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Kuranishi families for Hopf surfaces

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Abstract. The Kuranishi family for any (primary) Hopf surface is construced. It is proved that any two Hopf surfaces are deformations of each other. For every Hopf surface, the group of holomorphic line bundles which can be determined by a divisor is examined.

By a *Hopf surface* we mean any complex manifold homeomorphic to $S^1 \times S^3$. For any such surface we give an explicit construction of the Kuranishi family. Our construction is a generalization and a simplification of that obtained by Kodaira and Spencer [6] for some special Hopf surfaces. Moreover, we construct such an ascending chain of analytic families that, after a finite number of steps, contains any two given Hopf surfaces. This implies that any two of them are deformations of each other. We also examine the group of holomorphic line bundles which can be determined by a divisor.

A construction of the Kuranishi families for any Hopf surface was obtained independently by Wehler [10]. He used ideas different from the original ideas of Kodaira and Spencer, namely more algebraic ones. We give here our contruction as obtained in [1].

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Notation

general:

$\mathbf{N}$ the set of positive integers,
$\mathbf{Z}$ the group of integers,
$\mathbf{Z}_n$ the cyclic group of n elements,
$\mathbf{C}$ the field of complex numbers,
$\mathbf{C}^*$ the multiplicative group of non-zero complex numbers,
$\mathbf{C}^n$ the complex affine n-dimensional space,
$\mathbf{C}^2_*$ the domain $\mathbf{C}^2 \setminus \{(0, 0)\}$,
$\mathbf{P}^n$ the complex projective n-dimensional space,
$\langle g \rangle$ the infinite cyclic group generated by an element g;

for a complex manifold X:

$\pi_1(X)$	the fundamental group of X,
$a(X)$	the algebraic dimension of X: the transcendence degree over $\mathbf{C}$ of the field of all global meromorphic functions on X,
Θ	the tangent sheaf of X: the sheaf of tangent holomórphic vector fields on X,
h^ν	the dimension $\dim_{\mathbf{C}} H^\nu(X, \Theta)$,
$[D]$	the complex line bundle determined by a divisor D (when D is a divisor) or by the divisor $1\cdot D$ (when D is a 1-codimensional analytic set),
$\mathrm{Div}(X)$	the group of divisors on X,
$\mathrm{Pic}(X)$	the group of isomorphism classes of complex line bundles on X,
$\deg X$	the degree of a Hopf surface, see Definition 1.6.

1. By a *Hopf surface* we mean any complex manifold homeomorphic to $S^1 \times S^3$. The present paper is based on the following theorem of Kodaira.

THEOREM 1.1 ([4], § 10 and [5]). *For every Hopf surface X there exist numbers* $m \in \mathbf{N}$, $a, b, t \in \mathbf{C}$ *satisfying*

$$0 < |a| \leqslant |b| < 1 \quad \text{and} \quad (b^m - a)t = 0$$

such that X is biholomorphic to $\mathbf{C}^2_*/\langle\gamma\rangle$, *where* γ *is an automorphism of* $\mathbf{C}^2_*$ *given by*

$$\gamma(z_1, z_2) = (az_1 + tz_2^m, bz_2).$$

Conversely, for any m, a, b, t, as above, the corresponding group $\langle\gamma\rangle$ *acts freely and properly discontinuously on* $\mathbf{C}^2_*$ *and the complex manifold* $\mathbf{C}^2_*/\langle\gamma\rangle$ *is a Hopf surface.*

By this theorem we will assume that every Hopf surface under consideration is given as $\mathbf{C}^2_*/\langle\gamma\rangle$ for some γ as above.

Since the automorphism γ need not be linear, it is not always given by a matrix. In connection with it we introduce a notion which enable us to describe γ in a convenient way.

DEFINITION 1.2. A *quasimatrix* is an expression of the form

$$\begin{bmatrix} a & t(\ )^m \\ & b \end{bmatrix}$$

which is a short notation for a (not necessarily linear) mapping

$$(x, y) \mapsto (ax + ty^m, by).$$

Notice that the conditions of Theorem 1.1 mean that the automorphism γ may have two forms:

$$\text{for } t = 0: \quad \gamma = \begin{bmatrix} a & \\ & b \end{bmatrix}, \qquad \text{for } t \neq 0: \quad \gamma = \begin{bmatrix} b^m & t(\ )^m \\ & b \end{bmatrix}.$$

In connection with the above we assume the following

DEFINITION 1.3. A quasimatrix being of the first of the above forms will be called a *quasimatrix of degree* 0, and a quasimatrix of the second form will be called of *degree m*, where $m \in \mathbf{N}$.

By standard methods, using universal coverings and examination of coefficients of power series (compare with [8]) one obtains the following facts.

LEMMA 1.4. *Two given Hopf surfaces* $\mathbf{C}^2_*/\langle\varphi\rangle$ *and* $\mathbf{C}^2_*/\langle\gamma\rangle$ *are biholomorphic if and only if there exists a holomorphic automorphism* F *of* $\mathbf{C}^2$ *such that*

$$\gamma \circ F = F \circ \varphi.$$

If φ *and* γ *are linear, then* F *can be also linear.*

PROPOSITION 1.5. *Let* φ, γ *be automorphisms of* $\mathbf{C}^2_*$ *satisfying the conditions of Theorem* 1.1. *Then*

(i) *If the quasimatrices of* φ *and* γ *are of different degrees, then Hopf surfaces* $\mathbf{C}^2_*/\langle\varphi\rangle$ *and* $\mathbf{C}^2_*/\langle\gamma\rangle$ *are non-biholomorphic.*

(ii) *Let the quasimatrices of* φ *and* γ *be of the same degree* $m \in \mathbf{N}$ *and*

$$\varphi = \begin{bmatrix} a^m & t(\ )^m \\ & a \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} b^m & s(\ )^m \\ & b \end{bmatrix}.$$

Then $\mathbf{C}^2_*/\langle\varphi\rangle$ *is biholomorphic to* $\mathbf{C}^2_*/\langle\gamma\rangle$ *if and only if* $a = b$.

(iii) *Let the quasimatrices of* φ *and* γ *be of degree* 0 *and*

$$\varphi = \begin{bmatrix} a & \\ & b \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} a' & \\ & b' \end{bmatrix}.$$

Then $\mathbf{C}^2_*/\langle\varphi\rangle$ *and* $\mathbf{C}^2_*/\langle\gamma\rangle$ *are biholomorphic if and only if* $\varphi = \gamma$ *or* $a' = b$ *and* $a = b'$.

Note that by the first part of Proposition 1.5 we can speak not only about the degree of a quasimatrix but even about the degree of a Hopf surface, namely

DEFINITION 1.6. A Hopf surface $X = \mathbf{C}^2_*/\langle\gamma\rangle$ is of *degree m* if and only if the quasimatrix of γ is of degree m. We denote this by $\deg X = m$.

It is possible to characterize the degree of a Hopf surface X intrinsically by examining the algebraic dimension of X and the divisor of the canonical bundle on X, when the algebraic dimension is 0, but we will not use it here (see [2]).

2. Here, for any Hopf surface X, we examine the subgroup in $\mathrm{Pic}(X)$ which consists of those (classes of) line bundles on X that can be determined by a divisor.

First of all, Betti numbers show that for a Hopf surface X its algebraic dimension $a(X)$ can be equal only to 0 or 1.

THEOREM 2.1 (Kodaira [3], [4]). *Let a Hopf surface X be given by a quasimatrix*

$$\gamma = \begin{bmatrix} a & t(\ )^m \\ & b \end{bmatrix}.$$

Then:

(i) $a(X) = 1$ *if and only if* $t = 0$ *and* $a^k = b^l$ *for some* $k, l \in \mathbf{N}$,

(ii) *If* $a(X) = 1$ *then X contains infinitely many curves* (1-*dimensional analytic subsets*),

(iii) *If* $a(X) = 0$ *then X contains not more than two irreducible curves. Each of them is non-singular and elliptic.*

Note that any Hopf surface contains a curve. Let $X = \mathbf{C}^2_*/\langle\gamma\rangle$ where γ is as above. Then the equation $z_2 = 0$ is invariant under $\langle\gamma\rangle$, hence it determines an elliptic curve C, biholomorphic to $\mathbf{C}^*/\langle b\rangle$. If $t = 0$, then the equation $z_1 = 0$ also determines an elliptic curve C_o biholomorphic to $\mathbf{C}^*/\langle a\rangle$. Part (iii) of Theorem 2.1 says that for $a(X) = 0$ there are no curves other than C and C_o.

But in case $t \neq 0$ the only curve is C.

THEOREM 2.2 [2]. *A Hopf surface X contains exactly one irreducible curve if and only if* $\deg X > 0$.

Consider the canonical homomorphism

$$[\ ]\colon \operatorname{Div}(X) \to \operatorname{Pic}(X)$$

given by

$$D \mapsto [D].$$

In this section we want to describe the image $[\operatorname{Div}(X)]$ of $\operatorname{Div}(X)$ in $\operatorname{Pic}(X)$ and $[\ ]$, where X is any Hopf surface.

Since in the case $a(X) = 0$ the homomorphism $[\ ]$ is a monomorphism and we know the number of curves on X, we immediately have

LEMMA 2.3. *Let X be a Hopf surface with* $a(X) = 0$. *Then*

(i) *If* $\deg X = 0$, *then* $[\operatorname{Div}(X)] = \mathbf{Z}\cdot C \oplus \mathbf{Z}\cdot C_o$.

(ii) *If* $\deg X > 0$, *then* $[\operatorname{Div}(X)] = \mathbf{Z}\cdot C$.

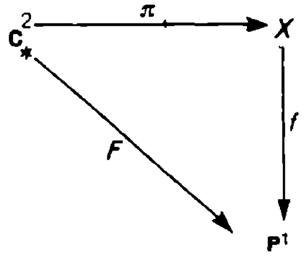
Now consider the case where $a(X) = 1$. Let $X = \mathbf{C}^2_*/\langle\gamma\rangle$ where

$$\gamma = \begin{bmatrix} a & \\ & b \end{bmatrix}$$

and $a^k = b^l$ for $k, l \in \mathbf{N}$. Denote $d := \gcd(k, l)$ (the greatest common divisor).

LEMMA 2.4. *Let X be as above. Then* $[\mathrm{Div}(X)] \cong \mathbf{Z}\oplus\mathbf{Z}_d$.

Proof. Without loss of generality we may assume that k and l are chosen the smallest possible. By assumption, there exists a meromorphic function on X, hence a holomorphic mapping $f\colon X \to \mathbf{P}^1$. Let $\pi\colon \mathbf{C}^2_* \to X$ denote the factor map. Then we have a commutative diagram of holomorphic surjections



where $F = f\circ\pi$ is given by

$$F(z_1, z_2) = z_1^k/z_2^l.$$

The only curves on X are fibres of f ([3], Theorem 4.3, and [4], Section 9).

For $w \in \mathbf{P}^1$ denote $C_w := f^{-1}(w) \subset X$. In particular we have $C_\infty = C$ and $C_0 = C_o$. The divisors kC_o, lC and C_w, for $w \neq 0, \infty$, are linearly equivalent, hence $[\mathrm{Div}(X)]$ is equal to the image G of $\mathbf{Z}\cdot C\oplus\mathbf{Z}\cdot C_o$ in $\mathrm{Pic}(X)$.

We obtain a short exact sequence of groups

$$0 \to \ker\varphi \to \mathbf{Z}\oplus\mathbf{Z} \xrightarrow{\varphi} G \to 0$$

where $\varphi(k, 0) = \varphi(0, l)$. Since G has a non-zero free part, $\ker\varphi = \mathbf{Z}$ or 0. The condition that k and l are the smallest possible means that the element $(k, -l) \in \mathbf{Z}\oplus\mathbf{Z}$ is a generator of $\ker\varphi$. The numbers k/d and l/d are mutually prime, thus there exist integers x and y such that

$$x\frac{k}{d} + y\frac{l}{d} = 1$$

hence the element $(k/d, -l/d)$ can be completed to a basis of $\mathbf{Z}\oplus\mathbf{Z}$. Since we may write

$$\ker\varphi = d\mathbf{Z}(k/d, -l/d),$$

we obtain that

$$G \cong \frac{\mathbf{Z}\oplus\mathbf{Z}}{0\oplus d\mathbf{Z}} = \mathbf{Z}\oplus\mathbf{Z}_d$$

QED.

COROLLARY 2.5. *Let X be a Hopf surface given by a quasimatrix*

$$\gamma = \begin{bmatrix} a & t(\ )^m \\ & b \end{bmatrix}.$$

Then we have the following table:

conditions for γ	[Div(X)]
$t \neq 0$	$\mathbf{Z}$
$t = 0$ and $a^k \neq b^l$, $(k, l \in \mathbf{N})$	$\mathbf{Z} \oplus \mathbf{Z}$
$t = 0$ and $a^k = b^l$, $\gcd(k, l) = 1$	$\mathbf{Z}$
$t = 0$ and $a^k = b^l$, $\gcd(k, l) = d > 1$	$\mathbf{Z} \oplus \mathbf{Z}_d$

Remark 2.6. Lemma 2.4 can be proved also in another way. By Kodaira [4], for any Hopf surface X, $\mathrm{Pic}(X) \cong \mathbf{C}^*$. By (2.2) in [2] we have a correspondence

$$\mathrm{Pic}(X) \ni [C_o] \leftrightarrow a \in \mathbf{C}^*, \qquad \mathrm{Pic}(X) \ni [C] \leftrightarrow b \in \mathbf{C}^*$$

hence [Div(X)] is isomorphic to the subgroup in $\mathbf{C}^*$ generated by a and b.

3. Note that the definition of a Hopf surface implies that the class of all Hopf surfaces is closed under deformations (any deformation of a Hopf surface is again a Hopf surface). In this section we will prove that any two Hopf surfaces are deformations of each other. We introduce a filtration in the class of all Hopf surfaces and construct an ascending chain of analytic families in such a way that Hopf surfaces from a given member of the filtration are some fibers of the analytic family which corresponds to that member. The filtration will be given by considering Hopf surfaces of degree less or equal to a given one.

In our construction we use a notion of a hyperquasimatrix being a generalization of that of matrix. On could avoid use of those two notions and denote all automorphisms in the conventional way, but it would only cause complication of terminology and notation.

DEFINITION 3.1. A *hyperquasimatrix* is an expression of the form

$$\begin{bmatrix} a & \sum_{\mu=1}^{m} t_\mu(\ )^\mu \\ & b \end{bmatrix}$$

which is a short notation for a mapping

$$(x, y) \mapsto (ax + \sum_{\mu=1}^{m} t_\mu y^\mu, by).$$

In what follows we assume that a, b, $t_1, \ldots, t_m$ are complex numbers and m is a positive integer. If $t_\mu \neq 0$ for some μ, then the greatest μ such that $t_\mu \neq 0$ is called the *degree* of the hyperquasimatrix. If $t_\mu = 0$ for all μ, then we say that the hyperquasimatrix is of *degree* 0.

Any hyperquasimatrix may be considered to be a point of $\mathbf{C}^{n+2}$ (where $n \geqslant m$) in the following way

$$(a, b, t_1, \ldots, t_m, 0, \ldots, 0).$$

Assume the following notation

$$D^* := \{z \in \mathbf{C}:\ 0 < |z| < 1\}, \qquad M^m := D^* \times D^* \times \mathbf{C}^m.$$

Let u be a hyperquasimatrix and $u \in M^m$. Consider an automorphism

$$g\colon\ M^m \times \mathbf{C}_*^2 \to M^m \times \mathbf{C}_*^2, \qquad (u, x) \mapsto (u, u(x)),$$

where $x = (z, w) \in \mathbf{C}_*^2$.

LEMMA 3.2. *The group $\langle g \rangle$ acts freely and properly discontinuously on $M^m \times \mathbf{C}_*^2$.*

Proof. The proof of this lemma is a slight modification of that of Lemma 1 in [8]. We give it here for the convenience of the reader.

Assume the contrary. Suppose that there exist a non-zero integer n and a point $x \in \mathbf{C}_*^2$ such that

$$(u, u^n(x)) = (u, x).$$

This implies that $u^n(x) = x$. Let

$$u = \begin{bmatrix} a & \sum_{\mu=1}^{m} t^{(\mu)}(\ )^{\mu} \\ & b \end{bmatrix}.$$

(In this proof we write $t^{(\mu)}$ instead of t_μ.) It is easy to see that then for $n > 0$ holds

$$a^n z + \sum_{\mu=1}^{m} t^{(\mu)} \sum_{\nu=0}^{n-1} a^{n-1-\nu} b^{\mu\nu} w^{\mu} = z, \qquad b^n w = w$$

and for $n < 0$

$$\frac{z}{a^n} - \sum_{\mu=1}^{m} t^{(\mu)} \sum_{\nu=0}^{n-1} \frac{w^{\mu}}{a^{n-\nu} b^{\mu(\nu+1)}} = z, \qquad \frac{w}{b^n} = w.$$

Hence in both cases it must be $w = 0$, whence $z = 0$ and it gives a contradiction. Therefore $\langle g \rangle$ acts freely.

To prove that $\langle g \rangle$ acts properly discontinuously it is sufficient to show that for any two compact subsets $K_1 \subset M^m$, $K_2 \subset \mathbf{C}_*^2$,

$$\operatorname{card}\{n \in \mathbf{Z}:\ g^n(K_1 \times K_2) \cap (K_1 \times K_2) \neq \emptyset\}$$

is finite. Since K_1 and K_2 are compact, there exist positive real numbers C, D such that for every $u \in K_1$,

$$|a| \leqslant C < 1, \qquad |b| \leqslant C < 1, \qquad |t| \leqslant D$$

and there exist real numbers $A > 0$, $B > 1$ such that for every $x \in K_2$,

$$A \leqslant |x| \leqslant B$$

where $|\ |$ is a norm in $\mathbf{C}^k$ defined as

$$|(z_1, \ldots, z_k)| := |z_1| + \ldots + |z_k|$$

for $(z_1, \ldots, z_k) \in \mathbf{C}^k$. Let

$$s_n^{(\mu)} := \sum_{\nu=0}^{n-1} a^{n-1-\nu} b^{\mu\nu}.$$

Then for $u \in K_1$, $x \in K_2$, $n \in \mathbf{N}$ we have

$$|u^n x| \leqslant |a|^n |z| + \sum_{\mu=1}^{m} |t^{(\mu)}| |s_n^{(\mu)}| |w|^\mu + |b|^n |w| \leqslant C^n B + mDnC^{n-1} B^m + C^n B \underset{n \to \infty}{\longrightarrow} 0.$$

Thus there exists $N \in \mathbf{N}$ such that for every $n \geqslant N$ we have $|u^n x| < A$.

Now we consider negative exponents of u. We show that there exists $N' \in \mathbf{N}$ such that for every $n \geqslant N'$ and $(u, x) \in K_1 \times K_2$ we have

$$|u^{-n} x| > B.$$

Assume that there exist a sequence of points (u_j, x_j) of $K_1 \times K_2$ and a sequence of integers

$$n_1 < n_2 < \ldots < n_j < \ldots$$

such that for every $j \in \mathbf{N}$,

$$|u_j^{-n_j} x_j| \leqslant B.$$

Let $y_j := u_j^{-n_j} x_j$. Then $x_j = u_j^{n_j} y_j$. Let $y_j =: (\tilde{z}_j, \tilde{w}_j)$ and

$$u_j = \begin{bmatrix} a_j & \sum_{\mu=1}^{m} t_j^{(\mu)} (\ )^\mu \\ & b_j \end{bmatrix}.$$

Then

$$x_j = u_j^{n_j} y_j = \big(a_j^{n_j} \tilde{z}_j + \sum_{\mu=1}^{m} t_j^{(\mu)} s_j^{(\mu)} \tilde{w}_j^\mu,\ b_j^{n_j} \tilde{w}_j\big)$$

where

$$s_j^{(\mu)} := \sum_{\nu=0}^{n_j - 1} a_j^{n_j - 1 - \nu} b_j^{\mu\nu}.$$

Hence

$$|x_j| \leqslant C^{n_j} B + mDn_j C^{n_j - 1} B^m + C^{n_j} B \underset{j \to \infty}{\to} 0.$$

This contradicts the assumption that $x_j \in K_2$ for every $j \in \mathbf{N}$. We have therefore obtained that the set

$$\{n \in \mathbf{Z}\colon\ g^n(K_1 \times K_2) \cap (K_1 \times K_2) \neq \emptyset\}$$

is contained in

$$\{n \in \mathbf{Z}\colon\ -N' < n < N\}.$$

QED.

Now fix $m \in \mathbf{N}$ and consider all Hopf surfaces of degree less or equal to m. We construct an analytic family which will contain all such Hopf surfaces as its fibres.

Let $\langle g \rangle$ be as in Lemma 3.2. From this lemma we infer that

$$V := M^m \times \mathbf{C}^2_*/\langle g \rangle$$

is a complex manifold. Let

$$\varkappa\colon\ M^m \times \mathbf{C}^2_* \to V$$

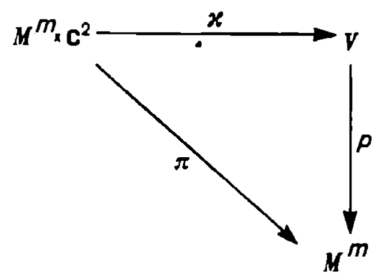
be the factor map and

$$\pi\colon\ M^m \times \mathbf{C}^2_* \to M^m$$

be the Cartesian projection. There holds

$$\pi \circ g = \pi$$

hence there exists a holomorphic map $p\colon V \to M^m$ such that the diagram



commutes. Since $\varkappa$ is a covering, p is a holomorphic submersion.

We want to show that (V, p, M^m) is an analytic family. For this purpose we must know that p is proper.

Definition 3.3 ([4], Section 10). Let B be a closed unit ball in $\mathbf{C}^2$, $B := \{(z_1, z_2) \in \mathbf{C}^2\colon |z_1|^2 + |z_2|^2 \leqslant 1\}$. A holomorphic automorphism h of $\mathbf{C}^2$ is called a *contraction* if $h^n(B)$ converges to $(0, 0)$ for $n \to \infty$.

Lemma 3.4. *Fibres of p are compact.*

Proof. In our situation a hyperquasimatrix u determines an automorphism u of $\mathbf{C}^2_*$, but by definition $u(0, 0) := (0, 0) \in \mathbf{C}^2$ we can extend u to an automorphism of $\mathbf{C}^2$. Lemma 3.2 implies that the (extended) automorphism u is a contraction. Hence we may assume by an appropriate choice of the system of global complex coordinates on $\mathbf{C}^2$ that the contraction u has the form of a quasimatrix (which satisfies conditions of Theorem 1.1) [7], [9]. Since any fibre $p^{-1}(u)$ of p has the form $p^{-1}(u) = \mathbf{C}^2_*/\langle u \rangle$, Theorem 1.1 implies that $p^{-1}(u)$ is a Hopf surface, hence is compact. QED.

Since p is a submersion with compact (and connected) fibres, it is proper. Thus (V, p, M^m) is an analytic family of Hopf surfaces. In this way we have

PROPOSITION 3.5. *For every $m \in \mathbf{N}$ there exists an analytic family such that all Hopf surfaces of degree less or equal to m are fibres of this family.*

Proof. Take the family (V, p, M^m) we have constructed. In order to obtain a Hopf surface of degree 0 as some fibre $p^{-1}(u)$, take for u a hyperquasimatrix

$$\begin{bmatrix} a & \sum_{\mu=1}^{m} t_\mu (\ )^\mu \\ & b \end{bmatrix}$$

in which $|a| \leqslant |b|$ and $t_\mu = 0$ for every μ. For a Hopf surface of a non-zero degree $k \leqslant m$, take u with $a = b^k$ and $t_\mu = 0$ for $\mu \neq k$. QED.

As a corollary we obtain

THEOREM 3.6. *Any two Hopf surfaces are deformations of each other.*

Remark 3.7. In [2] we constructed an analytic family which contains all Hopf surfaces, hence gives a stronger result. That family was obtained by means of glueing of infinitely many ones. Nevertheless, we decided to publish the present construction too, because it is simpler and in some situations it can be more convenient to have a family which is given as a factor space of an action of a group of automorphisms of a simple manifold, than as infinitely many families glued together into a big one.

4. By the *Kuranishi family* for a compact complex manifold X we mean an analytic family $\mathscr{V} = (V, p, M) = \{X_t\}_{t \in M}$ such that X is biholomorphic to X_{t_0} for some point $t_0 \in M$ and the Kodaira–Spencer map

$$\varrho_t\colon\ T_t M \to H^1(X, \Theta)$$

is an epimorphism for any t and an isomorphism for $t = t_0$.

In order to construct the Kuranishi family for any Hopf surface X, we must know h^1 of X. The following lemma shows that it is sufficient to know h^0.

LEMMA 4.1. *For any Hopf surface, $h^2 = 0$ and $h^1 = h^0$.*

Proof. By the Serre duality theorem, for a Hopf surface X we have

$$h^2 = \dim H^2(X, \Omega^0(TX)) = \dim H^0(X, \Omega^2(T^* X))$$

(where $\Omega^p(E)$ is the sheaf of holomorphic p-forms with coefficients in a holomorphic vector bundle E). To prove the remaining part of the lemma observe that transition functions of the bundle $T^*X \otimes \bigwedge^2 T^*X$ for a Hopf surface X determined by a quasimatrix

$$\begin{bmatrix} a & t(\ )^m \\ & b \end{bmatrix}$$

are given by a matrix

$$\begin{bmatrix} \frac{1}{a^2 b} & \\ -\frac{mt}{a^2 b^2} z_2^{m-1} & \frac{1}{ab^2} \end{bmatrix}.$$

Since the bundle induced on $\mathbf{C}_*^2$ is trivial, our task ist to find functions f, g holomorphic on $\mathbf{C}^2$ that satisfy

$$f(az_1 + tz_2^m, bz_2) = \frac{1}{a^2 b} f(z_1, z_2),$$

$$g(az_1 + tz_2^m, bz_2) = -\frac{mt}{a^2 b^2} z_2^{m-1} f(z_1, z_2) + \frac{1}{ab^2} g(z_1, z_2).$$

By the first equality, $f \equiv 0$, hence, by the second one, $g \equiv 0$. Thus, for any Hopf surface, $h^2 = 0$. Now we can find h^1.

It is easy to see that for every Hopf surface, its Chern classes satisfy $c_1 = c_2 = 0$. This is implied by the theorem of Künneth and the fact that c_2 is equal to the Euler characteristic with coefficients in a field. Hence the Riemann–Roch–Hirzebruch formula in our situation has a form

$$h^2 - h^1 + h^0 = 0.$$

Since $h^2 = 0$, we obtain that $h^1 = h^0$. QED.

Namba [8] determined the dimensions of complex Lie groups of holomorphic automorphisms of Hopf surfaces of degrees 0 and 1. Since those dimensions are equal to the dimensions h^0, we have to find h^0 only for Hopf surfaces of degree greater than 1.

LEMMA 4.2. *For every Hopf surface X of degree greater than* 1, $h^0 = 2$.

Proof. By assumption, X is determined by an automorphism γ of the form

$$\gamma = \begin{bmatrix} b^m & (\ )^m \\ & b \end{bmatrix}$$

where $m \geq 2$. Since this automorphism may be considered as a transition function for the manifold X, the tangent bundle of X has a transition matrix

$$\begin{bmatrix} b^m & mz_2^m \\ & b. \end{bmatrix}$$

Hence in order to obtain a global vector field on X, it is sufficient to find functions f, g holomorphic on $\mathbf{C}^2$ that satisfy the following conditions

$$(4.3)\qquad \begin{aligned} f(b^m z_1 + z_2^m, bz_2) &= b^m f(z_1, z_2) + mz_2^{m-1} g(z_1, z_2), \\ g(b^m z_1 + z_2^m, bz_2) &= bg(z_1, z_2). \end{aligned}$$

Let

$$f(z_1, z_2) = \sum_{k,l=0}^{\infty} f_{kl} z_1^k z_2^l, \qquad g(z_1, z_2) = \sum_{k,l=0}^{\infty} g_{kl} z_1^k z_2^l.$$

We show that $g(z_1, z_2) = g_{01} z_2$.

From the linear part of the second equality (4.3) we infer that $g_{10} = 0$. We will see that $g_{kl} = 0$ for every k, l with $k+l > 1$.

The second equality (4.3) may be written in the form

$$\sum_{k,l=0}^{\infty} g_{kl} \sum_{\nu=0}^{k} \binom{k}{\nu} b^{m+l} z_1^{k-\nu} z_2^{m\nu+l} = \sum_{k,l=0}^{\infty} g_{kl} b z_1^k z_2^l.$$

Hence on the left side of this equality the coefficient g_{kl} occurs at monomials $z_1^\alpha\ z_2^\beta$ with the following pairs (α, β) of exponents

$$(k, l),\ (k-1, m+l), \ldots, (k-\nu, \nu m+l), \ldots, (0, km+l).$$

For our purpose it is sufficient to consider only those pairs (α, β) in which $\alpha+\beta$ is minimal, hence $(\alpha, \beta) = (k, l)$ (we recall that $m \geq 2$). For this pair it should be

$$g_{kl} b^{km+l} z_1^k z_2^l = g_{kl} b z_1^k z_2^l.$$

But the equality $b^{km+l} = b$ is impossible when $k+l \geq 2$ (and $m \geq 2$), hence $g_{kl} = 0$.

It is clear that we need not consider other pairs of exponents because if two power series are equal then in particular their parts with the sums of exponents not greater than $k+l$ are equal, too.

Thus actually $g(z_1, z_2) = g_{01} z_2$. Therefore we have

$$\sum f_{kl} (b^m z_1 + z_2^m)^k b^l z_2^l = \sum f_{kl} b^m z_1^k z_2^l + m g_{01} z_2^m.$$

Now on the left side of this equality the coefficient f_{kl} stands precisely at the same monomials $z_1^\alpha\ z_2^\beta$ as g_{kl} did before.

Let $(\alpha, \beta) \neq (0, m)$. As before, we consider only that pair (α, β) for which $\alpha+\beta$ is minimal. Then it should be

$$f_{kl} b^{km+l} z_1^k z_2^l = f_{kl} b^m z_1^k z_2^l.$$

Hence f_{kl} can be non-zero only for $(k, l) = (1, 0)$ or $(0, m)$.

Now let $(\alpha, \beta) = (0, m)$. Then we have, for the coefficients

$$f_{10} + f_{0m} = f_{0m} + m g_{01}.$$

Hence

$$f_{10} = m g_{01}.$$

Thus two independent coefficients can be non-zero, namely $f_{10} = m g_{01}$ and f_{0m}. QED.

Therefore, together with the results of Namba [8], we obtain the following table:

the form of γ	h^0
$\begin{bmatrix} a & \\ & a \end{bmatrix}$	4
$\begin{bmatrix} b^m & \\ & b \end{bmatrix}$, where $m \geqslant 2$	3
any other	2

5. Before we construct the Kuranishi families, we briefly recall the definition of the Kodaira–Spencer map. Since for every Hopf surface we have $h^2 = 0$, there are no obstructions for the basis of the Kuranishi family to be non-singular, thus it is sufficient for us to consider only the case of a non-singular basis.

Assume we have a family $\mathscr{V} = (V, p, M) = \{X_t\}_{t \in M}$ over a non-singular basis M. Fix a point $t_0 \in M$ and consider the fibre X_{t_0}. Let D be a coordinate neighbourhood around t_0 in M and let $(t^1, \ldots, t^m)$ be local coordinates. Then we can assume D so chosen that

$$p^{-1}(D) = \bigcup_{j=1}^{J} U_j$$

where U_j are coordinate neighbourhoods with coordinates $(t, z_j) = (t^1, \ldots, t^m, z_j^1, \ldots, z_j^n)$ on each of them and

$$U_j = \{x = (t, z_j):\ t \in D,\ |z_j^\alpha| < \varepsilon_j\}$$

where ε_j are small positive numbers and $\alpha = 1, \ldots, n$. For $x \in U_j \cap U_k$ holds

$$z_j^\alpha(x) = f_{jk}^\alpha(t^1(x), \ldots, t^m(x), z_k^1(x), \ldots, z_k^n(x)) = f_{jk}^\alpha(t, z_k)$$

where f^{α}_{jk} are holomorphic functions. Let $U_{tj} := X_t \cap U_j$. We can use z^{α}_j as coordinates on U_{tj}. The transformation $z^{\alpha}_j = f^{\alpha}_{jk}(t, z_k)$ depends on t.

We define the Kodaira–Spencer map

$$\varrho_{t_o}\colon\ T_{t_o}M \to H^1(X_{t_o}, \Theta).$$

An element $v \in T_{t_o}M$ can be expressed as

$$v = \sum_{i=1}^{m} c_i \left.\frac{\partial}{\partial t^i}\right|_{t_o}.$$

Consider a vector field on $U_{t_oj} \cap U_{t_ok}$

$$\vartheta_{jk}(t_o) := \left[\sum_{i=1}^{m} c_i \left.\frac{\partial}{\partial t^i}\right|_{t_o} f^1_{jk}(t, z_k), \ldots, \sum_{i=1}^{m} c_i \left.\frac{\partial}{\partial t^i}\right|_{t_o} f^n_{jk}(t, z_k)\right].$$

Then define $\varrho_{t_o}(v)$ to be the cohomology class

$$[\{\vartheta_{jk}(t_o)\}_{jk}] \in H^1(X_{t_o}, \Theta)$$

of the cocycle obtained.

6. Now we are in position to construct the Kuranishi family for an arbitrary Hopf surface.

Consider a general situation. Assume the following three conditions.

(i) We have a set $\tilde{M}$ of automorphisms of $\mathbf{C}^2_*$ analytically parametrized by a domain $M \subset \mathbf{C}^n$. For $t \in M$, denote by $\tau \in \tilde{M}$ the corresponding automorphism. (One can say that the correspondence is given by "passing to the Greek letter".) By the analytical parametrization we mean that the mapping

$$\Phi\colon\ M \times \mathbf{C}^2_* \to \mathbf{C}^2_*; \quad (t, z) \mapsto \tau(z)$$

is holomorphic in t. Let for any $t \in M$, $\mathbf{C}^2_*/\langle\tau\rangle$ be a Hopf surface.

(ii) The group $\langle\eta\rangle$ generated by an automorphism

$$\eta\colon\ M \times \mathbf{C}^2_* \to M \times \mathbf{C}^2_*; \quad (t, z) \mapsto (t, \tau(z))$$

acts freely and properly discontinuously on $M \times \mathbf{C}^2_*$.

We introduce a complex manifold $V := M \times \mathbf{C}^2_*/\langle\eta\rangle$. Let

$$\varkappa\colon\ M \times \mathbf{C}^2_* \to V$$

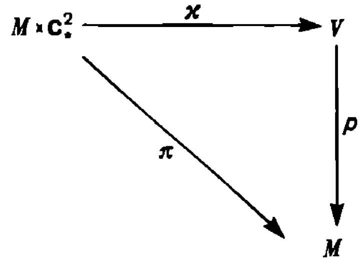
be the factor map and

$$\pi\colon\ M \times \mathbf{C}^2_* \to M$$

be the projection onto the first factor in a Cartesian product. We have

$$\pi \circ \eta = \pi$$

hence there exists a holomorphic map p: $V \to M$ such that the diagram



commutes. Since $\varkappa$ is a covering, p is a holomorphic submersion. Hence we obtain an analytic family of Hopf surfaces.

(iii) Let $S := \{(z^1, z^2) \in \mathbf{C}^2_*: |z^1|^2 + |z^2|^2 = 1\}$ be the unit sphere and let $\tau(S)$ be the image of S under τ. The condition says that for any $t \in M$, $S \cap \tau(S) = \emptyset$.

Later on we will show (Lemma 6.10) that in our applications we do not lose generality by assumption of Condition (iii).

THEOREM 6.1. *Assume conditions* (i), (ii), (iii). *Then there exists an analytic family* (V, p, M) *of Hopf surfaces such that for any* $t \in M$, *the kernel of* ϱ_t *consists of all vectors* $v_t = (v_t^1, \ldots, v_t^n) \in T_t M$ *for which there exists a holomorphic vector field* $w_t = (w_t^1, w_t^2)$ *on* $\mathbf{C}^2_*$ *such that*

$$\sum_\mu v_t^\mu \frac{\partial}{\partial t^\mu} \tau(z) = \tau'\big(w_t(z)\big) - w_t\big(\tau(z)\big)$$

for any $z \in \mathbf{C}^2_*$, *where* τ' *denotes the differential of* τ.

Proof. In the proof we give an explicit description of ϱ_t in our case. This description is a modification of that of Kodaira and Spencer in [6], § 15.

Let $t \in M$ and let

W_1 be a neighbourhood of S in $\{t\} \times \mathbf{C}^2_*$ (dependent of t),

W_2 be the domain in $\{t\} \times \mathbf{C}^2_*$ with the boundary equal to $S \cup \tau(S)$,

$W_3 := \tau(W_1)$.

Finally, let $U_i := \varkappa_t(W_i)$ where $\varkappa_t := \varkappa|_{\{t\} \times \mathbf{C}^2_*}$. Then we have

$$X_t = U_1 \cup U_2 \cup U_3 .$$

(In fact $U_1 = U_3$, hence even $X_t = U_1 \cup U_2$.) Since the covering $\varkappa_t$ is the factor map

$$\varkappa_t\colon \mathbf{C}^2_* \to X_t = \mathbf{C}^2_*/\langle\tau\rangle$$

and in any of the sets S, $\tau(S)$, W_2 there are no two points lying in one orbit of $\langle\tau\rangle$ (and $S \cap \tau(S) = \emptyset$), we can choose W_1 so small that the covering $\varkappa_t$

restricted to W_1 (and W_3) is trivial, hence there exist holomorphic maps

$$h_i(t)\colon\ U_i \to \{t\} \times \mathbf{C}^2$$

defined by

$$h_i(t) := \varkappa_t^{-1}\colon\ U_i \to W_i.$$

Use $z_i \in W_i$ as a local coordinate of $\varkappa_t(z_i) \in U_i$. Then the transition functions $f_{ik}(t, z_k)$ are as follows

$$f_{12}(t, z_2) = z_2, \quad f_{23}(t, z_3) = z_3, \quad f_{31}(t, z_1) = \tau(z_1).$$

By our assumption we can see that the transition functions depend holomorphically on $t \in M$. Let $v_t = (v_t^1, \ldots, v_t^n) \in T_t M$. As in the definition of ϱ_t, we have

$$\vartheta_{ik}(t_k) := \sum_{\mu} v_t^{\mu} \frac{\partial}{\partial t^{\mu}} f_{ik}(t, z_k)$$

where $t_k := \varkappa_t(t, z_k)$. We have

(6.2) $$\vartheta_{12}(t_2) = \vartheta_{23}(t_3) = 0.$$

But

$$\vartheta_{31}(t_1) = \sum_{\mu} v_t^{\mu} \frac{\partial}{\partial t^{\mu}} \tau(z)$$

generally is not equal to 0. For the sake of simplicity let us denote $\vartheta_{31}(t_1) =: v_t(z_1)$.

In this way we have obtained a 1-cocycle $\{\vartheta_{ik}(t)\}$, which determines its cohomology class $\vartheta_t = \varrho_t(v_t) \in H^1(X_t, \Theta)$.

Let $\mathcal{T}$ denote the sheaf of (C^∞-)differentiable sections of TX_t. Then the sheaves $\mathcal{T}/\Theta$ and $\bar{\partial}\mathcal{T}$ are isomorphic and we obtain a commutative diagram with exact rows

$$\begin{array}{ccccc}
H^0(X_t, \mathcal{T}/\Theta) & \longrightarrow & H^1(X_t, \Theta) & \longrightarrow & 0 \\
\| \wr & & \alpha\, \| \wr & & \\
Z^{01}_{\bar{\partial}}(X_t, TX_t) & \longrightarrow & H^{01}_{\bar{\partial}}(X_t, TX_t) & \longrightarrow & 0
\end{array}$$

where α is the Dolbeault isomorphism and $Z^{01}_{\bar{\partial}}(X_t, TX_t)$ and $H^{01}_{\bar{\partial}}(X_t, TX_t)$ mean the group of $\bar{\partial}$-closed TX_t-forms of type (0, 1) on X_t and the Dolbeault cohomology group respectively. Hence we have a $\bar{\partial}$-closed TX_t-form φ which represents ϑ_t. The form is given by sections $\lambda_i \in H^0(U_i, \mathcal{T})$ which satisfy

(6.3) $$\varphi|_{U_i} = \bar{\partial}\lambda_i \quad \text{and} \quad \lambda_k - \lambda_i = \vartheta_{ik} \quad \text{on} \quad U_i \cap U_k.$$

The form φ induces a $\bar{\partial}$-closed $T\mathbf{C}^2_*$-form $\tilde{\varphi} = (\tilde{\varphi}^1, \tilde{\varphi}^2)$ on $\{t\} \times \mathbf{C}^2_*$, which is $\langle\tau\rangle$-invariant in the sense that

$$\tau'(\tilde{\varphi}(t, z)) = \tilde{\varphi}(t, \tau(z)).$$

Any section λ_i induces a differentiable section $\tilde{\lambda}_i$ of $T\mathbf{C}^2_*$ on W_i. By (6.2), (6.3), $\tilde{\lambda} = \tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3$ is a differentiable section of $T\mathbf{C}^2_*$ on $W_1 \cup W_2 \cup W_3$ and

$$\tilde{\varphi} = \bar{\partial}\tilde{\lambda}; \quad \tau'\tilde{\lambda}(t, z_1) - \tilde{\lambda}(t, z_3) = v_t(z_1)$$

where $z_3 = \tau(z_1)$. Hence $\tilde{\lambda}$ extends to a global section $\tilde{\lambda}(t, z)$ of $T\mathbf{C}^2_*$ on $\{t\} \times \mathbf{C}^2_*$ such that

$$\tilde{\varphi} = \bar{\partial}\tilde{\lambda} \quad \text{and} \quad \tau'\tilde{\lambda}(t, z) - \tilde{\lambda}(t, \tau(z)) = v_t(z). \tag{6.4}$$

If also $\varphi = \bar{\partial}\mu$ on V_t for some μ, then μ induces $\tilde{\mu}$ on $\{t\} \times \mathbf{C}^2_*$ such that

$$\tilde{\varphi} = \bar{\partial}\tilde{\mu} \quad \text{and} \quad \tau'\tilde{\mu}(t, z) - \tilde{\mu}(t, \tau(z)) = 0 \tag{6.5}$$

hence the difference

$$w(t, z) := \tilde{\lambda}(t, z) - \tilde{\mu}(t, z)$$

is holomorphic in z and

$$\tau' w(t, z) - w(t, \tau(z)) = v_t(z). \tag{6.6}$$

Conversely, if there exists a holomorphic vector field $w(t, z)$ on $\{t\} \times \mathbf{C}^2_*$ satisfying (6.6), then

$$\tilde{\mu}(t, z) = \tilde{\lambda}(t, z) - w(t, z)$$

satisfies (6.5) and $\tilde{\mu}$ determines μ on V_t such that $\varphi = \bar{\partial}\mu$. QED.

From now on we identify t and τ. The construction of the Kuranishi families will be given as in Condition (ii) of Theorem 6.1. For different types of Hopf surfaces we will choose different domains M which will parametrize corresponding sets of automorphisms of $\mathbf{C}^2_*$.

We divide the class of all Hopf surfaces in five subclasses, namely we assume that a Hopf surfaces X is of the form $\mathbf{C}^2_*/\langle\gamma\rangle$, where

$$A\colon \gamma = \begin{bmatrix} a & \\ & a \end{bmatrix},$$

$$B\colon \gamma = \begin{bmatrix} a & \\ & b \end{bmatrix}, \qquad a \neq b^m \text{ for any } m \in \mathbf{N},$$

$$C\colon \gamma = \begin{bmatrix} b^m & \\ & b \end{bmatrix} \qquad \text{for some } m \in \mathbf{N}\setminus\{1\},$$

$$D\colon \gamma = \begin{bmatrix} b^m & (\ )^m \\ & b \end{bmatrix} \qquad \text{for some } m \in \mathbf{N}\setminus\{1\},$$

$$E\colon \gamma = \begin{bmatrix} a & 1 \\ & a \end{bmatrix}.$$

(The quasimatrices satisfy the conditions of Theorem 1.1.)

(A) Kodaira and Spencer [6] studied this case only for $|a| < \frac{2}{3}$ where a is the same as in the matrix

$$\begin{bmatrix} a & \\ & a \end{bmatrix}.$$

(Precisely speaking, they considered $|a| > \frac{3}{2}$, but this makes no difference, because the matrices

$$\begin{bmatrix} a & \\ & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a^{-1} & \\ & a^{-1} \end{bmatrix}$$

are the two generators of the same infinite cyclic group.) Nevertheless, it is easy to generalize their construction to the case $|a| < 1$. For this, pick a real number

$$\varepsilon := \tfrac{1}{2}(1 - |a|)$$

and consider the domain

$$M' := \{t \in GL(2, \mathbf{C}) \colon |\sigma| < 1 - \varepsilon, \ |\Delta| < \varepsilon^2\}$$

where for

$$t = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

we have

$$\sigma := \tfrac{1}{2}(\alpha + \delta), \qquad \Delta := \tfrac{1}{4}(\alpha - \delta)^2 + \beta\gamma.$$

Next, let M be the set of those matrices $t \in M'$ that are not equivalent to matrices of the form

$$\begin{bmatrix} b^n & \\ & b \end{bmatrix}$$

where $n \in \mathbf{N} \setminus \{1\}$. Clearly M is open in M'.

For any $t \in M$, the eigenvalues of t satisfy

$$|\sigma \pm \sqrt{\Delta}| < 1$$

thus the group $\langle \eta \rangle$ (as in Condition (ii) of Theorem 6.1) acts freely and properly discontinuously on $M \times \mathbf{C}_*^2$ and we obtain an analytic family (V, p, M). By Jordan's theorem and Kodaira's Theorem 1.1, this is a family of Hopf surfaces.

To prove that this is the Kuranishi family, we must know how $\ker \varrho_t$ looks like. In our situation the thesis of Theorem 6.1 can be formulated in a simpler way.

COROLLARY 6.7. *In our situation* ker ϱ_t *consists of all* $v_t \in T_t M$ *for which there exists a* 2×2 *complex matrix* w_t *such that* $v_t = tw_t - w_t t$, v_t *being considered as a matrix*

$$v_t = \begin{bmatrix} v_t^1 & v_t^2 \\ v_t^3 & v_t^4 \end{bmatrix}.$$

Proof. By Hartogs' theorem, the functions w_t^j are holomorphic at any point of $\mathbf{C}^2$, hence they define a holomorphic mapping $w_t = (w_t^1, w_t^2)$: $\mathbf{C}^2 \to \mathbf{C}^2$. As the desired matrix, take the differential of w_t. QED.

We check that for

$$t = \begin{bmatrix} \sigma & \\ & \sigma \end{bmatrix},$$

ker $\varrho_t = 0$. In the remaining cases it is easy to see that ker ϱ_t is equal to the tangent space $T_t M_{\sigma,\Delta}$, where $M_{\sigma,\Delta}$ is a 2-dimensional submanifold in M given by the equations

$$\alpha+\delta-2\sigma = 0, \quad (\alpha-\delta)^2+4\beta\gamma-4\Delta = 0.$$

Hence, dim ker $\varrho_t = 2$.

We already know that for $t \in M$,

$$h^1(\mathbf{C}_*^2/\langle t\rangle) = \begin{cases} 4; & \text{if } t = \begin{bmatrix} \sigma & \\ & \sigma \end{bmatrix}, \\ 2; & \text{else,} \end{cases}$$

hence ϱ_t is an epimorphism for any t and an isomorphism for

$$t = \begin{bmatrix} \sigma & \\ & \sigma \end{bmatrix},$$

in particular for

$$t = \begin{bmatrix} a & \\ & a \end{bmatrix},$$

where a is the number fixed at the beginning of the solution of Case (A).

Remark 6.8. It was essential to take the subset M in M' because if we wanted to obtain a similar family over M', such a family would not be complete at points of $M'\setminus M$. In fact, for those points, $h^1 = 3$ and ϱ_t is merely a monomorphism. As we will see later, those points should correspond to Hopf surfaces of degree greater than one, which can not appear in a family determined by matrices. Such a mistake was made by Kodaira and Spencer in [6]. They considered a family over M' and their family is not complete. For points of $M'\setminus M$ the proof of Theorem 15.4 in [6] does not work. But, as we could see, this mistake is not very serious, because we can take such a smaller neighbourhood of the point of the basis, that for the fibre over that point we have the Kuranishi family constructed.

(B) Let $D^* := \{z \in \mathbf{C}: 0 < |z| < 1\}$. Define an open subset $N \subset D^* \times D^*$ by

$$N := \{(\alpha, \beta) \in D^* \times D^*: \underset{m \in \mathbf{N}}{\forall} \alpha \neq \beta^m, \beta \neq \alpha^m\}.$$

(N is open in $D^* \times D^*$, because the complement is a union of a locally finite family of closed subsets of $D^* \times D^*$, hence it is closed itself.) If we identify an automorphism

$$g = \begin{bmatrix} \alpha & \\ & \beta \end{bmatrix}$$

with a point $(\alpha, \beta) \in D^* \times D^*$, then the assumption $g \in N$ makes sense. Then, for M take any connected neighbourhood of g in N. Lemma 3.2 implies that Condition (ii) of Theorem 6.1 is fulfilled, hence we obtain the appropriate analytic family (V, p, M). (If in a matrix

$$\begin{bmatrix} \alpha & \\ & \beta \end{bmatrix}$$

there happens $|\alpha| > |\beta|$, then it is easy to check that this matrix gives the same Hopf surface, as

$$\begin{bmatrix} \beta & \\ & \alpha \end{bmatrix}$$

does.) Since for any $t \in M$, $\dim T_t M = h^1(X_t) = 2$, we should only show that $\ker \varrho_t = 0$ for any t. In our situation the thesis of Theorem 6.1 has the following form.

COROLLARY 6.9. $\ker \varrho_t$ *consists of all* $v_t \in T_t M$ *for which there is a* 2×2 *complex matrix* w_t *such that* $v_t = tw_t - w_t t$, v_t *being considered as a matrix*

$$v_t = \begin{bmatrix} v_t^1 & \\ & v_t^4 \end{bmatrix}.$$

Proof. Is a consequence of the proof of Corollary 6.7. QED.

From this we easily obtain that $v_t = 0$, hence $\ker \varrho_t = 0$ for any t.

Warning. It would not be sufficient to consider a set

$$\{(\alpha, \beta): 0 < |\alpha| < |\beta| < 1 \text{ and } \alpha \neq \beta^m \ \forall m\}$$

because the fact $a \neq b$ does not imply $|a| \neq |b|$, so the case $|\alpha| = |\beta|$ must be included.

(C) Any quasimatrix

$$\begin{bmatrix} \alpha & \lambda(\ )^m \\ & \beta \end{bmatrix}$$

may be viewed as a point $(\alpha, \beta, \lambda) \in \mathbf{C}^3$. By considerations similar to those at the beginning of the solution of Case (B), we see that we may take M as such a neighbourhood of

$$t_o = \begin{bmatrix} b^m & \\ & b \end{bmatrix}$$

in a set

$$N := \{t = (\alpha, \beta, \lambda) \in \mathbf{C}^3 \colon \ 0 < |\alpha| < |\beta| < 1\}$$

that M contains no point t with $\alpha = \beta^m$ for $n \neq m$. The Hopf surfaces $\mathbf{C}^2_*/\langle g \rangle$ and $\mathbf{C}^2_*/\langle h \rangle$ where

$$g = \begin{bmatrix} \alpha & \lambda(\ )^m \\ & \beta \end{bmatrix}, \quad h = \begin{bmatrix} \alpha & \\ & \beta \end{bmatrix},$$

$\alpha \neq \beta^m$ and $\lambda \neq 0$, are biholomorphic (in Lemma 1.4, substitute

$$F := \begin{bmatrix} 1 & \dfrac{\lambda}{\beta^m - \alpha}(\ )^m \\ & 1 \end{bmatrix}).$$

Thus we should show that

$$\dim \ker \varrho_t = \begin{cases} 0 & \text{if } \alpha = \beta^m \text{ and } \lambda = 0, \\ 1 & \text{else.} \end{cases}$$

For this purpose we want to apply Theorem 6.1. Lemma 3.2 implies that Condition (ii) is fulfilled. The following lemma shows that so is Condition (iii).

LEMMA 6.10. *If in*

$$t = \begin{bmatrix} \alpha & \lambda(\ )^m \\ & \beta \end{bmatrix}$$

there holds $|\lambda| < 1$, *then* $S \cap t(S) = \emptyset$.

Proof. Let $z = (z_1, z_2) \in S$. This means that $|z_1|^2 + |z_2|^2 = 1$. We show

$$|\alpha z_1 + \lambda z_2^m|^2 + |\beta z_2|^2 < 1.$$

The left side is not greater than

$$|\alpha^2 z_1^2| + 2|\alpha \lambda z_1 z_2^m| + |\lambda^2 z_2^{2m}|.$$

Consider an inequality

$$Ax^2+Bx+C \leqslant 0,$$

where

$$x := |\lambda|, \quad A := |z_2^{2m}|, \quad B := 2|\alpha z_1 z_2^m|, \quad C := |\alpha^2 z_1^2|-1.$$

We obtain two roots

$$x_1 < 0, \quad x_2 = \frac{|\alpha z_1|+1}{|z_2|^m}.$$

It should be $|\lambda| < x_2$. There holds

$$\frac{|\alpha|\,|z_1|+1}{|z_2|^m} \geqslant |\alpha|\,|z_1|+1 \geqslant 1$$

hence it is sufficient to assume $|\lambda| < 1$. QED.

By Proposition 1.5, we know that we may assume it without loss of generality. We do it, but to simplify our notation we will write $\lambda = 1$.

Now it is easy to check that we can formulate Theorem 6.1 as follows.

COROLLARY 6.11. *A vector v_t belongs to* ker ϱ_t *if and only if there is a holomorphic vector field w_t on $\mathbf{C}^2$ such that*

$$v_t(z) = t'(w_t(z)) - w_t(t(z))$$

for any $z \in \mathbf{C}^2_*$, v^t being considered as a quasimatrix

$$v_t = \begin{bmatrix} v_t^1 & v_t^2(\ )^m \\ & v_t^4 \end{bmatrix}.$$

From this corollary, by the power series methods we obtain that for $\alpha = \beta^m$ and $\lambda = 0$ it must be $v_t = 0$, hence $\ker \varrho_t = 0$; and for the other t the vector v_t has a non-zero coordinate v_t^2, hence $\dim \ker \varrho_t = 1$.

(D) Define

$$N := \{(\alpha, \beta) \in \mathbf{C}^2\colon\ 0 < |\alpha| < |\beta| < 1\}$$

and consider a quasimatrix

$$t = \begin{bmatrix} \alpha & 1(\ )^m \\ & \beta \end{bmatrix}$$

as a point $(\alpha, \beta) \in N$. Let

$$t_o := \begin{bmatrix} b^m & (\ )^m \\ & b \end{bmatrix}$$

for some fixed $m \neq 1$. We find a neighbourhood M of t_o in N which consists of points (α, β) such that $\alpha \neq \beta^n$ for any $n \neq m$. In order to show that $\ker \varrho_t = 0$ for any $t \in M$ we have

COROLLARY 6.12. *A vector $v_t \in \ker \varrho_t$ if and only if there is a holomorphic vector field w_t on $\mathbf{C}^2$ such that*

$$v_t(z) = t'(w_t(z)) - w_t(t(z))$$

for any $z \in \mathbf{C}^2_*$, v_t being considered as a matrix

$$v_t = \begin{bmatrix} v_t^1 & \\ & v_t^4 \end{bmatrix}.$$

Proof. By checking. QED.

From this by the power series methods we obtain that for any $t \in M$ it must be $v_t = 0$, hence $\ker \varrho_t = 0$.

(E) Let

$$t_o = \begin{bmatrix} a & 1 \\ & a \end{bmatrix}.$$

Consider the Kuranishi family for the Hopf surface given by a matrix

$$\begin{bmatrix} a & \\ & a \end{bmatrix}.$$

Case (A). The basis of this family is a domain $M \subset GL(2, \mathbf{C})$ with elements denoted by

$$t = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Pick any $\lambda \neq 0$ such that

$$t'_o := \begin{bmatrix} a & \lambda \\ & a \end{bmatrix} \in M.$$

By Proposition 1.5 we may identify t'_o with t_o. Consider a 2-dimensional subset of M

$$M' := \{t \in M \colon \beta = \lambda, \ \alpha = a\}.$$

Restrict the family (V, p, M) from Case (A) to the set M' and obtain a family (V', p, M'). By Case (A) we know that for any $t \in M'$, $h^1(\mathbf{C}^2_*/\langle t \rangle) = 2$. We check that there holds

COROLLARY 6.13. *A vector $v_t \in \ker \varrho_t$ if and only if there is a 2×2 complex matrix w_t such that*

$$v_t = t w_t - w_t t$$

where

$$v_t = \begin{bmatrix} 0 & 0 \\ v_t^3 & v_t^4 \end{bmatrix}.$$

From this we easily obtain that it must be $v_t = 0$ for any $t \in M'$, hence $\ker \varrho_t = 0$.

This completes our construction.

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