COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 2

SOME PROPERTIES OF WEAKLY ALMOST PERIODIC MAPPINGS ON COMPACT SPACES

BY

TOMASZ DOWNAROWICZ (WROCŁAW)

In this paper we consider weakly almost periodic (w.a.p.) continuous mappings of a compact Hausdorff space X. In the theory of Markov operators on $C(X)$, where X is metrizable, it is known that almost all trajectories converge to the Foguel boundary $F \subset X$ (see [9]). For deterministic operators we obtain a characterization of F in terms of limit points of trajectories. We also show that for w.a.p. mappings on zero-dimensional spaces the center is equal to the set of recurrent points. If $F = X$ for a w.a.p. mapping on a zero-dimensional space X, then also the center is equal to X.

1. Terminology and notation. Let X be a compact Hausdorff space and $\varphi\colon X \to X$ be a continuous mapping. The formula $T_\varphi f = f \circ \varphi$ defines a deterministic Markov operator on the space $C(X)$ of complex-valued continuous functions on X. A nonempty closed subset $A \subset X$ is called *invariant* if $\varphi(A) \subset A$ and it is called *minimal* (*invariant*) if there are no proper invariant subsets of A. It is known (see, e.g., [3]) that a point $x \in X$ belongs to M_φ^1, the union of all minimal sets, if and only if x is *uniformly recurrent*, i.e., such that for every neighbourhood U of x there exists a natural k such that for every natural n we have

$$\{\varphi^{n+i}(x)\colon\ i = 1, 2, 3, \ldots, k\} \cap U \neq \emptyset.$$

A point $y \in X$ is called *recurrent* if for every neighbourhood U of y we have

$$\{\varphi^i(y)\colon\ i = 1, 2, 3, \ldots\} \cap U \neq \emptyset.$$

The set of recurrent points will be denoted by M_φ^2. Obviously, $M_\varphi^1 \subset M_\varphi^2$. By a *center* M_φ of φ we mean the closure of the union of supports of all φ-invariant probability (Radon) measures on X. Since each minimal set supports some invariant probability (Radon) measure, we have $M_\varphi^1 \subset M_\varphi$. The *Foguel boundary* F_φ is defined as the zero set of all l.s.c. functions f $(0 \leqslant f \leqslant 1)$ such that $f \circ \varphi^n \to 0$ pointwise on X. The above definitions of M_φ and F_φ agree with the usual notions of the center and the Foguel boundary for general Markov operators on $C(X)$ (see, e.g., [9]). We say that a system (X, φ) is *conservative* if $F_\varphi = X$.

A Markov operator T on $C(X)$ is called *weakly* (*strongly*) *almost periodic* if the orbits $\{T^n f\colon n \geqslant 0\}$ are relatively compact for the weak (uniform) topology in $C(X)$. For a system (X, φ) the above definition of w.a.p. is equivalent to the condition that each element of the pointwise closure S_φ of the family $\{\varphi^n\}_{n=1}^{\infty}$ is a continuous mapping. (S_φ is always compact for the pointwise convergence, monothetic subgroup of mappings on X; see, e.g., [1], Theorem 3.1.) It is easily seen that

$$M_\varphi^2 = \{x\colon \exists \psi \in S_\varphi,\ x = \psi(x)\} = \bigcup_{\psi \in S_\varphi} M_\psi^1.$$

By $O_\varphi(x)$ we denote the set $\{\psi(x)\colon \psi \in S_\varphi\}$. Obviously, $O_\varphi(x)$ is compact. The system (X, φ) for which $X = x \cup O_\varphi(x)$ for some $x \in X$ is called *point transitive* [5].

2. Weakly almost periodic systems. Let (X, φ) be a w.a.p. system. In [8] Sine has proved that $M_\varphi^1 = M_\varphi$. Now the deLeeuw–Glicksberg decomposition will be used. For every $f \in C(X)$ we have $f = Ef + (I-E)f$, where Ef is in $C_p(\varphi)$, the minimal closed subspace of $C(X)$ containing all eigenfunctions of T_φ pertaining to eigenvalues of modulus 1, and $(I-E)f$ belongs to $C_0(\varphi)$, the subspace of $C(X)$ consisting of all functions g having zero in the weak closure of the orbit $\{g \circ \varphi^n\colon n \geqslant 0\}$ (see [1]). It is not difficult to see that in the deterministic case we have $E = T_\varepsilon$ for some retraction ε onto M_φ (cf. [5], Theorem 2.4). Let $\tilde{S}_\varphi$ be the set of all limit points of the sequence φ^n for the topology of pointwise convergence on X. By [1], $\varepsilon \in \tilde{S}_\varphi$ and $\tilde{S}_\varphi \subset S_\varphi$. The limit points of trajectories $\psi(x)$ ($\psi \in \tilde{S}_\varphi$) need not be cluster points of $O_\varphi(x)$ in the topological sense but, obviously, by the w.a.p. assumption, each topological cluster point of $O_\varphi(x)$ is such a limit point.

LEMMA 1. *Let (X, φ) be a w.a.p. point transitive system generated by a recurrent point $x \in X$. Then $M_\varphi^2 = X$ and φ is a homeomorphism.*

Proof. We have $x = \psi(x)$ for some $\psi \in S_\varphi$. But also

$$\psi\varphi^n(x) = \varphi^n\psi(x) = \varphi^n(x) \qquad \text{for } n = 1, 2, \ldots$$

By continuity, ψ is the identity on X, and thus each $y \in X$ is recurrent. We have $\psi = \lim \varphi^{n_\alpha}$ pointwise for some net n_α. Let φ^{-1} denote some cluster point of the net $\varphi^{n_\alpha - 1}$. Since φ^{-1} is continuous and $\varphi\varphi^{-1} = \varphi^{-1}\varphi = \psi$, φ is a homeomorphism.

PROPOSITION 1. *Let (X, φ) be an s.a.p. system. Then $M_\varphi^2 = M_\varphi$.*

Proof. We need only to prove that $M_\varphi^2 \subset M_\varphi$. Suppose $x = \psi(x)$, $\psi \in S_\varphi$. If $\psi = \varphi$, then $\{x\}$ is a minimal set, so $x \in M_\varphi$. Therefore we may assume that $\psi = \varphi\chi = \chi\varphi$ for some $\chi \in S_\varphi$. Moreover, $\psi | O_\varphi(x)$ must be the identity on $O_\varphi(x)$, so

$$\chi | O_\varphi(x) = [\varphi | O_\varphi(x)]^{-1}.$$

But by the s.a.p. assumption, the family S_φ is equicontinuous, whence $S_\varphi | O_\varphi(x)$ is a group. This implies that $\psi | O_\varphi(x) = \varepsilon | O_\varphi(x)$ is the only idempotent, so $x = \varepsilon(x)$ and $x \in M_\varphi$ by the previous remarks.

LEMMA 2. *Let (X, φ) be an arbitrary system and U some clopen set containing M_φ^1. Then for each $x \in U^c$ the set $O_\varphi(x)$ contains a non-recurrent point y.*

Proof. Suppose x is recurrent. Since $O_\varphi(x)$ is an invariant set, it is not disjoint with M_φ^1, and thus with U. Moreover, it is easy to see that for every natural n there exists $k(n)$ such that

$$\varphi^{k(n)}(x) \in U^c, \qquad \varphi^{k(n)+i}(x) \in U \text{ for } i = 1, 2, \ldots, n.$$

Let y be some cluster point of the sequence $\varphi^{k(n)}(x)$. We have now $y \in U^c$ and $\varphi^i(y) \in U$ for $i = 1, 2, \ldots,$ thus y is not recurrent.

THEOREM 1. *For every w.a.p. system on a zero-dimensional space X we have $M_\varphi^2 = M_\varphi$.*

Proof. Suppose there exists a recurrent point $x \notin M_\varphi$. Consider the system $(O_\varphi(x), \varphi)$. Since, by the w.a.p. assumption, $M_\varphi^1 = M_\varphi$ is closed and $O_\varphi(x)$ is zero-dimensional, Lemma 2 is valid. The application of Lemma 1 leads to a contradiction.

Remark. For w.a.p. systems the condition $M_\varphi^2 = M_\varphi$ $(= M_\varphi^1)$ is equivalent to the equality $M_\varphi = M_\psi$ $(M_\varphi^1 = M_\psi^1)$ for all $\psi = S_\varphi$.

It should be noted that symbolic systems constitute an important class of zero-dimensional systems (see [3]).

THEOREM 2. *Let X be a finite graph. Then for every w.a.p. system (X, φ) we have $M_\varphi^2 = M_\varphi$.*

Proof. Suppose the assertion is not true. Then we find $x \in M_\varphi^2 \setminus M_\varphi$ and a neighbourhood U of x such that $\bar{U} \cap M_\varphi = \emptyset$. For every n there exists $k(n)$ with

$$\varphi^{k(n)}(x) \in \bar{U}, \qquad \varphi^{k(n)+i}(x) \in \bar{U}^c \text{ for } i = 1, 2, \ldots, n.$$

Let y be some cluster point of the sequence $\varphi^{k(n)}(x)$. We have $y \notin M_\varphi$, so there is $U_y \ni y$ with $\bar{U}_y \cap M_\varphi = \emptyset$. We also have $\varphi^i(y) \in U^c$ for $i = 1, 2, \ldots,$ and thus $O_\varphi(y) \subset U^c$. Now, by the recurrence of x, $O_\varphi(y)$ cannot contain the points $\varphi^n(x)$, $n = 1, 2, \ldots,$ so $O_\varphi(y)$ is nowhere dense in $O_\varphi(x)$, and thus in X. This implies that the set $O_\varphi(y) \cap U_y$ is zero-dimensional. Consider the system $(O_\varphi(y), \varphi)$. As in the proof of Theorem 1 the application of Lemmas 1 and 2 leads to a contradiction.

Our Example 1 will show that for infinite-dimensional spaces the assertion of Theorem 2 need not hold. The question asked in [5] (p. 84) about not minimal point transitive w.a.p. systems on manifolds is related to the question whether Theorem 2 holds for finite-dimensional manifolds: for any point transitive w.a.p. system on a space X with no isolated points we have $M_\varphi^2 = X$ by Lemma 1, and whenever Theorem 2 holds X is minimal.

The inclusion $M_T \subset F_T$, which holds for all Markov operators T (see [9]), is now, for w.a.p. systems, an easy consequence of the characterization of M_φ^1 cited in Section 1 and of the following result which describes F_φ in terms of trajectories (cf. Theorem 17 in [9] for more general Markov operators).

PROPOSITION 2. *Let φ: $X \to X$ be continuous. Then the Foguel boundary is the closure of the limit points of trajectories*:

$$F_\varphi = \{\psi(x)\colon\ x \in X,\ \psi \in \tilde{S}_\varphi\}^-.$$

Proof. Denote by F_1 the set on the right. Let $y \notin F_1$ and U be a neighbourhood of y such that $\bar{U} \cap F_1 = \emptyset$. The function

$$f = 1_W, \quad \text{where } W = \bigcup_{k=0}^{\infty} \varphi^{-k}(U),$$

is l.s.c. and subinvariant. Suppose there exists $x \in X$ with $f \circ \varphi^{n'}(x) = 1$ for some subsequence n'. This would mean that $\varphi^{n'}(x) \in \varphi^{-k'}(U)$ for some natural k', so $\varphi^{n'+k'}(x) \in U$. But $n'+k' \to \infty$. Pick up a subnet n_α such that φ^{n_α} converges pointwise to $\psi \in \tilde{S}_\varphi$. Then $\psi(x) \in F_1$ by the definition of F_1. On the other hand, $\psi(x) \in \bar{U}$, which is a contradiction. It follows that $f \circ \varphi^n \to 0$. Since $f(y) = 1$, $y \notin F_\varphi$. We have proved that $F_\varphi \subset F_1$. The converse inclusion is obvious: notice that whenever $g \circ \varphi^n \to 0$, g vanishes at all limit points.

It is not hard to construct an example showing that the set

$$\{\psi(x)\colon\ x \in X,\ \psi \in \tilde{S}_\varphi\}$$

need not be closed.

Remark. It is obvious now that $M_\varphi^2 \subset F_\varphi$.

THEOREM 3. *A w.a.p. system is conservative if and only if M_φ^2 is dense in X.*

Proof. The sufficiency is an immediate consequence of Proposition 2. To prove the necessity assume that $V_1 \neq \emptyset$ is open. The function

$$f_1 = 1_{W_1}, \quad \text{where } W_1 = \bigcup_{i=0}^{\infty} \varphi^{-i}(V_1),$$

is l.s.c. and subinvariant. By conservativeness, there exists $x_1 \in V_1$ with

$$f_1 \circ \varphi(x_1) = f_1(x_1) = 1$$

(see a $\Rightarrow$ b in [2], p. 29; the metrizability assumption is unnecessary). For some $i_1 \geq 1$ we have $x_1 \in \varphi^{-i_1}(V_1)$, so there is a neighbourhood V_2 of x_1 with $\bar{V}_2 \subset V_1$ and $\varphi^{i_1}(\bar{V}_2) \subset V_1$. Analogously, replacing V_1 by V_2 we take f_2 and choose x_2 to find V_3. By induction we obtain a sequence of nonempty open sets V_n with

$$\bar{V}_{n+1} \subset V_n \quad \text{and} \quad \varphi^{i_n}(\bar{V}_{n+1}) \subset V_n.$$

The set

$$V_\infty = \bigcap_{n=1}^{\infty} \bar{V}_n$$

is closed and nonempty with $\varphi^{i_n}(V_\infty) \subset V_n$ $(n \geq 1)$. Pick up a subnet i_α of i_n for which φ^{i_α} converges pointwise to some continuous $\psi \in S_\varphi$. It is obvious now that V_∞ is ψ-invariant, whence $V_\infty \cap M_\psi \neq \emptyset$ and the proof is complete.

COROLLARY. *A w.a.p. system (X, φ), where X is a zero-dimensional space or a finite graph, is conservative if and only if $M_\varphi = X$. It is then an s.a.p. homeomorphism.*

The last part follows from Theorem 2.4 in [5].

It should be noted that the structure of s.a.p. homeomorphisms is fully described by the celebrated Halmos–von Neumann theorem: they act as translations in disjoint unions of compact monothetic groups.

Remark. In metric spaces Theorem 3 holds without w.a.p. assumption: we may choose V_n with diameters converging to zero; then V_∞ turns out to be a ψ-invariant point, and so a recurrent one (although ψ need not be continuous). The theorem can be expressed as follows:

If the set of limit points is dense in a compact metric dynamical system, then the set of recurrent points is also dense.

A question is whether the same is true without metrizability (**P 1336**).

We end the section with a lemma which will be used in later examples to verify the w.a.p. property.

LEMMA 3. *Let (X, φ) be a point transitive system generated by a point $x_0 \in X$. The following conditions are equivalent:*

(i) *(X, φ) is a w.a.p. system;*

(ii) *S_φ is commutative;*

(iii) $\psi_1 \psi_2(x_0) = \psi_2 \psi_1(x_0)$ *for any two* $\psi_1, \psi_2 \in S_\varphi$.

Proof. (i)$\Rightarrow$(ii) holds for arbitrary systems: It is obvious that φ commutes with all $\psi \in S_\varphi$. Let $\psi_1, \psi_2 \in S_\varphi$. We have $\psi_1 = \lim \varphi^{n_\alpha}$ pointwise on X. But, by (i), ψ_2 is continuous, whence

$$\psi_2 \psi_1 = \lim \psi_2 \varphi^{n_\alpha} = \lim \varphi^{n_\alpha} \psi_2 = \psi_1 \psi_2.$$

The implication (ii)$\Rightarrow$(iii) is trivial. To prove (iii)$\Rightarrow$(i) assume that $\psi_1 \in S_\varphi$

and $y_\alpha = \psi_\alpha(x_0) \to y$ in X. By compactness, there is a subnet $\psi_{\alpha'}$ convergent pointwise to some $\psi_2 \in S_\varphi$. We have $\psi_2(x_0) = y$. Now

$$\psi_1(y_{\alpha'}) = \psi_1 \psi_{\alpha'}(x_0) = \psi_{\alpha'} \psi_1(x_0)$$

by (iii), so

$$\psi_1(y_{\alpha'}) \to \psi_2 \psi_1(x_0) = \psi_1 \psi_2(x_0) = \psi_1(y),$$

which proves the continuity of ψ_1.

3. An example. There exists a w.a.p. system for which $M_\varphi^2 \neq M_\varphi$.

EXAMPLE 1. Let φ be the left shift on the Hilbert cube I^N, i.e.,

$$\varphi(z_1, z_2, \ldots) = (z_2, z_3, \ldots).$$

We choose a certain element $a \in I^N$ and define X as $O_\varphi(a)$. Let

$$0 < \alpha_m < 1 \quad \text{and} \quad \prod_{m=1}^{\infty} \alpha_m > 0$$

(thus $\alpha_m \to 1$). There is a sequence of natural numbers $w_m \geqslant 2$ with $\alpha_m^{w_m} \to 0$. Set

$$\begin{aligned} a_1 &= (1,\ \alpha_1,\ \alpha_1^2,\ \ldots,\ \alpha_1^{w_1},\ \alpha_1^{w_1-1},\ \ldots,\ \alpha_1,\ 1,\ \alpha_1,\ \alpha_1^2,\ \ldots,\ \alpha_1^{w_1},\ \ldots), \\ a_2 &= (1,\ 1,\ 1,\ \ldots,\ 1,\ 1,\ \ldots,\ 1,\ \alpha_2,\ \alpha_2,\ \alpha_2,\ \ldots,\ \alpha_2,\ \ldots), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

so that each $a_m \in I^N$ is a periodic sequence with the period

$$o_m = 2w_m o_{m-1} \qquad (o_0 = 1),$$

a_m is constant on the blocks of length o_{m-1} and its values are

$$1,\ \alpha_m,\ \alpha_m^2,\ \ldots,\ \alpha_m^{w_m},\ \alpha_m^{w_m-1},\ \ldots,\ \alpha_m.$$

Take

$$a = \prod_{m=1}^{\infty} a_m$$

multiplied coördinatewise. Since at every coordinate the product is finite, $a > 0$ everywhere. It is not hard to see that a is recurrent, but not uniformly recurrent: the zero element of I^N (denoted by $\mathit{0}$) is easily seen to be in $O_\varphi(a)$, and hence $O_\varphi(a)$ is not minimal. The system (X, φ) is by definition point transitive. It remains to show that φ is w.a.p. on X. We use the condition (iii) of Lemma 3. Let $x \in X$. Then, by metrizability,

$$x = \lim \varphi^{n_k}(a).$$

By an easy diagonal argument and by compactness we may choose a subsequence n_k with the following properties:

(1) $n_k = n_m (\bmod o_m)$ whenever $k \geqslant m$. We put $p_m = (n_m)_{o_m}$.

(2) $r_k = \prod_{m=k+1}^{\infty} \varphi^{n_k}(a_m)$ converges in k at each coordinate.

The limit element is denoted by r. Clearly,

(3) $0 \leqslant p_m < o_m$ and $p_m = p_{m-1} (\bmod o_{m-1})$.

We have now $\varphi^{n_k}(a_m) = \varphi^{p_m}(a_m)$ for $k \geqslant m$, and thus

$$\varphi^{n_k}(a) = \prod_{m=1}^{\infty} \varphi^{n_k}(a_m) = \prod_{m=1}^{k} \varphi^{p_m}(a_m) r_k.$$

Since the first factor decreases monotonically, we obtain

$$x = \prod_{m=1}^{\infty} \varphi^{p_m}(a_m) r.$$

Now we observe that r is constant, i.e., $r = (\varrho, \varrho, \ldots)$. In fact, if q_m and n_k are sequences of natural numbers such that the limit r' of r'_k,

$$r'_k = \prod_{m=k+1}^{\infty} \varphi^{q_m + n_k}(a_m),$$

exists, then r' is constant. This easily follows from the observation that the quotient of any two neighbouring coordinates of r'_k is contained in the interval

$$\Big[\prod_{m=k+1}^{\infty} \alpha_m, \prod_{m=k+1}^{\infty} \alpha_m^{-1} \Big]$$

convergent to $\{1\}$. So far the points of X have been described: they are of the form

$$x = \varrho \prod_{m=1}^{\infty} \varphi^{p_m}(a_m),$$

where ϱ denotes a real factor from $[0, 1]$ and the numbers p_m satisfy (3). Moreover, if $x \neq \mathbf{0}$, then all coordinates of x are nonzero and the above representation is unique. The first assertion follows from the observation that the quotient of any two neighbouring coordinates of x is in the interval

$$\Big[\prod_{m=1}^{\infty} \alpha_m, \prod_{m=1}^{\infty} \alpha_m^{-1} \Big].$$

To prove the uniqueness suppose that

$$\varrho \prod_{m=1}^{\infty} \varphi^{p_m}(a_m) = \varrho' \prod_{m=1}^{\infty} \varphi^{p'_m}(a_m) \neq \mathbf{0}.$$

Let m_0 be the least number for which $p_{m_0} \neq p'_{m_0}$. We have

$$\varrho \prod_{m=m_0}^{\infty} \varphi^{p_m}(a_m) = \varrho' \prod_{m=m_0}^{\infty} \varphi^{p'_m}(a_m) \neq 0.$$

Notice that in the sequence on the left the block of length o_{m_0-1}, for which $\varphi^{p_{m_0}}(a_{m_0})$ takes the value $\alpha_{m_0}^{w_{m_0}}$, has smaller values than the neighbouring blocks (from the assumption $w_m \geqslant 2$ all sequences $\varphi^{p_m}(a_m)$ for $m > m_0$ are constant on these three blocks). But, since $p_{m_0} \neq p'_{m_0}$, $\varphi^{p'_{m_0}}(a_{m_0})$ has at the same positions another value, and thus one of the neighbouring blocks is smaller than the middle one. This contradiction implies $p_m = p'_m$ for all m and, consequently, $\varrho = \varrho'$. Now let $\psi = S_\varphi$ and $x \in X$,

$$x = \varrho_x \prod_{m=1}^{\infty} \varphi^{q_m}(a_m) \neq 0.$$

By metrizability and compactness of X there exists a sequence of iterates φ^{n_k} such that $\varphi^{n_k}(a) \to \psi(a)$, $\varphi^{n_k}(x) \to \psi(x)$, and for which properties (1), (2) and the following condition are satisfied:

(2′) $r'_k = \prod_{m=k+1}^{\infty} \varphi^{q_m+n_k}(a_m)$ converges in k at each coordinate.

The limit r' has been shown to be constant. We have

$$\varphi^{n_k}(x) = \varrho_x \prod_{m=1}^{\infty} \varphi^{q_m+n_k}(a_m) = \varrho_x \prod_{m=1}^{k} \varphi^{q_m+p_m}(a_m) r'_k.$$

Thus

$$\psi(x) = \varrho' \varrho_x \prod_{m=1}^{\infty} \varphi^{q_m+p_m}(a_m)$$

and, as previously,

$$\psi(a) = \varrho \prod_{m=1}^{\infty} \varphi^{p_m}(a_m).$$

We show that $\varrho = \varrho'$. First note that each sequence a_m has the following property: for any two natural i and j,

$$\alpha_m \frac{a_m(i)}{a_m(0)} \leqslant \frac{a_m(i+j)}{a_m(j)} \leqslant \alpha_m^{-1} \frac{a_m(0)}{a_m(i)},$$

where $a_m(i)$ denotes the $(i+1)$-st coordinate of a_m $(a_m(0) = 1)$. Since

$$\prod_{m=1}^{\infty} a_m(q_m) = \varrho_x^{-1} x(0) \neq 0,$$

the product $\prod_{m=k+1}^{\infty} a_m(q_m)$ tends to 1 when $k \to \infty$. Thus, by the above inequalities,

$$\frac{\prod_{m=k+1}^{\infty} a_m(q_m+n_k)}{\prod_{m=k+1}^{\infty} a_m(n_k)} \to 1.$$

In our notation the quotient tends to

$$\frac{r'(0)}{r(0)} = \frac{\varrho'}{\varrho}.$$

From the uniqueness of the representation of nonzero elements of X it follows that each $\psi \in S_\varphi$ which is nonzero at a is of the form $\psi = \varrho\psi_p$, where $\varrho \in (0, 1]$ and ψ_p is defined on X for $p = (p_1, p_2, \ldots)$ satisfying (3) by the formula

$$\psi_0(x) = \begin{cases} \boldsymbol{0} & \text{if } x = \boldsymbol{0}, \\ \varrho_x \prod_{m=1}^{\infty} \varphi^{q_m+p_m}(a_m) & \text{if } x = \varrho_x \prod_{m=1}^{\infty} \varphi^{q_m}(a_m) \neq \boldsymbol{0}. \end{cases}$$

Note also that for any such p the sequence $\varphi^{p_m}(a)$ converges to $\psi_p(a)$, and thus, whenever the last is nonzero, $\varphi^{p_m} \to \psi_p$ (each convergent subnet of φ^{p_m} has the same limit). So $\psi_p \in S_\varphi$. The above formula implies easily the required condition $\psi_1\psi_2(a) = \psi_2\psi_1(a)$ for $\psi_1, \psi_2 \in S_\varphi$ which are nonzero at a. The same holds trivially when both mappings are zero at a. The remaining case is $\psi_1(a) \neq \boldsymbol{0} = \psi_2(a)$. We have $\psi_1 = \varrho\psi_p$, $\psi_p(a) \neq \boldsymbol{0}$, for some ϱ and p. Set

$$p'_m = \begin{cases} o_m - p_m & \text{if } p_m \neq 0, \\ 0 & \text{if } p_m = 0. \end{cases}$$

Since the p'_m satisfy (3), we have $\psi_{p'} \in S_\varphi$. Moreover, $\psi_{p'}(a)$ and $\psi_p(a)$ are, by the definition of a, equal at the first coordinate, thus $\psi_{p'} a \neq \boldsymbol{0}$. We have

$$\psi_p\psi_{p'}(a) = \prod_{m=1}^{\infty} \varphi^{o_m}(a_m) = a.$$

Suppose now $\psi_2\psi_1(a) \neq \boldsymbol{0}$. Then

$$\begin{aligned} \psi_2\psi_1(a) &= \psi_2\psi_1(\psi_p\psi_{p'}(a)) \\ &= \psi_p\psi_{p'}(\psi_2\psi_1)(a) = \psi_p(\psi_2\psi_1)\psi_{p'}(a) \\ &= \psi_p\psi_2\varrho\psi_p\psi_{p'}(a) = \varrho\psi_p\psi_2(a) = \boldsymbol{0}. \end{aligned}$$

The second and the third equalities follow from the previously proved commutation law for mappings which are nonzero at a. This contradiction implies $\psi_2 \psi_1(a) = 0 = \psi_1 \psi_2(a)$, the last equality being obvious. An application of Lemma 3 shows that (X, φ) is a w.a.p. system.

In this example it is easy to verify that $M_\varphi = \{0\}$. By Lemma 1 we have $M_\varphi^2 = X$, and thus the system (X, φ) is conservative. This shows that we cannot omit the assumptions made on the space X in the Corollary.

4. Some remarks on the Rosenblatt theorem. In the paper of Rosenblatt ([6], Lemma 12) it is shown that if T is a w.a.p. irreducible Markov operator, then T^n converges in the weak operator topology if and only if the peripheral point spectrum of T (the eigenvalues of modulus 1) is $\{1\}$. By the same proof the Rosenblatt theorem remains valid if irreducibility is replaced by the weaker condition $M_T = X$. Since $M_\varphi = \varepsilon(X)$ for a w.a.p. system (X, φ), the peripheral point spectra of φ and $\varphi|M_\varphi$ are equal. Thus we easily infer that the peripheral point spectrum of φ is $\{1\}$ if and only if all minimal sets are singletons, i.e., $\varphi|M_\varphi$ is the identity on M_φ. If moreover $X = F_\varphi$ is a zero-dimensional space or a finite graph, then, by the Corollary and above remarks, the Rosenblatt theorem holds trivially. Example 1 shows that the assumption $M_\varphi = X$ cannot be replaced by the conservativity assumption for infinite-dimensional spaces. We show in Example 2 that also for zero-dimensional w.a.p. systems (even with a singleton center) φ^n may diverge on F_φ if $X \neq F_\varphi$. There exists a nonconservative zero-dimensional w.a.p. system (which is in fact a symbolic system) with peripheral point spectrum $\{1\}$ in which $M_\varphi \neq F_\varphi$ (see [8], Example 2). We construct such a system with φ^n diverging on F_φ.

EXAMPLE 2. Set

$$p = 0.\ 1\quad 0\,0\,1\quad 0\quad 0\,0\,0\,1\quad 0\quad 0\quad 0\,0\,0\,0\,1 \ldots = 0.\ \alpha_0\ \alpha_1\ \alpha_2 \ldots,$$

$$q = 0.\ \alpha_0\ 0\ 0\ \alpha_0\ \alpha_1\ 0\ 0\ 0\ \alpha_0\ \alpha_1\ \alpha_2\ 0\ 0\ 0\ 0\ \alpha_0\ \alpha_1\ \alpha_2\ \alpha_3 \ldots$$

in the dyadic representation and consider the mapping

$$\varphi(x) = 2x \pmod 1 \qquad (\varphi(0.\ \beta_1 \beta_2 \beta_3 \ldots) = 0.\ \beta_2 \beta_3 \beta_4 \ldots)$$

restricted to

$$X = O_\varphi(q).$$

To prove the w.a.p. property of (X, φ) note at first that $Y = O_\varphi(p)$ is a φ-invariant subset of X, (Y, φ) is a w.a.p. system analogous to that in Example 2 in [8] and the only limit points of (Y, φ) are 0, 1/2, 1/4, 1/8, ... Observe that 1 appears in p exactly at the same positions as α_0 does in q. Thus $\varphi^{n'}(p) \to 1/2$ iff $\varphi^{n'}(q) \to p$ for every subsequence n'. By similar arguments we obtain

(1) $\lim \varphi^{n'}(q) = p/2^t \Leftrightarrow \lim \varphi^{n'}(p) = 1/2^{t+1}$;

(2) $\lim \varphi^{n'}(q) = 0$ or $1/2^t \Rightarrow \lim \varphi^{n'}(p) = 0$;

(3) $\lim \varphi^{n'}(q) = \varphi^t(p) \Leftrightarrow \lim \varphi^{n'-t}(q) = p \Leftrightarrow \lim \varphi^{n'-t}(p) = 1/2$ $(t \geqslant 1)$.

It is also not very difficult to see that

(4) the set of limit points equals $Y \cup \{p/2^t: t = 1, 2, 3, \ldots\}$.

Putting the facts (1)–(4) together we obtain

$$X = Y \cup \{p/2^t: t = 1, 2, 3, \ldots\} \cup \{\varphi^n(q): n = 1, 2, 3, \ldots\}$$

and $\varphi^{n'}$ converges pointwise to some $\psi \in S_\varphi$, whenever $\varphi^{n'}(q)$ converges. Thus we may consider only subsequences of iterates instead of subnets. By Lemma 3 the only remaining thing is to check if for any two subsequences of iterates $\varphi^{n'}$ and $\varphi^{m'}$ which converge at q the iterated limits of $\varphi^{m'+n'}(q)$ commute. We consider four cases:

(a) one of the limits, say, $\lim \varphi^{n'}(q)$, is zero or $1/2^t$ $(t \geqslant 1)$;

(b) $\lim \varphi^{n'}(q) = \varphi^t(p)$, $\lim \varphi^{m'}(q) = \varphi^s(p)$ $(t, s \geqslant 0)$;

(c) $\lim \varphi^{n'}(q) = p/2^t$, $\lim \varphi^{m'}(q) = p/2^s$ $(t, s \geqslant 1)$;

(d) $\lim \varphi^{n'}(q) = \varphi^t(p)$, $\lim \varphi^{m'}(q) = p/2^s$ $(t \geqslant 0,\ s \geqslant 1)$.

In the case (a) we have $\lim\limits_{m'} \lim\limits_{n'} \varphi^{m'+n'}(q) = 0$. On the other hand, by (2), $\lim \varphi^{n'}(p) = 0$, and thus, since (Y, φ) is w.a.p., $\varphi^{n'}$ converges to zero on the whole Y. Clearly, $\lim \varphi^{m'}(q)$ is a limit point. If it is in Y, we are done, if not, it must be $p/2^s$ (by (4)). Then $\lim \varphi^{n'-s}(q)$ is still of the form $1/2^{t'}$ or zero and

$$\lim_{n'} \lim_{m'} \varphi^{n'+m'}(q) = \lim_{n'} \lim_{m'} \varphi^{(n'-s)+(m'+s)}(q) = \lim_{n'} \varphi^{n'-s}(p) = 0$$

(by (2)). We omit similar arguments for (b), (c) and (d) in which the facts (3), (1) and (3) are used, respectively.

In this example the center is $\{0\}$ (this is easy to see by Theorem 1) and the peripheral point spectrum is $\{1\}$. The Foguel boundary is (by (4) and Proposition 2) equal to

$$Y \cup \{p/2^t: t = 1, 2, 3, \ldots\}.$$

Clearly, φ^n does not converge on the Foguel boundary. Also the system (F_φ, φ) is not conservative.

I am due to thank A. Iwanik for his help during the preparation of this paper.

REFERENCES

[1] K. deLeeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. 105 (1961), pp. 63–97.

[2] S. R. Foguel, *The ergodic theory of positive operators on continuous functions*, Ann. Scuola Norm. Sup. Pisa 26 (1973), pp. 19–34.

[3] W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ., Vol. 36, 1955.

[4] B. Jamison, *Ergodic decompositions induced by certain Markov operators*, Trans. Amer. Math. Soc. 117 (1965), pp. 451–468.

[5] J. Montgomery, R. Sine and E. Thomas, *Some topological properties of weakly almost periodic mappings*, Topology Appl. 11 (1980), pp. 69–85.

[6] M. Rosenblatt, *Equicontinuous Markov operators*, Teor. Verojatnost. i Primenen. 9 (1964), pp. 205–222.

[7] R. Sine, *Geometric theory of a single Markov operator*, Pacific J. Math. 27 (1968), pp. 155–166.

[8] — *Convergence theorems for weakly almost periodic Markov operators*, Israel J. Math. 19 (1974), pp. 246–255.

[9] — *Sample path convergence of stable Markov process. II*, Indiana Univ. Math. J. 25 (1976), pp. 23–43.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY
WROCŁAW, POLAND

Reçu par la Rédaction le 20. 3. 1982;
en version modifiée le 12. 7. 1982