CERTAIN SUFFICIENT CONDITIONS
TO BE A COMPLEX PROJECTIVE SPACE

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1. Introduction. Let \((M, g)\) be an \(n\)-dimensional connected Riemannian manifold with metric tensor \(g\). By \(\Delta = g^{ij} V_i V_j\) we mean the Laplacian acting on the space of \(C^\infty\)-functions \(f\) on \((M, g)\), where \(V\) is the operator of covariant derivation with respect to the connection induced by \(g\), and we assume that the indices \(h, i, j, \ldots\) run over the range \(\{1, 2, \ldots, n\}\). Recently, in [1], [3], [5], [7], the eigenvalues \(\lambda\) of the Laplacian \((\Delta f + \lambda f = 0)\) were calculated by an interesting method and the relations between \(\lambda\) and the curvature on \((M^2, g)\) and the Einstein manifold \((M^n, g)\), \(n \geq 3\), were studied.

Let \((M, J, g)\) be a real \(n\)-dimensional \((n = 2m)\) Kähler manifold with complex structure tensor \(J\) and Kähler metric tensor \(g\).

The main purpose of this paper is to show the following result which will be proved by a method similar to that used in [1].

Theorem 1. Let \((M, J, g)\) be a closed and connected Kähler manifold of complex dimension \(m\). If the equation \(\Delta f + \lambda f = 0\) admits a non-zero solution and \((2R_{ij} - \lambda g_{ij}) f^i f^j\) is positive semi-definite, then each eigenvalue satisfies

\[
\lambda \geq 4(m+1) \kappa_0,
\]

where \(f_i = J_i f, \) and \(\kappa_0\) is the minimum of all sectional curvatures. The equality holds iff \((M, J, g)\) is holomorphically isometric to the complex \(m\)-dimensional projective space \((CP^m, J, g_0)\) with the Fubini-Study metric \(g_0\) of constant holomorphic sectional curvature \(4\kappa_0\).

Remark. The first eigenvalue of the Laplacian on a complex projective space \((CP^m, J, g_0)\) with the Fubini-Study metric of constant holomorphic sectional curvature \(k\) is \((m+1)k\).

The second purpose of the paper is to obtain the following theorem by means of the integral formula on a \(K\)-conformal Killing tensor.

Theorem 2. Let \((M, J, g)\) be a compact Kähler-Einstein manifold. If \((M, J, g)\) admits a Killing vector field \(u^i\) such that \(V_i \bar{u}^i = J^{ij} V_i u_j\) is not constant and \(u^i u^j u^k P_{ijk} \geq 0\), then \((M, J, g)\) is holomorphically isometric to \((CP^m, J, g_0)\).
2. Preliminaries. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold (connected and \(C^\infty\)) with metric tensor \(g\). In \((M, g)\), if we put \(f_{ji} = V_j V_i f\), where \(f\) is a \(C^\infty\)-function, then \(f_{ji}\) denote the components of the Hessian \(\text{Hess}(f)\). Hence \(\Delta f = g^{ji} f_{ji}\). If \(f\) satisfies \(\Delta f + \lambda f = 0\), then it is called the eigenfunction corresponding to the eigenvalue \(\lambda\). By \(R_{kji}^h\), \(R_{ji}\), and \(R\) we denote the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature, respectively.

To prove Theorem 1, we need the following lemmas obtained by Simon [5], [6]:

**Lemma 2.1.** Let \(f\) be a \(C^\infty\)-function. Then \(f\) satisfies the equation

\[
\frac{1}{2} \Delta (f_{ji} f^{ji}) = 2 \sum_{i < j} \kappa_{ij} (\sigma_i - \sigma_j)^2 + f^{ji} V_j V_i (\Delta f) + \]

\[+ V_k f_{ji} f^k j^i + f^{ji} f^{k} (2 V_i R_{jk} - V_k R_{ji}),\]

where \(\sigma_1, \ldots, \sigma_n\) are the eigenvalues of the Hessian (with the corresponding orthonormal eigenvectors \(E_1, \ldots, E_n\)) and \(\kappa_{ij}\) is the sectional curvature of the plane \(\{E_i, E_j\}_{i \neq j}\).

**Lemma 2.2.** Let \((M, g)\) be closed \((n \geq 2)\). Let \(f\) and \(h\) be \(C^\infty\)-functions. Then

\[
\int f_{ji} h^{ji} \, do - \int \Delta f \Delta h \, do + \int R^{ji} f_j h_i \, do = 0,
\]

where \(do\) means the volume element of \((M, g)\).

**Lemma 2.3.** Let \((M, g)\) be closed \((n \geq 2)\). Then each eigenfunction \(f\) corresponding to the eigenvalue \(\lambda\) satisfies the equation

\[
(n - 1) \lambda \int ||df||^2 \, do = n \int R^{ji} f_j f_i \, do + \sum_{i < j} (\sigma_i - \sigma_j)^2 \, do.
\]

Next, we recall the definition of a Kähler manifold \((M, J, g)\) and identities which are necessary in the sequel. In \((M, J, g)\), the following identities hold:

\[
J_i^r J_r^j = -\delta_i^j, \quad g_{rs} J_i^r J_j^s = g_{ij},
\]

\[V_k J_i^j = 0, \quad J_i j = -J_i j, \quad R_i^r J_r^j = -\frac{1}{2} R^r_{rs} J_i^s.
\]

where \(J_i j = J_i^j g_{ij}\).

The holomorphically projective curvature tensor \(P_{kji}^h\) is given by

\[
P_{kji}^h = R_{kji}^h + \frac{1}{n+2} (R_{kl} \delta_j^k - R_{lj} \delta_k^j + S_{kl} J_j^h - S_{ji} J_k^h + 2 S_{kj} J_i^h),
\]

where \(S_{ji} = J_i^r R_{ri}\). A necessary and sufficient condition for \(P_{kji}^h = 0\) is that the manifold is a space of constant holomorphic sectional curvature \(c\), i.e., a space whose curvature tensor \(R_{kji}^h\) takes the form

\[
R_{kji}^h = \frac{c}{4} (g_{ji} \delta_k^h - g_{kl} \delta_j^h + J_{ji} J_j^k - J_{ki} J_j^h - 2 J_{kj} J_i^h).
\]
The following theorem, that was announced by Obata [4] and proved by Tanno [9], plays very important roles in the proofs of our theorems.

**Theorem A.** Let \((M, J, g)\) be a complete connected Kähler manifold of complex dimension \(m\). In order for \((M, J, g)\) to admit a non-constant function \(f\) satisfying the following system of differential equations of order three for some positive constant \(c\):

\[
4V_k V_f f_i + c(2f_k g_{ji} + f_j g_{ki} + f_i g_{kj} - J_f J_j - J_i J_k) = 0,
\]

where \(J_f = J_f^r f\), it is necessary and sufficient that \((M, J, g)\) is holomorphically isometric to the complex \(m\)-dimensional projective space \((\mathbb{C}P^m, J, g_0)\) with the Fubini-Study metric of constant holomorphic sectional curvature \(c\).

**Lemma 2.4** ([12], p. 88). If \(f\) in a compact Kähler manifold, the form \((2R_{ij} - \lambda g_{ij}) J^i J^j\) is positive semi-definite, then \(J^i = J^i f\) is a Killing vector for a solution of the equation \(\Delta f + \lambda f = 0\).

3. **Proof of Theorem 1.** First we define \(B(f)_{kji}\) as

\[
B(f)_{kji} = V_k V_j f_i + \frac{\lambda}{2(n+2)}(2f_k g_{ji} + f_j g_{ki} + f_i g_{kj} - J_f J_j - J_i J_k),
\]

whence

\[
||B(f)_{kji}||^2 = ||V_k V_j f_i||^2 - \frac{2}{n+2} \lambda^2 ||df||^2.
\]

On the other hand, by the assumption that \((2R_{ij} - \lambda g_{ij}) J^i J^j\) is positive semi-definite, Lemma 2.4 implies that \(J^i\) is Killing. So we have

\[
V_k V_j J^i + f^r R_{r k j i} = 0.
\]

Transvecting (3.3) with \(J^k\), we obtain

\[
J^r V_r V_j J^i - f^r R_{r k j i} = 0.
\]

Applying \(V^j\) to the above equation, we get

\[
J^r J^i (V_r V^j V_f f_i + R^r f^s f_i R_{j s} f^j) - f^r R_{r k j i} f^j (V_k R_{r i} - V_r R_{k i}) = 0.
\]

Moreover, contracting (3.3) with \(J^j g^{k i}\), we obtain

\[
V^j V_f f_i + f^j R_{j s} = 0,
\]

whence

\[
f^{k i} J^r J^j (-V_r (f^j R_{j s}) + R^r f^s f_i - R_{j s} f^j) - f^{k i} f^r R_{r k j i} f^j (V_k R_{r i} - V_r R_{k i}) = 0.
\]

Consequently, we have

\[
f^{k i} f^r (2V_k R_{r i} - V_r R_{k i}) = 0.
\]
Therefore, by (3.2) and $\Delta \sigma + \lambda \sigma = 0$, the formula from Lemma 2.1 can be rewritten as follows:

\begin{equation}
0 = \int \left[ 2 \sum_{i<j} \kappa_{ij} (\sigma_i - \sigma_j)^2 - \lambda \|f_{ij}\|^2 + \|B(f)_{kij}\|^2 + \frac{2}{n+2} \lambda^2 \|df\|^2 \right] \, do.
\end{equation}

In this way, making use of (3.4) and the Ricci identity, we obtain

\begin{equation}
R_{ji} f^j f^i = \frac{1}{2} \lambda \|df\|^2,
\end{equation}

which together with Lemma 2.2 yields

\begin{equation}
0 = \int \left[ \|f_{ij}\|^2 - \lambda \|df\|^2 + R_{ji} f^j f^i \right] \, do = \int \left[ \|f_{ij}\|^2 - \frac{1}{2} \lambda \|df\|^2 \right] \, do.
\end{equation}

Similarly, putting (3.6) into the equation in Lemma 2.3, we get

\begin{equation}
(n-1) \lambda \int \|df\|^2 \, do = \frac{n}{2} \lambda \int \|df\|^2 \, do + \int \sum_{i<j} (\sigma_i - \sigma_j)^2 \, do,
\end{equation}

whence

\begin{equation}
(n-2) \lambda \int \|df\|^2 \, do = 2 \int \sum_{i<j} (\sigma_i - \sigma_j)^2 \, do.
\end{equation}

Putting (3.7) and (3.8) into (3.5), we obtain

\begin{align*}
0 &= \int \left[ 2 \sum_{i<j} \kappa_{ij} (\sigma_i - \sigma_j)^2 + \|B(f)_{kij}\|^2 - \frac{n-2}{2(n+2)} \lambda^2 \|df\|^2 \right] \, do \\
&= \int \left[ \sum_{i<j} \left( 2 \kappa_{ij} - \frac{\lambda}{n+2} \right) (\sigma_i - \sigma_j)^2 + \|B(f)_{kij}\|^2 \right] \, do \\
&\geq \int \left[ \sum_{i<j} \left( 2 \kappa_0 - \frac{\lambda}{n+2} \right) (\sigma_i - \sigma_j)^2 + \|B(f)_{kij}\|^2 \right] \, do,
\end{align*}

which implies $2 \kappa_0 - \lambda/(n+2) \leq 0$, i.e.,

\begin{equation}
\lambda \geq \frac{2(n+2)}{\kappa_0} = 4(m+1) \kappa_0.
\end{equation}

If the equality holds, we have

\begin{equation}
0 = B(f)_{kij} = V_k V_j f_i + \kappa_0 (2f_k g_{ji} + f_j g_{ki} + f_i g_{kj} - f_j J_{ki} - f_i J_{kj}).
\end{equation}

Applying Theorem A, we see that $(M, J, g)$ is holomorphically isometric to the complex projective space $(CP^n, J, g_0)$. Thus, the proof of the theorem is completed.

4. Proof of Theorem 2. In this section, making use of the integral formula on a $K$-conformal Killing tensor which was introduced in [11] by
Yamaguchi, i.e., a \( K \)-conformal Killing tensor is a skew symmetric tensor field \( u_{ij} \) satisfying

\[
V_i u_{jk} + V_j u_{ik} = 2 \varrho_{ik} g_{ij} - \varrho_i g_{jk} - \varrho_j g_{ik} + 3 ( \tilde{\varrho}_i J_{jk} + \tilde{\varrho}_j J_{ik} ),
\]

where

\[
\varrho_i = \frac{1}{n+2} V^r u_{ri} \quad \text{and} \quad \tilde{\varrho}_i = J^r_i \varrho_r,
\]

we study the relation between the \( K \)-conformal Killing tensor \( u_{ij} \) and the holomorphically projective curvature tensor \( P_{kij}^h \) and we prove Theorem 2.

In [11], the following theorem is proved:

**Theorem B.** In a compact Kähler manifold \( (M, J, g) \), \( \dim M = n = 2m \), the following integral formula is valid for any skew symmetric tensor \( u_{ij} \):

\[
\int [(V^r u_{ij} - R^r_{ij} u_{jr} - R^r_{ij} u_{rs} + n \varrho_{ij} + \varrho_{ji} - 3 \tilde{\varrho}^r_{ij} J_{ij} + 3 \tilde{\varrho}_r J_{jr}) u^{lj} + \frac{1}{2} ||A_{ijk}||^2] \, d\sigma = 0,
\]

where

\[
\varrho_{ij} = \frac{1}{n+2} V_i V^r u_{rj},
\]

\[
\tilde{\varrho}_{ij} = J^r_j \varrho_r = \frac{1}{3(n+2)} J^{rs} (V_i V_j u_{rs} + V_i V_r u_{js}),
\]

and

\[
A_{ijk} = V_i u_{jk} + V_j u_{ik} - 2 \varrho_{ik} g_{ij} + \varrho_i g_{jk} + \varrho_j g_{ik} - 3 ( \tilde{\varrho}_i J_{jk} + \tilde{\varrho}_j J_{ik}).
\]

Let us show the following

**Theorem 4.1.** Let \( (M, J, g) \) be a compact Kähler manifold with a parallel Ricci tensor. If \( (M, J, g) \) admits a Killing vector field \( u^i \) and \( u^j u^k P_{kij} \geq 0 \) is satisfied, then \( V_j u_i \) is a closed \( K \)-conformal Killing tensor field.

**Proof.** We now put

\[
T_{ij} = V^r u_{ij} - R^r_{ij} u_{jr} - R^r_{ij} u_{rs} + n \varrho_{ij} + \varrho_{ji} - 3 \tilde{\varrho}^r_{ij} J_{ij} + 3 \tilde{\varrho}_r J_{jr}.
\]

Let us calculate \( u^j T_{ij} \). As \( u^i \) is a Killing vector field, we have

\[
V_i V_j u_k + u_r^i R_{rjk} = 0,
\]

which means that \( V_j u_k \) is closed. Putting \( V_i u_j = u_{ij} \), we get

\[
\varrho_{ij} = -\frac{1}{n+2} u^r_{ij} R_{rij},
\]

\[
\tilde{\varrho}^r_{ij} J_{ij} = -\frac{1}{n+2} u^r_{ij} R_{rs} J^{rs} J_{ij}, \quad \tilde{\varrho}_r J^r_j = -\frac{1}{n+2} u^r_{ij} R_{rs} J^s_i J^r_j,
\]

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whence

\[(4.1) \quad u^{ij} T_{ij} = \frac{3}{2} u^{ij} u^{rs} R_{ijrs} - u^{ij} R_i^r u_{jr} + (n - 1) u^{ij} \xi_{ij} - 3u^{ij} (\bar{\xi}_r J_{ij} - \bar{\xi}_i J_j^r) \]

\[= \frac{3}{2} \left[ u^{ij} u^{rs} R_{ijrs} + \frac{2}{n + 2} \left( u^{ij} u^r_j R_{ir} + u^{ij} u^s_j J_{ir}^s J_{ij} R_{ts} + \right) \right. \]

\[\left. + u^{ij} u^{si} J_{ir}^s J_{sj} R_{is} \right]. \]

On the other hand, after a straightforward calculation we get

\[u^{kj} u^{ih} P_{kjih} = u^{kj} u^{ih} R_{kjih} + \frac{2}{n + 2} (u^{kj} u^i_j R_{ki} + u^{kj} u^{ih} J_{ki}^r J_{ij} R_{ri} + u^{kj} u^{ih} J_{ki}^s J_{ij} R_{is}) \]

Therefore, comparing (4.1) with this formula, we obtain

\[u^{ij} T_{ij} = \frac{3}{2} u^{kj} u^{ih} P_{kjih}. \]

We can see by Theorem B that \(u_{ij}\) is a closed \(K\)-conformal Killing tensor field.

As a corollary to Theorem 4.1, we can prove Theorem 2.

Proof of Theorem 2. According to Theorem 4.1, we see that \(V_i u_i\) is a closed \(K\)-conformal Killing tensor field. So, we get

\[2 V_i u_{jk} - V_k u_{ji} = 2 \xi_k g_{ij} - \xi_i g_{jk} - \xi_j g_{ik} + 3 (\bar{\xi}_r J_{jk} + \bar{\xi}_j J_{ik}). \]

Since the left-hand side is equal to \(3 V_i u_{jk} + V_i u_{kj} + V_k u_{ij}\) and \((M, J, g)\) is an Einstein manifold, we obtain

\[V_i u_{jk} = - \frac{R}{n(n + 2)} (u_k g_{ij} - u_j g_{ik} + 2 \bar{\xi}_i J_{jk} + \bar{\xi}_j J_{ik} + \bar{\xi}_k J_{ji}). \]

If we put \(f = V_i \bar{u}^i\), then it is easy to see that \(f\) is a solution of (2.1) because \(V_i f = -(2R/n) \bar{u}_i\). Hence our proof is completed by Theorem A.

As a consequence of Theorem 1, we have

Corollary. Let \((M, J, g)\) be a closed and connected Kähler-Einstein manifold of complex dimension \(m\). If the equation \(\Delta f + (R/m) f = 0\) admits a non-zero solution, then

\[\kappa_0 \leq \frac{R}{4m(m + 1)}. \]

The equality holds iff \((M, J, g)\) is holomorphically isometric to the complex \(m\)-dimensional projective space \((\mathbb{C} P^m, J, g_0)\).

Remark. Using Theorem 2 we can give another proof of the above corollary. Namely, since the equation \(\Delta f + (R/m) f = 0\) admits a non-zero
solution, $f^i_\mu ( = j^{ir}_\mu f_r)$ is a Killing vector field for which $P_i f^i_\mu$ is not constant. Consequently, if $f^{kj}_h f^l_\mu P_{klih}$ $\geq 0$, then we obtain the desired result by means of the properties of $P_{klih}^h$ and Lemma 2.3.

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