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ON THE FORM OF COUNTABLE ADMISSIBLE ORDINALS

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In this note we give a short self-contained proof of the following

THEOREM. *Let a be a countable admissible ordinal. Then the following are equivalent*:

(a) $a = \omega_1^A$ *for some a-finite* $A \subseteq \omega$;

(b) (i) $L_a \models V = HC$, *where* $V = HC$ *is the statement "everything is countable"*,

(ii) a *is recursively accessible.*

We had obtained this result before discovering the proof in the Marek paper [1], which draws upon the much more general theory of stable sets developed in that paper, and also uses results from other papers.

Proof. (a) $\Rightarrow$ (b). Assume that $a = \omega_1^A$ for some a-finite A.

(i) Suppose that $\beta < a$; then there is an A-recursive relation R such that

$$\exists f\colon f\colon (\omega, R) \simeq (\beta, <).$$

By the admissibility of L_a, we have $R \in L_a$, and hence $f \in L_a$. Thus

$$L_a \models \text{“}\beta \text{ is countable”}.$$

Now

$$L_a \models \text{“every set is equipotent to an ordinal”};$$

hence $L_a \models V = HC$.

(ii) Since $\lim(a)$, we have $A \in L_\gamma$ for some $\gamma < a$; then $\omega_1^A \leqslant \gamma^+$ (the next admissible ordinal after γ). On the other hand, $\gamma^+ \leqslant a$. Hence $a = \gamma^+$, i.e., a is recursively accessible.

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[1] W. Marek, *Stable sets, a characterization of β_2-models of full second order arithmetic and some related facts*, Fundamenta Mathematicae 82 (1974), p. 175-189.

(b) ⇒ (a). Suppose that a is recursively accessible and $L_a \models V = HC$. Then $a = \gamma^+$ for some $\gamma < a$, and

$$L_a \models \exists g\colon \gamma \to \omega \ \&\ g \text{ is 1-1, onto.}$$

From g we define $R \in L_a$, with $R \subseteq \omega^2$ such that $(\omega, R) \simeq (\gamma, <)$. Clearly, $\omega_1^R \leqslant a$. On the other hand, γ is R-recursive, so $\gamma < \omega_1^R$, hence $\gamma^+ \leqslant \omega_1^R$. Thus $a = \omega_1^R$.

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