty(f; \varrho_1, \varrho_2).$$

The proof of the lemma is an immediate consequence of the maximum principle and the fact that the function $f(z_1, z_2)$ is holomorphic in D_1 if $z_2 \in D_2$ is fixed and vice-versa.

Proof of Theorem 6.1. Suppose f is holomorphic on $D_1 \times D_2$ satisfying (6.3). Let $c > 0$ be arbitrary. Since

$$\lim_{(r_1, r_2) \to (1,1)^*} (1-r_1)^{n_1} (1-r_2)^{n_2} \log^+ M_\infty(f; r_1, r_2) = 0$$

there exists R_1, R_2, $0 < R_j < 1$, such that

$$M_\infty(f; r_1, r_2) \leqslant \exp[c(1-r_1)^{-n_1}(1-r_2)^{-n_2}] \tag{6.4}$$

for all r_1, r_2, $R_i \leqslant r_i < 1$.

Consider $\|f\|_{2c}$ given by (6.1). We split the integral up over the four sets $E_1 = [0, R_1] \times [0, R_2]$, $E_2 = [0, R_1] \times [R_2, 1)$, $E_3 = [R_1, 1) \times [0, R_2]$, and $E_4 = [R_1, 1) \times [R_2, 1)$. Let

$$I_j = \int_{E_j} \exp[-2c(1-r_1)^{-n_1}(1-r_2)^{-n_2}] M_\infty(f; r_1, r_2) dr_1\, dr_2.$$

On E_1, $M_\infty(f; r_1, r_2) \leqslant M_\infty(f; R_1, R_2)$ and thus $I_1 < \infty$. By (6.4)

$$I_4 \leqslant \int_{E_4} \exp[-c(1-r_1)^{-n_1}(1-r_2)^{-n_2}] dr_1\, dr_2 < \infty.$$

Therefore it remains to show that I_2 and I_3 are finite,

$$\begin{aligned} I_2 &= \int_0^{R_1} \int_{R_2}^{1} \exp[-2c(1-r_1)^{-n_1}(1-r_2)^{-n_2}] M_\infty(f; r_1, r_2) dr_1\, dr_2. \\ &\leqslant R_1 \int_{R_2}^{1} \exp[-2c(1-r_1)^{-n_2}] M_\infty(f; R_1, r_2) dr_2. \end{aligned}$$

By hypothesis, $\lim\limits_{r_2 \to 1} (1-r_2)^{n_2} \log^+ M_\infty(f; R_1, r_2) = 0$. Hence there exists $R_2' \geqslant R_2$ such that

$$M_\infty(f; R_1, r_2) \leqslant \exp[c(1-r_2)^{-n_2}]$$

for all $r_2 \geqslant R_2'$. Therefore

$$\begin{aligned} I_2 \leqslant R_1 M_\infty(f; R_1, R_2') \int_{R_2}^{R_2'} \exp[-2c(1-r_2)^{-n_2}] dr_2 + \\ + R_1 \int_{R_2'}^{1} \exp[-c(1-r_2)^{-n_2}] dr_2 \end{aligned}$$

which is finite. Similarly, $I_3 < \infty$. Thus $\|f\|_{2c} < \infty$ for all $c > 0$.

Conversely, suppose $\|f\|_c < \infty$ for all $c > 0$. Let $0 < r_i < 1$, $i = 1, 2$ be arbitrary. Set $\varrho_i = (1 + r_i)/2$. Then

$$\begin{aligned} \|f\|_c &\geqslant \int_{r_1}^{\varrho_1} \int_{r_2}^{\varrho_2} \exp[-c(1-s_1)^{-n_1}(1-s_2)^{-n_2}] M_\infty(f; s_1, s_2) ds_1\, ds_2 \\ &\geqslant M_\infty(f; r_1, r_2) \exp[-c(1-\varrho_1)^{-n_1}(1-\varrho_2)^{-n_2}](\varrho_1 - r_1)(\varrho_2 - r_2). \end{aligned}$$

Therefore

$$M_\infty(f; r_1, r_2) \leqslant \frac{4\|f\|_c}{(1-r_1)(1-r_2)} \exp\left[\frac{2^n c}{(1-r_2)^{n_1}(1-r_2)^{n_2}}\right],$$

where $n = n_1 + n_2$. Hence, for fixed r_2,

$$\limsup_{r_1 \to 1} (1-r_1)^{n_1} \log^+ M_\infty(f; r_1, r_2) \leqslant \frac{2^n c}{(1-r_2)^{n_2}} \tag{6.5}$$

with an analogous result if r_1 is fixed, and

$$\limsup_{(r_1, r_2)\to(1,1)} (1-r_1)^{n_1}(1-r_2)^{n_2} \log^+ M_\infty(f; r_1, r_2) \leqslant 2^n c. \tag{6.6}$$

Since (6.5) and (6.6) hold for all $c > 0$, f satisfies (6.3).

We now show that if $f \notin N_*(D)$, where $D = \bigtimes_{i=1}^{k} D_i$, each D_i irreducible, then f satisfies (6.3). Thus $N_*(D) \subset F_*(D)$.

PROPOSITION 6.3. *Let* $D = \bigtimes_{i=1}^{k} D_i$. *If* $f \in N_*(D)$, *then* f *satisfies* (6.3).

Proof. Let S be a nonempty proper subset of $\{1, \ldots, k\}$. We adopt the following notation: $D_S = \bigtimes_{i\in S} D_i$, $D_{\bar S} = \bigtimes_{i\notin S} D_i$. Similarly for B_S, $B_{\bar S}$. Points in D_S, B_S will be denoted by z_S, t_S respectively. Also, if $r = (r_1, \ldots, r_k)$, $r_S = (r_i)$, $i \in S$. If $w_{\bar S}$ is an arbitrary but fixed point of $D_{\bar S}$, f_S will denote the function on D_S defined by

$$f_S(z_S) = f(z),$$

where $z_i = z_i$ if $i \in S$, $z_i = w_i$ if $i \notin S$. $d\lambda_S$ and $d\lambda_{\bar S}$ will denote the appropriate measures on B_S and $B_{\bar S}$ respectively. Finally, P^S and $P^{\bar S}$ denote the Poisson Kernels on $D_S \times B_S$ and $D_{\bar S} \times B_{\bar S}$ respectively.

Let $f \in N_*(D)$. Then by (3.2),

$$\log^+ |f(z)| \leqslant P_z[\varphi],$$

where $\varphi \in L^1(B)$. Let $u_{\bar S}$ be an arbitrary but fixed point in $B_{\bar S}$ and fix r_j, $0 < r_j < 1$, $j \in S$. For $t_S \in B_S$, set

$$\varphi_S(t_S) = \int_{B_{\bar S}} P(r_{\bar S} u_{\bar S}, t_{\bar S})\, \varphi(t)\, d\lambda_{\bar S}.$$

Then φ_S is nonegative and measurable on B_S with

$$\varphi_S(t_S) \leqslant \left[\prod_{j\notin S} \left(\frac{1+r_j}{1-r_j}\right)^{n_j}\right] \varphi_S^*(t_S),$$

where

$$\varphi_S^*(t_S) = \int_{B_{\bar S}} \varphi(t)\, d\lambda_{\bar S}(t_{\bar S}).$$

Since $\int_{B_S} \varphi_S^*\, d\lambda_S = \int_B \varphi d\lambda$, $\varphi_S^* \in L^1(B_S)$.

Let $u \in B$ be arbitrary, $0 < r_i < 1$ where for $i \notin S$, r_i is fixed. Then

$$\begin{aligned}\log^+ |f(r_1 u_1, \ldots, r_k u_k)| &\leqslant \int_B P((r)u, t)\varphi(t)\, d\lambda(t)\\ &= \int_{B_S} P^S(r_S u_S, t_S)\varphi_S(t_S)\, d\lambda_S(t_S)\\ &\leqslant A \int_{B_S} P^S(r_S u_S, t_S)\varphi_S^*(t_S)\, d\lambda_S(t_S),\end{aligned}$$

where $A = \prod_{i \notin S}\left(\frac{1+r_i}{1-r_i}\right)^{n_i}$. Therefore

$$\log^+ |f(r_1 u_1, \ldots, r_k u_k)| \leqslant A h_S(r_S u_S),$$

where h_S is the Poisson integral over B_S of the L^1 function φ_S^*. Since the function h_S is independent of u_i, $i \notin S$,

$$\log^+ M_\infty(f; (r)) \leqslant A M_\infty(h_S; r_S).$$

Therefore by Theorem 4.2 (a),

$$\lim_{r_S \to (1)} \left(\prod_{i \in S}(1-r_i)^{n_i}\right)\log^+ M_\infty(f; (r)) = 0.$$

Thus f satisfies (3.6).

As a consequence of the above, $N_*(D) \subset F_*(D)$. Furthermore, as in Theorem 5.2, if $f \in F_*$ and $f_{(r)}$ is defined by $f_{(r)}(z) = f(r_1 z_1, \ldots, r_k z_k)$, then $f_{(r)} \to f$ in F_* as $(r) \to (1)$. Thus N_* is dense in F_* and as for irreducible domains, the F_* topology restricted to N_* is weaker than the metric topology of N_*.

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