ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIX, 3–4 (1987), pp. 375–386

R. EULER (Grenoble)

ODD CYCLES AND A CLASS OF FACETS OF THE AXIAL 3-INDEX ASSIGNMENT POLYTOPE

Abstract. We study the facial structure of the polytope associated with the axial 3-index assignment problem $AP3_n$. Any solution of this problem is the incidence vector of an n-element set within an n^3-element ground set E_n. These n-sets are the bases of a 3-matroid-intersection, the circuits of which are all of cardinality 2. Thus $AP3_n$ can be put into relation to a particular vertex packing problem. It is well known (cf. [5]) that odd cycles without chords induce facets of the polytope defined to be the convex hull of the incidence vectors of vertex packings. We show how odd cycles give rise to a particular class of facets of the axial 3-index assignment polytope.

1. Introduction. The *axial* 3-*index* (or 3-*dimensional*) *assignment problem*, denoted by $AP3_n$ in the sequel, can be stated as follows:

Given a ground set $N = \{1, \ldots, n\}$ and real weights c_{ijk} for $i, j, k \in N$ find a real vector x which satisfies the conditions

$$(1)\qquad \begin{aligned} &\sum_{i\in N}\sum_{j\in N} x_{ijk} = 1 \quad \text{for every } k\in N,\\ &\sum_{i\in N}\sum_{k\in N} x_{ijk} = 1 \quad \text{for every } j\in N,\\ &\sum_{j\in N}\sum_{k\in N} x_{ijk} = 1 \quad \text{for every } i\in N,\\ &x_{ijk}\in\{0, 1\} \quad \text{for all } i, j, k\in N, \end{aligned}$$

and which minimizes (or maximizes)

$$\sum_{i\in N}\sum_{j\in N}\sum_{k\in N} c_{ijk}\, x_{ijk}.$$

$AP3_n$ is called axial in distinction to the planar 3-index assignment problem whose defining system of equations is the following:

$$(2)\qquad \begin{aligned} &\sum_{i\in N} x_{ijk} = 1 \quad \text{for all } j, k\in N,\\ &\sum_{j\in N} x_{ijk} = 1 \quad \text{for all } i, k\in N,\\ &\sum_{k\in N} x_{ijk} = 1 \quad \text{for all } i, j\in N. \end{aligned}$$

There are various applications for these problems such as traffic assignment in communication satellites (cf. [1]), transportation of goods of several types (cf. [6]), or timetabling (cf. [4]). Usually, these problems are solved by branch and bound (see, e.g., [7]). The idea of combining it with cutting plane methods has turned out to be very efficient in solving large scale 0-1 programming problems. However, such an approach requires the knowledge of at least a partial description of an associated polytope P, defined to be the convex hull of all feasible 0-1 vectors. Only recently, work has been done to (partially) describe the polytopes associated with 3-index assignment problems: Euler et al. [3] have considered the planar case, whereas Balas and Saltzman [2] treat the axial problem. Before stating their main results let us introduce some basic notions.

Let $E_n = \{e_{ijk}\colon 1 \leqslant i, j, k \leqslant n\}$ be a set of n^3 elements and L be a *feasible array*, i.e., an $(n \times n)$-array over N, the cells of which contain every element (or symbol) k from N at most once. Obviously, we can identify L with a subset E_L of E_n containing exactly those elements e_{ijk} for which symbol k is contained in cell ij of L. Therefore, feasible arrays are in a 1-1 correspondence with subsets of E_n. The *incidence vector* of such a subset E_L is the vector x^L having n^3 components x^L_{ijk} which are defined to be 1 if the element e_{ijk} is contained in E_L and to be 0 if this is not the case. With any feasible array we can thus associate its incidence vector in a unique way. It is easily seen that in this way the solutions of the planar problem correspond exactly to the *Latin squares* (of order n), i.e., those feasible arrays the cells of which contain exactly one symbol such that each row and each column contain each symbol of type k exactly once. If now L is defined to contain in each row and column exactly one symbol so that each symbol $k \in N$ occurs exactly once, the associated incidence vector x^L is seen to represent a solution of $\mathrm{AP3}_n$ and vice versa. Such an L will be called a *permutation square* (of order n), an example of which is shown in Fig. 1.

Our main interest lies in describing the polytope P_n, defined to be the convex hull of all incidence vectors of permutation squares, by a system of

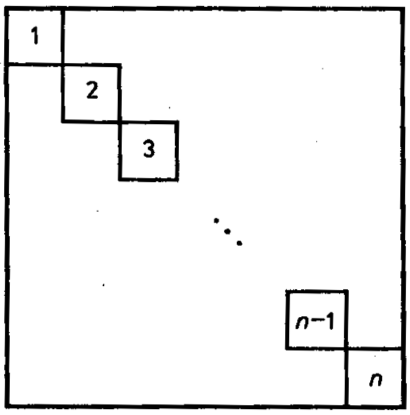


Fig. 1

linear inequalities, but since such a complete description is very difficult to obtain (if at all), we will focus on a partial description. For this let us first introduce some basic notation from polyhedral theory. A *polyhedron* $P \subseteq \mathbf{R}^m$ is the intersection of finitely many half-spaces in $\mathbf{R}^m$. A *polytope* is a bounded polyhedron or, equivalently, the convex hull of finitely many points. The *dimension* of a polyhedron P, denoted by dim P, is the maximum number of affinely independent points in P minus 1. A linear inequality $a^T x \leqslant a_0$, where $a \in \mathbf{R}^m - \{0\}$ and $a_0 \in \mathbf{R}$, is said to be *valid* for P if

$$P \subseteq \{x \in \mathbf{R}^m\colon\ a^T x \leqslant a_0\}.$$

A subset F of P is called a *face* of P if there exists an inequality $a^T x \leqslant a_0$, valid for P, such that

$$F = \{x \in P\colon\ a^T x = a_0\}.$$

We also say that the inequality $a^T x \leqslant a_0$ *defines* F. A face F is *proper* if $F \neq P$. A proper, nonempty face maximal with respect to set-inclusion is called a *facet*. Any polyhedron P has a representation of the form

$$P = \{x \in \mathbf{R}^m\colon\ Ax \leqslant b,\ Dx = d\}.$$

If D has full rank and $\{x \in \mathbf{R}^m\colon\ Dx = d\}$ equals the affine space spanned by P, which we denote by aff(P), then $Dx = d$ is called a *minimal equation system* for P.

Consider now the clutter $\mathscr{B}_n$ of all those subsets of E_n, the feasible arrays associated with constitute the permutation squares. This clutter induces an independence system $(E_n, \mathscr{I}_n)$ in a natural way, where $\mathscr{I}_n$ is the collection of all subsets of members of $\mathscr{B}_n$. Any *circuit*, i.e., a minimal dependent subset of E_n, of this independence system has cardinality 2, which enables us to associate with $(E_n, \mathscr{I}_n)$ an undirected graph $G_n = (E_n, \mathscr{C}_n)$, where the edge set $\mathscr{C}_n$ stands for the collection of circuits. Of special interest within G_n are the *maximal cliques*, i.e., complete subgraphs, as well as the induced *odd cycles*, i.e., chordless cycles of length $2p+1$ for $p \geqslant 2$. These

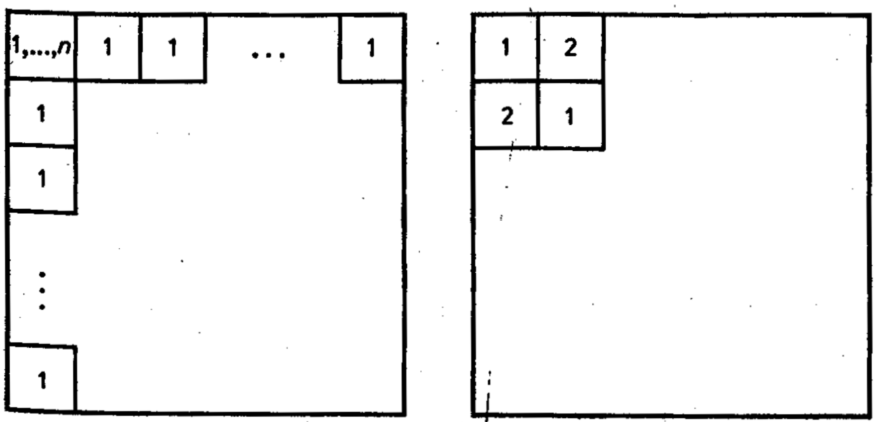


Fig. 2

subgraphs are known to induce facets of the *monotonization* of P_n, the convex hull of the incidence vectors of members of $\mathscr{I}_n$. Balas and Saltzman [2] have identified three classes of maximal cliques within G_n and shown that they are exhaustive. Moreover, two classes among them have been shown to define facets of our polytope P_n, whereas the cliques of the third class give rise to a minimal equation system for P_n. Two instances of the first two classes are represented in Fig. 2 by their associated feasible arrays.

For the planar problem (cf. [3]) only one class of maximal cliques could be identified, which not only induce improper faces but also give rise to a minimal equation system. Note also that in this case the associated circuits can have cardinality greater than two. Moreover, a class of odd cycles could be identified, the corresponding inequalities all define facets of the associated polytope. The feasible array of such an odd cycle is illustrated in Fig. 3.

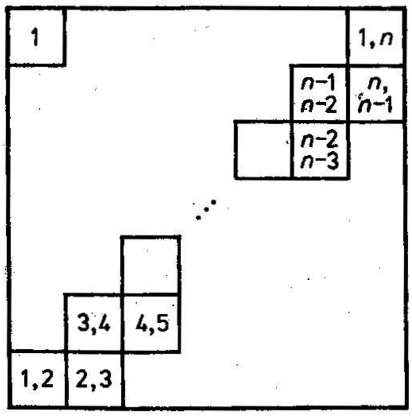


Fig. 3

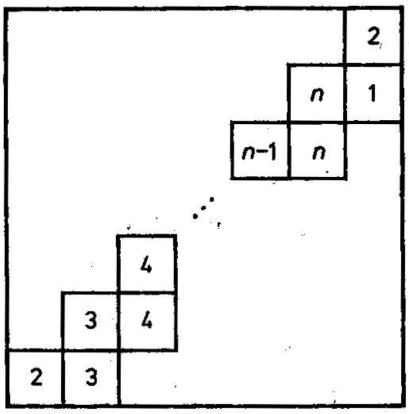


Fig. 4

It is the aim of this paper to show how facet-defining inequalities for P_n can be obtained from odd cycles in G_n. A particular instance of such an odd cycle is shown in Fig. 4.

2. A minimal equation system for P_n. Let us consider the equation system (1) defining the set of solutions to AP3$_n$. The underlying matrix A has the following form (assuming a lexicographically increasing order of the triples ijk):

1 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1	...	1 1 1 1 1 1 1 1 1
11 ... 1 11 ... 1 11 ... 1	11 ... 1 11 ... 1 11 ... 1	...	11 ... 1 11 ... 1 11 ... 1
11111111111 ... 11111	1111111111 ... 11111	...	1111111111 ... 11111

We will show in the sequel that $A'x = 1$, A' being obtained by deletion of rows $n+1$ and $2n+1$, is a minimal equation system for P_n. Such a proof has already been given in [2] but our proof is closely related to the notion of a permutation square and, besides, it will be very important for the remaining part of this paper.

THEOREM 1. *The system $A'x = 1$ is a minimal equation system for P_n.*

Proof. Let $a^T x \leqslant a_0$ be a linear inequality such that

$$P_n \subseteq \{x \in \boldsymbol{R}^n\colon\ a^T x = a_0\}.$$

We have to show the existence of a vector λ having n^3 components such that $A^T \lambda = a$. Let us define such a λ as follows:

$$\begin{aligned} \lambda_{1k} &:= a_{11k} && \text{for } k = 1, \ldots, n,\\ \lambda_{21} &:= 0, && \\ \lambda_{2j} &:= a_{1j1} - a_{111} && \text{for } j = 2, \ldots, n,\\ \lambda_{31} &:= 0, && \\ \lambda_{3i} &:= a_{i11} - a_{111} && \text{for } i = 2, \ldots, n. \end{aligned}$$

We have to show that the equality

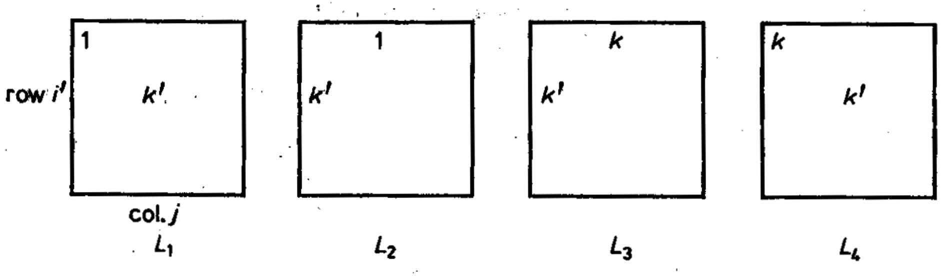
$$a_{ijk} = \lambda_{1k} + \lambda_{2j} + \lambda_{3i} \tag{3}$$

holds for all $i, j, k \in N$. We observe that by the definition of λ this equality holds for all triples ijk, at least two of which are equal to 1. Two main cases remain:

Case 1. Exactly one of the indices is equal to 1, say the index i. Then we have to show that

$$a_{1jk} + a_{111} = a_{11k} + a_{1j1}.$$

For this consider the following feasible arrays $L_1, \ldots, L_4$ (the occurring symbol k' is chosen to be unequal to k and 1):



We observe that L_1 and L_2, resp. L_3 and L_4, can be completed identically to full permutation squares $L_1', \ldots, L_4'$, which according to our assumption on the vector a satisfy the inequality $a^T x \leqslant a_0$ with equality. Therefore, we infer with the notation $a(L) := a^T x^L$ that

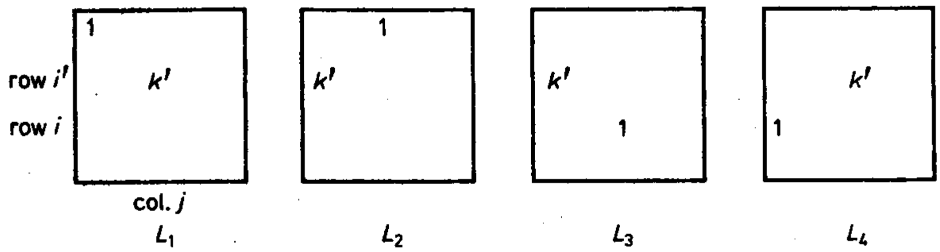
$$a(L_1') - a(L_2') = a(L_1) - a(L_2) = 0 \quad \text{and} \quad a(L_3') - a(L_4') = a(L_3) - a(L_4) = 0$$

implying

$$0 = a(L_1) - a(L_2) + a(L_3) - a(L_4) = a_{1jk} + a_{111} - a_{1j1} - a_{11k},$$

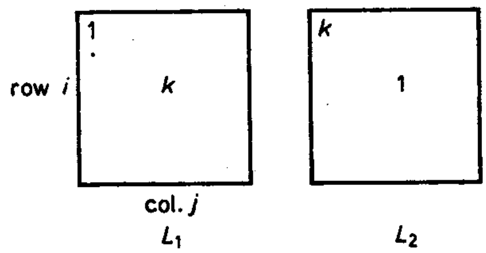
which proves case 1.

The case $j = 1$, $i, k \neq 1$ follows along the same line by transposition and for $k = 1$, $i, j \neq 1$ the consideration of the four arrays (with $k' \neq 1$ and $i' \neq 1$, i)



yields the proof.

Case 2. All three indices i, j, k are unequal to 1. In view of (3) and the latter case we have only to consider the two arrays



which, clearly, can also be completed identically to full permutation squares. This completes our proof.

Corollary 1. *We have*

$$\dim P_n = n^3 - 3n + 2 = (n-1)^2 (n+2).$$

3. A further class of facet-defining inequalities for P_n. Starting from an odd cycle of the type illustrated in Fig. 4 and using Padberg's lifting theorem (cf. [5]) we can obtain facet-defining inequalities for the monotonization of P_n. It seems that quite a number of distinct classes of such inequalities can be found this way. However, in view of the (lower-dimensional) polytope P_n it still has to be shown that these inequalities define faces of P_n, which are different from those induced by the clique inequalities as given in [2]. The class of such inequalities we found arises from two feasible arrays L_1 and L_2 (illustrated in Fig. 5) by "summation", i.e., they have the form

$$x(L_1) + x(L_2) \leqslant (n-1). \tag{4}$$

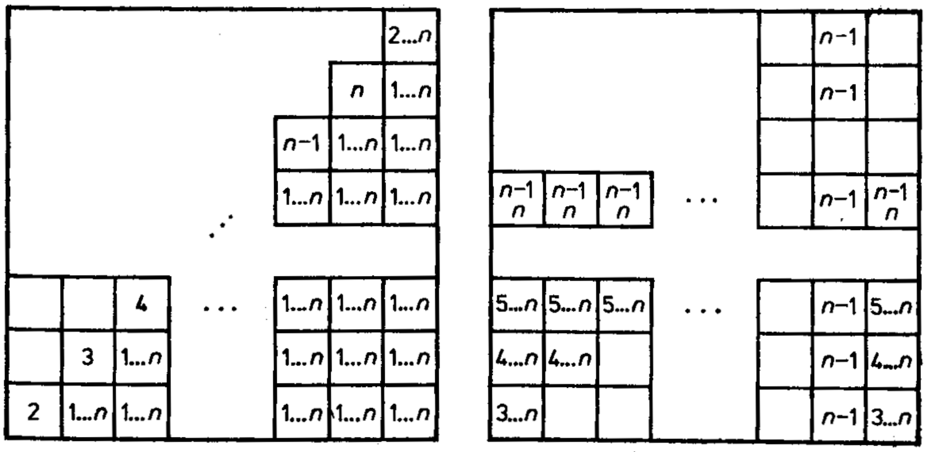


Fig. 5

Proposition 1. *The inequality* $x(L_1)+x(L_2) \leqslant (n-1)$ *is valid for* P_n.

Proof. Suppose there exists a permutation square L such that $x^L(L_1)+x^L(L_2) \geqslant n$.

1. L does not contain the symbol $n-1$ in cells $4(n-1), \ldots, n(n-1)$ nor a symbol k out of the set $\{n-i+3, \ldots, n\}$ in cell in for $i=4, \ldots, n$. Then, by a well-known result from the theory of permutation matrices, L has to contain the symbol $n-1$ in cell $3(n-2)$. But then, again, it has also to contain the symbol n in cell $2(n-1)$ as well as the symbols $n-i+2$ in cells ij for $i=4, \ldots, n$ and $j=n-i+1$, respectively. At the end, L has to contain the symbol 1 in cell $1n$, which leads to a contradiction with our assumption.

2. If the symbol $n-1$ within L is contained in one of the cells $i(n-1)$, where $i \in \{4, \ldots, n\}$, then L has to contain two symbols in the first 3 rows and $n-2$ columns. Note that the corresponding coefficients of our inequality are zero. By our assumption, L must contain a symbol k from $\{n-i+3, \ldots, n\}$ in cell in for some $i \in \{4, \ldots, n\}$. But then L has even to contain three symbols in the first 3 rows and $n-2$ columns, which contradicts our assumption.

3. The same observation holds if L contains the symbol $n-1$ in one of the cells $4n, \ldots, nn$. Moreover, the assumption that L contains two or more symbols in the subrectangle formed by rows 1, 2, 3 and columns $1, 2, \ldots, n-2$ (symbol $n-1$ in cell $3(n-2)$ excluded) leads to a contradiction.

4. If L contains exactly one element in this subrectangle, then it has to contain a symbol k from $\{n-i+3, \ldots, n\}$ in cell in for some $i \in \{4, \ldots, n\}$. If this is the symbol n, then L contains at least two symbols whose coefficient in our inequality is zero, leading as in case 2 to a contradiction. Therefore, an element, say k, in column n must be unequal to n (and as we have already seen also unequal to $n-1$). Now the symbol within the subrectangle must be in row 1, since otherwise at least one further such element exists. But then L

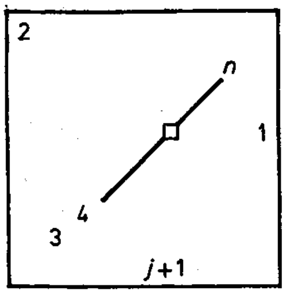
is forced to contain the symbol n in cell $2(n-1)$ and the symbol $n-1$ in cell $3(n-2)$. By our assumption, L must also contain the symbol $n-2$ in cell $4(n-3)$, so the symbol in its n-th column must be unequal to $n-2$, $n-1$, n. Therefore, in line 5, L has to contain the symbol $n-3$ in its $(n-4)$-th column, and continuing this way we arrive at the conclusion that the symbol 3 has to occur in cell $(n-1)2$ as well as in cell nn, a contradiction.

5. Finally, the assumption that there is no element in L with a corresponding zero coefficient in our inequality leads us to case 1.

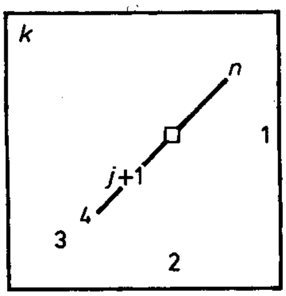
Now let $F = \{x \in P_n\colon\ x(L_1)+x(L_2) = (n-1)\}$ and let $a^T x \leqslant a_0$ be any other inequality valid for P_n such that $F = \{x \in P_n\colon\ a^T x = a_0\}$. Obviously, there is a permutation square L whose incidence vector is not contained in F. Therefore, to show that F is a facet of P_n we have only to prove the existence of a scalar α and a real vector λ such that $a^T = \lambda^T A + \alpha c^T$, where c is the sum of the incidence vectors of L_1 and L_2. For this we make use of our vector λ as introduced in Theorem 1 and show the desired equality first for all those triples ijk for which the coefficient in inequality (4) is zero. In particular, we will exhibit for every subcase to be considered a permutation square contained in F to which the partial permutation squares already considered in the proof of Theorem 1 can always be completed.

For the case where at least two of the indices i, j, k are equal to one the result is obvious.

Case 1. $i = 1$, $j, k \neq 1$, $j \leqslant n-1$ and jk is unequal to $(n-1)(n-1)$. Here the following two permutation squares (one for $k = 2$, one for $k \neq 2$) yield the proof (in the second one the symbol k has been replaced by $j+1$):

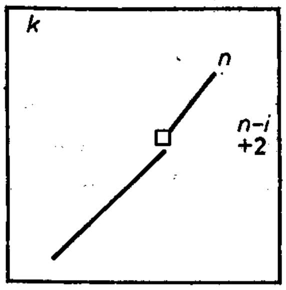


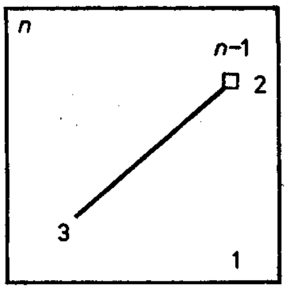
col. j



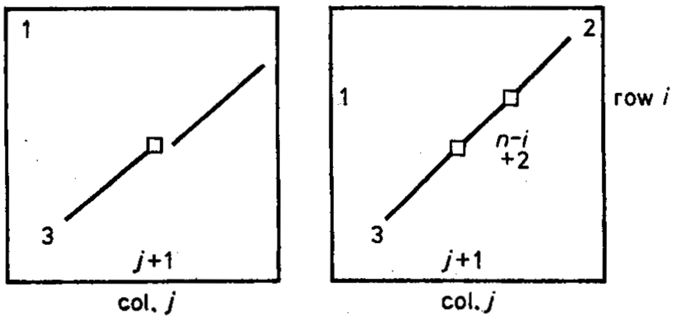
col. j

for $j = 1$, $i, k \neq 1$ the following two permutation squares (one for $k \leqslant n-i+2$ and one for $i = 3$ and $k = n$) are appropriate:

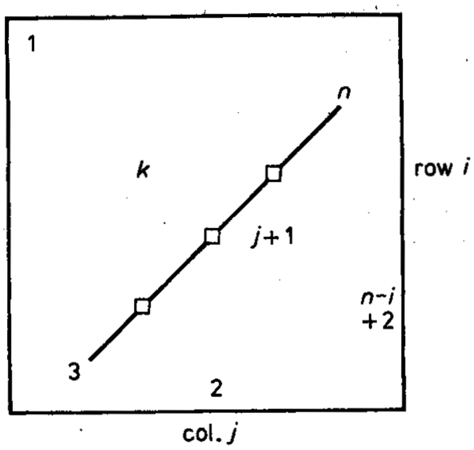




Finally, for $k = 1$, i and $j \neq 1$ as well as $i+j \leqslant n+1$ the following two permutation squares yield the proof:



Case 2. All three indices are unequal to one; here the essential example for a possible completion is given by the following permutation square:



This completes our proof for all these triples ijk having a zero coefficient in (4).

Let us now pass to the remaining ones. We have to show the equations

(5) $$a_{ijk} = \lambda_{1k} + \lambda_{2j} + \lambda_{3i} + \alpha,$$

respectively,

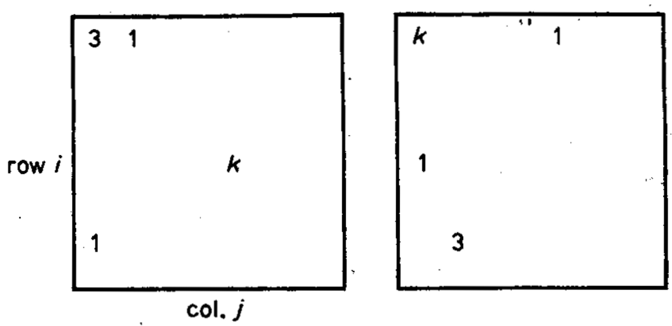
(6) $$a_{ijk} = \lambda_{1k} + \lambda_{2j} + \lambda_{3i} + 2\alpha$$

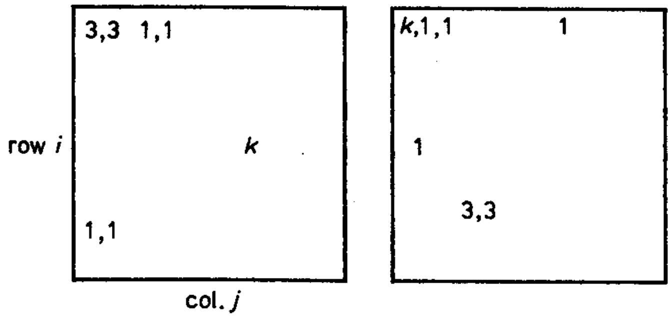
for appropriate triples ijk.

For this we choose α as follows:

$$\alpha = a_{(n-1)23} + a_{(n-1)11} + 2a_{111} - a_{121} - a_{113},$$

which shows equality (5) for $ijk = (n-1)23$. For the other triples we have to treat the following two pairs of arrays:





arising from (5) and (6), respectively. In the sequel we will consider 6 further cases and indicate how the above-mentioned pairs of arrays can be "decomposed" further into identically completable, partial permutation squares (or into pairs of arrays which are easily seen to be decomposable in this way). We will, however, restrict ourselves to just presenting the triples of indices defining these partial permutation squares.

Case 3. $ij = 1n$ and $k = 2, \ldots, n$, or $i+j = n+1$ and $k = n-i+2$:
$[1nk,\ n13-11k,\ nn3]$;
$[(n-1)11,\ nn3-n13,\ (n-1)n1]$;
$[123,\ (n-1)n1-(n-1)23,\ 1n1]$;
$[111,\ ij1-i11,\ 1j1]$;
$[ijk,\ 1n1-ij1,\ 1nk]$;
$[113,\ 121-111,\ 123]$.

Case 4. $i \geqslant 4$, $j \in \{1, \ldots, n-1\}$, $k \in \{1, \ldots, n\} - \{n+2-i\}$ if $i+j > n+1$ and for $k \in \{n+3-i, \ldots, n\}$ if $i+j < n+1$; $k \neq n-1$ if $j = n-1$:
$[1(n+1-i)(n+2-i),\ ijk-i(n+1-i)(n+2-i),\ 1jk]$;
$[(n-1)11,\ i(n+1-i)(n+2-i)-i11,\ (n-1)(n+1-i)(n+2-i)]$;
$[123,\ (n-1)(n+1-i)(n+2-i)-(n-1)23,\ 1(n+1-i)(n+2-i)]$;
$[113,\ 121-111,\ 123]$;
$[111,\ 1jk-11k,\ 1j1]$.
This decomposition is essentially the same as that with $i+j < n+1$.

Case 5. $i+j = n+1$, $k = n+3-i, \ldots, n$ and $i > 3$, or $i+j > n+1$ and $k = n+2-i$:
$[113,\ 121-111,\ 123]$;
$[111,\ 1jk-11k,\ 1j1]$;
$[11n,\ ijk-i1n,\ 1jk]$;
$[(n-1)1n,\ 123-11n,\ (n-1)23]$;
$[i1n,\ (n-1)11-i11,\ (n-1)1n]$.

Case 6. $i \geqslant 4$, $j = n$, $k \in \{2, \ldots, n-i+2\}$:
$[111,\ ink-i11,\ 1nk]$;
$[113,\ 121,\ 1nk,\ (n-1)11-111,\ 11k,\ 1n1,\ (n-1)23]$,
where the second expression in brackets can be treated exactly as in case 3; for $k = 1$ we obtain the following decomposition:

[113, $in1-i11$, $1n3$];
[121, $(n-1)11-111$, $(n-1)21$];
[$(n-1)21$, $1n3-(n-1)23$, $1n1$].

Case 7. $i \geqslant 4$ and $j=n$:
[113, 121, $(n-1)11$, $ink-11k$, $1n1$, $i11$, $(n-1)23$];
[113, 121, $(n-1)11-111$, 111, $(n-1)23$],
where the first expression in brackets can be treated exactly as in case 4.

Case 8. $i \geqslant 4$, $j=n-1$ and $k=n-1$:
[123, $(n-1)(n-1)1-(n-1)23$, $1(n-1)1$];
[113, $121-111$, 123];
[$11n$, $i(n-1)(n-1)-i1n$, $1(n-1)(n-1)$];
[$21n$, $1(n-1)(n-1)-11(n-1)$, $2(n-1)n$];
[$2(n-1)n$, $(n-1)11-21n$, $(n-1)(n-1)1$];
[113, 121, $i1n$, $(n-1)11-11n$, 111, $i11$, $(n-1)23$],
where the last expression in brackets can be treated as in case 4.

Finally, the remaining cases in rows 1–3 are easily treated in a similar way. Since there exists a permutation square L satisfying inequality (4) strictly, our α must be strictly greater from zero. Altogether we have thus shown:

THEOREM 2. *Inequality* (4) *defines a facet of* P_n.

What we still have to show is that F, the face of P_n induced by inequality (4), is distinct from those faces which are induced by

(i) the "trivial" inequalities $x_{ijk} \geqslant 0$, $i, j, k \in N$,

(ii) the inequalities $x(L) \leqslant 1$, where L is a clique of type 1 as illustrated in Fig. 2,

(iii) the corresponding inequality arising from type 2,

all of which have been shown by Balas and Saltzman [2] to define facets of P_n. For this it suffices to exhibit a permutation square L' for every face F' such that x^L is a member of F but not of F'.

For (i) this is easy to see since for each triple ijk there is a permutation square L such that $x^L \in F$ and $x^L_{ijk}=1$.

As to (ii) we note that any clique of this type is uniquely defined by a pair of indices ij defining the cell which contains the symbols $1, \ldots, n$ as well as a single index k, by which row i and column j are filled up. First, if $i+j = n+1$, $ij \neq 1n$ and $k \neq 1, j+1$, then L as illustrated in Fig. 6 has the desired properties. If, however, $k=1$ ($k=j+1$), then putting the symbol 1 in cell $1(j-1)$ and the symbol j in cell in (the symbol $j+1$ in cell $(i+1)n$ and the symbol j in cell $1j$) yields a permutation square of the desired type. The case $ij=1n$ is easily treated in a similar way. Second, if $i+j \neq n+1$, then the permutation square shown in Fig. 7 has the required properties provided $k \neq n-i+2$ and $j+1$ (mod n). Obviously, by exchanging the symbol 1 for $n-i+2$, for 2 or for n we always obtain a permutation square as required.

The same can be done for $k = j+1 \pmod n$, and so any member of the first class of cliques is shown to induce a facet of P_n which is distinct from F. Quite along the same lines this is shown for the members of the second class.

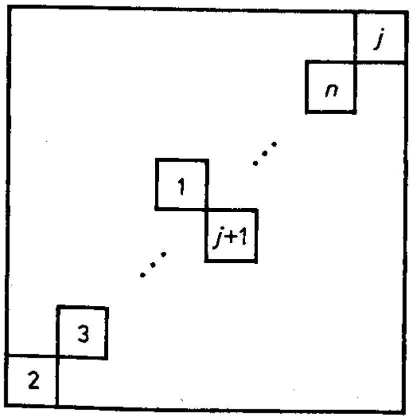


Fig. 6

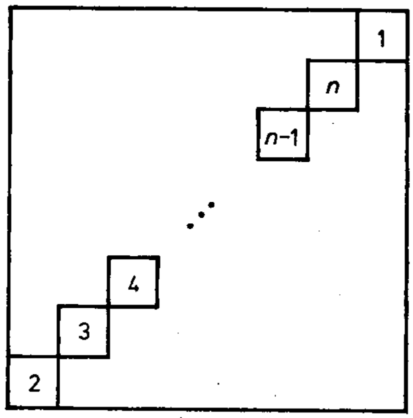


Fig. 7

To conclude we only mention that the total number of inequalities of type (4) is exponential in n (permutations of rows, columns and symbols yield distinct facets!). Moreover, it seems to be worthwhile to keep n fixed and further generalize this class of facet-defining inequalities for polyhedra $P_{n'}$, where n' is strictly greater than n.

References

[1] E. Balas and P. Landweer, *Traffic assignment in communication satellites*, Oper. Res. Letters 2 (1983), pp. 141–147.

[2] E. Balas and M. Saltzman, *Some facets of the three-index assignment polytope*, Management Science Research Report No. MSRR-511 (R), GSIA, Carnegie-Mellon University 1984.

[3] R. Euler, R. E. Burkard and R. Grommes, *On Latin squares and the facial structure of related polytopes*, Discrete Math. 62 (1986), pp. 155–181.

[4] C. C. Gotlieb, *The construction of class-teacher timetables*, pp. 73–77 in: *Proceedings of the IFIP Congress*, 1962.

[5] M. W. Padberg, *On the facial structure of set packing polyhedra*, Math. Programming 5 (1973), pp. 199–215.

[6] E. D. Schell, *Distribution of a product by several properties*, pp. 615–642 on: *Second Symposium in Linear Programming*, 2, DCS: Controller HQ. U.S. Air Force, Washington, D.C., 1955.

[7] M. Vlach, *Branch and bound method for the three-index assignment problem*, Ekonom.-Mat. Obzor 3 (1967), pp. 181–191.

LABORATOIRE ARTEMIS (IMAG)
UNIVERSITÉ SCIENTIFIQUE ET MÉDICALE DE GRENOBLE
FRANCE