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GROUPS OF SQUARES AND HALF-EQUIVALENCES IN GEOMETRY

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Introduction. In many geometrical considerations one uses an appropriate equivalence of segments or triangles as a primitive notion. This is the case of Euclidean geometry treated as a theory of equidistance relation, affine geometry as a theory of parallelity, ordered affine geometry as a theory of directed parallelity, similarity geometry, and so on. But, besides there is also another kind of notions that are useful in formalization of geometrical theories: orthogonality of segments, converse similarity, converse parallelity and many similar. They all are somehow similar – namely their square is an equivalence. And even more – their third power is equal to them. It turns out that any such relation – it will be called a *half-equivalence* – induces two equivalences of geometrical importance – its square and the sum of it and its square.

Next, for any equivalence $\approx\ \subseteq (X^n)^2$ consider the group of all bijections f of X satisfying

$$(\forall x_1, \ldots, x_n)\ x_1 \ldots x_n \approx f(x_1), \ldots, f(x_n)$$

and denote it by $\mathrm{Izt}(\approx)$. So then if $\|$ is an affine parallelity, then $\mathrm{Izt}(\|)$ is the group Dil of dilatations, for $\equiv$ being an equidistance $\mathrm{Izt}(\equiv)$ is the group of isometries, and so on. When the plane we investigate is built over a Euclidean, Pythagorean and 2-formally real field, then the class of squares in every such group forms a subgroup of $\mathrm{Izt}(\approx)$ of index 2. Moreover, when we define the congruence $\equiv_G$ by the formula

for any permutation group G of X,

$$x_1, \ldots, x_n \equiv_G^n y_1, \ldots, y_n \Leftrightarrow (\exists f \in G)[f(x_1) = y_1 \wedge \ldots \wedge f(x_n) = y_n],$$

we see that $\equiv_{\mathrm{Izt}^2(\approx)}$ is again the relation of clear geometrical sense. We can define here formally the analog of orthogonality, $\equiv^n_{G \setminus G^2}$; it turns out to be a half-equivalence.

In the first part of the paper we formalize the notions introduced above. These observations allow us to investigate the group $\mathrm{Izt}(\perp \cup \parallel)$, where $\perp$ is a Euclidean orthogonality (perhaps the most interesting, very little known though very simple), and to formulate an axiom system for the class of these groups. We show then how to reconstruct the geometry from them.

1. Half-equivalences and groups of squares. The relation $R \subseteq (X^n)^2$ will be called a *half-equivalence* provided it satisfies

(i) $R \circ R \circ R = R$,

(ii) $R = R^{-1}$,

(iii) $(\forall x_1, \ldots, x_n)(\exists y_1, \ldots, y_n)[\langle\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_n\rangle\rangle \in R]$,

where $\circ$ denotes the composition of relations, i.e., for $x, y \in X^n$,

$$\langle x, y\rangle \in R \circ R \Leftrightarrow (\exists z \in X^n)[\langle x, z\rangle, \langle z, y\rangle \in R].$$

Write $\Delta(Y) = \{\langle y, y\rangle : y \in Y\}$, $R^2 = R \circ R$, $R^3 = R \circ R^2$, and so on.

LEMMA 1. *If R is a half-equivalence, then $R \circ R$ and $R \cup R \circ R$ are equivalences.*

Proof. From (iii) and (ii) it follows that

$$\Delta(X^n) \subseteq R^2 \subseteq R \cup R^2.$$

This proves reflexivity. Furthermore,

$$(R \circ R)^{-1} = R^{-1} \circ R^{-1} = R^2 \quad \text{and} \quad (R \cup R^2)^{-1} = R^{-1} \cup (R^2)^{-1} = R \cup R^2.$$

This proves symmetry. Finally, transitivity follows from (i).

Let G be a transformation group of X, $Z \subseteq G$. For natural n define the relation $\equiv_Z^n \subseteq (X^n)^2$ by

$$\langle a_1, \ldots, a_n\rangle \equiv_Z^n \langle b_1, \ldots, b_n\rangle :\Leftrightarrow (\exists f \in Z)[f(a_1) = b_1 \wedge \ldots \wedge f(a_n) = b_n].$$

For a given group G denote by G^2 the set of squares in G, i.e., $G^2 = \{f \cdot f : f \in G\}$. Now denote by $\perp_G^n$ the relation $\equiv_{G \setminus G^2}^n$.

LEMMA 2. *If G is a transformation group of X, and G^2 is a subgroup of G of index 2, then $\perp_G^n$ is a half-equivalence and*

$$((\perp_G^n)^2 \cup \perp_G^n) = (\equiv_G^n), \quad (\perp_G^n)^2 = \equiv_{G^2}^n.$$

Proof. Assume that $a_1, \ldots, a_n \in X$ and

$$\langle a_1, \ldots, a_n\rangle (\perp_G^n)^3 \langle b_1, \ldots, b_n\rangle.$$

Then there are $f_1, f_2, f_3 \in G \setminus G^2$ with $f_1 f_2 f_3(a_i) = b_i$ for $i = 1, \ldots, n$. Consequently, $f_1 f_2 \in G^2$ and $f_1 f_2 f_3 \in G \setminus G^2$. Thus

$$\langle a_1, \ldots, a_n\rangle \perp_G^n \langle b_1, \ldots, b_n\rangle.$$

The converse inclusion is trivial. If $f(a_i) = b_i$ $(i = 1, \ldots, n)$ and $f \in G \setminus G^2$, then

$f^{-1} \in G \backslash G^2$ and $f^{-1}(b_i) = a_i$. This proves the symmetry of $\perp_G^n$. Obviously, the domain of $\perp_G^n$ is X^n just because $G \backslash G^2 \neq \emptyset$. Assume that

$$\langle a_1, \ldots, a_n \rangle \equiv_{G^2}^n \langle b_1, \ldots, b_n \rangle.$$

Then, for some $g = h^2$, $g(a_i) = b_i$ $(i = 1, \ldots, n)$. Consider any f from $G \backslash G^2$. We have

$$\langle a_1, \ldots, a_n \rangle \perp_G^n \langle f(a_1), \ldots, f(a_n) \rangle \perp_G^n \langle b_1, \ldots, b_n \rangle.$$

If $\langle a_1, \ldots, a_n \rangle (\perp_G^n)^2 \langle b_1, \ldots, b_n \rangle$, then for some $f_1, f_2 \in G \backslash G^2$ we have $f_1 f_2(a_i) = b_i$. But $f_1 f_2 \in G^2$. Hence we have proved that $\equiv_{G^2}^n = (\perp_G^n)^2$. Now the thesis follows from the equality $G = G^2 \cup (G \backslash G^2)$.

Finally, for a given equivalence $\approx$ on X^n define $\mathrm{Izt}(\approx)$ to be the class of all bijections f of X satisfying

$$(\forall x_1, \ldots, x_n \in X)[\langle x_1, \ldots, x_n \rangle \approx \langle f(x_1), \ldots, f(x_n) \rangle].$$

2. Group of orthogonalizations. The Euclidean geometry we shall deal with is the theory of all Euclidean planes over 2-formally real and Euclidean fields. Every such plane is the structure

$$A = \langle U, \perp^A \rangle = \langle F^2; \perp_F \rangle,$$

where we define

$$\langle a_1 a_2 \rangle \langle b_1 b_2 \rangle \perp_F \langle c_1 c_2 \rangle \langle d_1 d_2 \rangle \Leftrightarrow (a_1 - b_1)(c_1 - d_1) + (a_2 - b_2)(c_2 - d_2) = 0.$$

It is known [2] that such a theory is definitionally equivalent to the standard formalization of Euclidean geometry in terms of an equidistance relation. Analytically, an equidistance $\equiv$ is defined by

$$\langle a_1 a_2 \rangle \langle b_1 b_2 \rangle \equiv \langle c_1 c_2 \rangle \langle d_1 d_2 \rangle \Leftrightarrow (a_1 - b_1)^2 + (a_2 - b_2)^2 = (c_1 - d_1)^2 + (c_2 - d_2)^2.$$

Now modify the orthogonality relation $\perp^A$ so as for $a, b, c, d \in U$

$$ab \mathbin{\underline{\perp}}^A cd \Leftrightarrow (ab \perp^A cd \wedge a \neq b \wedge c \neq d) \vee (a = b \wedge c = b).$$

It is commonly known that $\underline{\perp}^A$ is a half-equivalence, and $(\underline{\perp}^A)^2$ is the relation of parallelity $\|^A$ of nondegenerate segments. From Lemma 1 it follows that $\perp^A \cup \|^A$ is an equivalence as well. Write $H^A = \underline{\perp}^A \cup (\underline{\perp}^A)^2$ and consider the group $\mathrm{Izt}(H^A)$. It will be called the *group of orthogonalizations* in A. For short, we shall write $\mathrm{OSim}(A)$ for $\mathrm{Izt}(H^A)$, and the index A will be omitted in the sequel.

LEMMA 3. *The class of lines in A and the class of circles in A are invariant under* $\mathrm{OSim}(A)$.

Proof. Take two points $a, b \in U$, $a \neq b$. Then the set

$$S\binom{a}{b} = \{x\colon axHxb\} = \{x\colon ax \parallel xb \vee ax \perp xb\}$$

is the sum of the line through a and b and the circle with diameter ab with a and b excluded. Analogously,

$$O(a, b) = \{x\colon abHax\} = \{x\colon ab \parallel ax \vee ab \perp ax\}$$

is the sum of the line through a and b and the line orthogonal to it at a. Clearly, $\mathrm{OSim} \subseteq \mathrm{Aut}(H)$, so the classes $\left\{S\binom{a}{b}\colon a \neq b\right\}$ and $\{O(a, b)\colon a \neq b\}$ are invariant under OSim. But the line $L(a, b)$ through a and b is equal to

$$L(a, b) = \big(O(a, b) \cap O(b, a)\big) \cup \{a, b\} = \left(O(a, b) \cap S\binom{a}{b}\right) \cup \{a, b\};$$

analogously, the circle $C(a, b)$ with diameter ab is equal to

$$\left(S\binom{a}{b} \setminus O\binom{a}{b}\right) \cup \{a, b\}.$$

Hence the classes of lines and the classes of circles are invariant.

LEMMA 4. *OSim is 2-rigid, i.e., if* $f \in \mathrm{OSim}$, $a, b \in U$, $f(a) = a$ *and* $f(b) = b$, *then* $a = b$ *or* $f = \mathrm{id}$.

Proof. Let $a \neq b$, $f(a) = a$, $f(b) = b$, and $f \in \mathrm{OSim}$. First we show that if $c \in C(a, b)$ or $c \in O(a, b) \setminus L(a, b)$, then $f(c) = c$. Let $c \in O(a, b) \setminus L(a, b)$. Clearly,

$$f(c) \in O(a, b) \setminus L(a, b).$$

Assume $c \neq f(c)$. Thus $bc \parallel bf(c)$ is impossible, and so $bc \perp bf(c)$. Analogously, $f\big(f(c)\big) = c$.

Let $q \in L(c, b) \cap C(a, b)$, $q \neq b$. Then

$$f(q) \in L\big(b, f(c)\big) \cap C(a, b).$$

The equality $q = f(q)$ is impossible; otherwise $c = f(c)$. The same considerations give us $f\big(f(q)\big) = q$. Thus we should have $cf(q) \parallel qf(c)$ or $cf(q) \perp qf(c)$. But this is inconsistent. Now let $p \in L(a, b)$. Consider

$$C(a, p) \cap \big(O(a, b) \setminus L(a, b)\big) \quad \text{and} \quad C(b, p) \cap \big(O(b, a) \setminus L(a, b)\big).$$

One of these sets is nonempty – let q be a point in it. Then $f(q) = q$, and next $f(p) = p$. Finally, for arbitrary $x \in L(a, b)$ we look for $p \in L\binom{a}{b}$, $px \perp ab$. Then $f(p) = p$ and $f(x) = x$, which completes the proof.

We have shown in fact in Lemma 3 that OSim is a subgroup of similarities. Now we shall explain the structure of OSim better. Denote by $\mathrm{Dil}(A)$ the class of dilatations (i.e., translations and homotheties) in A; that is, $\mathrm{Dil}(A) = \mathrm{Izt}(\|^A)$. Clearly, since $\| \subseteq H$, we have $\mathrm{Dil}(A) \subseteq \mathrm{OSim}(A)$. Next, consider a rotation ϱ such that ϱ^2 is a central symmetry σ_p with center p. We see that $\varrho \in \mathrm{OSim}$. For arbitrary a we have simply

$$a\varrho(a) \equiv \varrho(a)\varrho^2(a) = \varrho(a)\sigma_p(a).$$

Therefore $ap \perp pf(a)$. Next ϱ is an isometry, and therefore it preserves $\|$. For arbitrary a and b take q with $ab \| pq$, $p \neq q$. Then

$$f(a)f(b) \| pf(q) \perp pq \quad \text{and} \quad ab \perp f(a)f(b).$$

Now we can prove the theorem which states that OSim consists exactly of dilatations and such rotations and their compositions.

THEOREM 1. $f \in \mathrm{OSim}$ *if and only if there exist a rotation* ϱ *and a dilatation* g *such that* ϱ^2 *is a central symmetry and* $f = g$ *or* $f = g\varrho$.

Proof. Take any points a, b, $a \neq b$. Let $f \in \mathrm{OSim}$ and put $c = f(a)$ and $d = f(b)$. We have $ab \| cd$ or $ab \perp cd$. Assume $ab \| cd$. Then there is g in Dil with $g(a) = c$ and $g(b) = d$. By Lemma 4, $f = g$. Now let $ab \perp cd$. Take any ϱ with ϱ^2 being a central symmetry and put $\varrho(a) = a'$, $\varrho(b) = b'$. Then $a'b' \perp ab$ and thus $a'b' \| cd$. Therefore, for some $g \in \mathrm{Dil}$, $g(a') = c$ and $g(b') = d$. Again by Lemma 4 we obtain $f = g$. The converse implication is trivial.

As a corollary we obtain now

COROLLARY 1. $\mathrm{OSim}(A)$ *is a subgroup of positive similarities in* A.

COROLLARY 2. *The only involutions in* OSim *are central symmetries.*

COROLLARY 3. *For every bijection* f *of the universe we have*

$$f \in \mathrm{OSim} \Leftrightarrow (\forall a, b)[ab \| f(a)f(b)] \vee (\forall a, b)[ab \perp f(a)f(b)].$$

COROLLARY 4. $\mathrm{OSim}^2 = \mathrm{Dil}$.

Proof. By Corollary 3, $\mathrm{OSim}^2 \subseteq \mathrm{Dil}$. Now let $f \in \mathrm{Dil}$. If f is a translation, then there is a translation u such that $f = u^2$. Therefore, assume $f(p) = p$ for some p. Consider ϱ such that $\varrho^2 = \sigma_p$. Take any q, $q \neq p$. By the conditions imposed on the field there is a homothety g with center p such that $g^2(q) = f(q)$ or $g^2(q) = \sigma_p(f(q))$. In the first case $f = g^2$, and in the second case $f = (g\varrho)^2$.

COROLLARY 5. OSim^2 *is a subgroup of* OSim *of index* 2.

It suffices to notice that $f \in \mathrm{OSim} \setminus \mathrm{OSim}^2$ iff it satisfies

$$(\forall a, b)[ab \perp f(a)f(b)].$$

COROLLARY 6. $\perp^A = \perp^2_{\mathrm{OSim}}$ *and* $H^A \equiv^2_{\mathrm{OSim}}$.

Remark. Sometimes the Euclidean geometry is presented as a theory of geometrical relations defined over complex fields [4]. In such models positive similarities are described as linear transformations of the underlying field. Then the group OSim consists of all functions f such that for some a, b we have $f(x) = ax+b$ for every x and $\bar{a} = a$ or $\bar{a} = -a$, $a \neq 0$.

3. Axiomatic description of orthogonalizations. Consider the following axiom system:

OS1: $(\forall x, y, z)[x\cdot(y\cdot z) = (x\cdot y)\cdot z]$.

OS2: $(\forall x)(\exists y)[x\cdot y = y\cdot x = 1]$.

OS3: $(\forall x)[x\cdot 1 = 1\cdot x = x]$.

OS4: $(\forall x, y, z)[x\cdot y = y\cdot x \wedge x\cdot z = z\cdot x \rightarrow x = 1 \vee yz = zy]$.

OS5: $(\forall x, y)(\exists z)[z^2 = x^2 y^2]$.

OS6: $(\exists x, y)[x^2 = y^4 = 1 \wedge xy^2 \neq y^2 x]$.

OS7: $(\forall x, y, z)[x^2 = y^2 = z^2 = 1 \rightarrow x = 1 \vee y = 1 \vee z = 1 \vee (xyz)^2 = 1]$.

OS8: $(\forall x, y)[x^2 = y^2 = 1 \wedge x, y \neq 1 \rightarrow (\exists z)[z^2 = 1 \wedge xz = zy]]$.

OS9: $(\forall x, y, z)[x^2 = y^2 = z^2 = 1 \wedge x, y, z \neq 1 \rightarrow ((\exists u)[u^2 x = xu^2 \wedge u^2 y = zu^2 \wedge u^2 \neq 1] \rightarrow (\exists v)[v^2 z = zv^2 \wedge v^2 y = xv^2])]$.

OS10: $(\forall x)[x^8 = 1 \rightarrow x^4 = 1]$.

Let G be any model of OS1–OS9. Then G is a group. Denote by $\Pi(G)$ the class of involutions in G. For every $g \in G$ let λ_g be the inner automorphism correlated with g, restricted to $\Pi(G)$. So λ_g is a function defined by $\lambda_g(a) = gag^{-1}$ for all involutions a of G. Now fix some model G of OS1–OS9.

LEMMA 5. (i) $(\exists a, b \in \Pi(G))[a \neq b]$.

(ii) $(\forall a \in \Pi(G))(\exists \alpha)[\alpha^2 = a]$.

(iii) $(\forall a, b \in \Pi(G))[ab = ba \rightarrow a = b]$.

(iv) $(\forall a, b \in \Pi(G))(\forall \alpha)[\lambda_\alpha(a) = a \wedge \lambda_\alpha(b) = b \rightarrow a = b \vee \alpha = 1]$.

(v) $(\forall \alpha)[\lambda_\alpha = \mathrm{id} \rightarrow \alpha = 1]$.

Proof. (i) follows from OS6. Just take a and b with $a^2 = b^4 = 1$, $ab^2 \neq b^2 a$. Then $a \neq b^2$; $a, b^2 \neq 1$. So $a, b^2 \in \Pi(G)$.

(ii) By OS6 there exists q with $p = q^2 \in \Pi(G)$. Let $a \in \Pi(G)$. Take α from OS8 such that $\alpha p \alpha^{-1} = a$ and put $\beta = \alpha q \alpha^{-1}$. Then $\beta^2 = a$.

(iii) Assume that $a, b \in \Pi(G)$, $ab = ba$; let moreover $a \neq b$. Clearly, $aa = aa$ and, by (ii), $a = \alpha^2 \neq 1$ for some α. Then there exists β from OS9 such that $\beta b = b$ and $\beta a = b\beta$. Thus $a = b$.

(iv) Assume $\alpha \in G$, $a, b \in \Pi(G)$, $\alpha a = a\alpha$, $\alpha b = b\alpha$. Then $\alpha = 1$ or $ab = ba$ by OS4, which implies $\alpha = 1$ or $a = b$.

(v) is a consequence of (i) and (iv).

Now we shall look at squares in G. We know that $\Pi(G) \subseteq G^2$. Moreover, by (v), $\lambda(G) \cong G$.

THEOREM 2. (i) G^2 *is a subgroup of* G.

(ii) $\langle \Pi(G), \equiv^2_{\lambda(G^2)} \rangle$ *is the affine space of dimension* $\geqslant 2$ *such that its dilatation group is equal to* $\lambda(G^2)$.

Proof. (i) is trivial in view of OS5.

(ii) We should only check appropriate axioms of the space or – even easier – we should check that G^2 satisfies axioms of dilatation groups presented in [1]. But this is trivial since G and G^2 have the same involutions, and the rest of axioms consist of universal sentences. The only problem is with the dimension axiom. Take any $p, a \in \Pi(G) = \Pi(G^2)$. Find α with $\alpha^2 = a$ and put $q = \lambda_\alpha(p)$. We claim that $\sim ap \equiv_{G^2} aq$. Otherwise, for some β, $\lambda_{\beta^2}(a) = a$ and $\lambda_{\beta^2}(p) = q$. By Lemma 5 (iv), $\beta^2 = \alpha$. But then $\beta^4 = a$ and $\beta^8 = 1$. Thus, by OS10, $\beta^4 = 1$, which is impossible.

To obtain an axiom system of groups of orthogonalizations we should guarantee now that $(G:G^2) = 2$ and the orthogonality we have obtained is the Euclidean one. Assume first that the following holds:

OS11: $(\forall x, y)[x^4 = 1 \wedge x^2 \neq 1 \rightarrow (\exists z)[y = z^2 \vee y = z^2 x]]$.

LEMMA 6. *If* $G \models$ OS1–OS11, *then* $(G: G^2) = 2$.

Proof. We must show that $f, g \notin G^2 \rightarrow fg \in G^2$ since the rest is trivial by the axiom OS5. Take $f \notin G^2$. Next consider ϱ such that $\varrho^2 \in \Pi(G)$. There exists h_1 such that $f = h_1^2 \varrho$. Analogously, there exists h_2 with $g = \varrho h_2^2$ (from OS11 take z with $g = z^2 \varrho$ and put $h_2 = \varrho^{-1} z \varrho$). Now we have $fg = h_1^2 \varrho^2 h_2^2 \in G^2$.

Now it makes sense to investigate $\perp^2_{\lambda(G)}$ as a half-equivalence. We recall that $\perp^2_{\lambda(G)}$ is equal to $\equiv^2_{\lambda(G \setminus G^2)}$. Denote by $E(G)$ the structure $\langle \Pi(G); \perp^2_G \rangle$. Let $\|_G = (\perp^2_G)^2$. We know that for every g in G either g is in $\mathrm{Dil}(E(G))$ or, for ϱ such that $\varrho^2 \in \Pi(G)$, ϱg is in $\mathrm{Dil}(E(G))$. Write

$$P(G) = \{f \in G: f^4 = 1 \wedge f^2 \neq 1\}, \qquad V(G) = \Pi(G) \cdot \Pi(G).$$

Clearly, $V(G)$ is a commutative subgroup of G consisting of translations. G itself is generated by $G^2 \cup P(G)$. Now consider the group $O(G)$ generated by $P(G)$. We should obtain $O(G) = P(G) \cup V(G) \cup \Pi(G)$.

OS12: $(\forall x, y)[x^4 = 1 \wedge y^2 = 1 \rightarrow (xy)^4 = 1 \vee x^2 = 1]$.

LEMMA 7. *If* $\alpha, \beta \in P(G)$, *then* $\alpha\beta \in \Pi(G)$ *or* $\alpha\beta \in V(G)$.

Proof. Assume first $\alpha^2 = \beta^2 = p \in \Pi(G)$. Now we prove that $\alpha = \beta \vee \alpha = \beta^{-1}$. We have $\alpha p = \alpha^3 = p\alpha$ and $\beta p = p\beta$. Thus $\alpha\beta = \beta\alpha$. So $(\alpha\beta)^2 = \alpha^2\beta^2 = 1$ and $\alpha\beta p = p\alpha\beta$. Therefore, $\alpha\beta = 1$ and $\alpha = \beta^{-1}$ or $\alpha\beta = p$ and $\alpha\beta = \beta^2$, $\alpha = \beta$. Now assume $\alpha^2 = p$, $\beta^2 = q$, $p, q \in \Pi(G)$. Consider r such that $pr = rq$ and $\gamma = r\alpha r$. Then $\gamma^2 = q$, so $\gamma = \beta$ or $\gamma = \beta^{-1}$. Let $\gamma = \beta$. We should show that $(\alpha\beta)^2 = 1$, that is, $\alpha\beta = \beta^{-1}\alpha^{-1}$. We have $\alpha\beta = \alpha\gamma = \alpha r \alpha r$ and $\beta^{-1}\alpha^{-1}$

$= r\alpha^{-1} r\alpha^{-1}$. By OS12, $(\alpha r)^4 = 1$. Thus we obtain the assertion. If $\gamma = \beta^{-1}$, we prove that $\alpha\beta q \in \Pi(G)$. But $\beta q = \beta^{-1}$ since $\beta q \beta = \beta^4 = 1$ and the thesis follows from the first part.

COROLLARY 7. (i) $P(G)\cdot\Pi(G) \cup \Pi(G)\cdot P(G) \subseteq P(G)$.

(ii) $P(G)\cdot V(G) \cup V(G)\cdot P(G) \subseteq P(G)$.

Now define in $\Pi(G)$ the relation B_G by the formula

$$B_G(a, b, c) :\Leftrightarrow (\exists g)[g^2 = b \wedge ga = cg] \wedge a, b, c \in \Pi(G).$$

We see that it is defined quite similarly as the midpoint relation (coinciding with the usual affine midpoint relation) $b = a \oplus c \Leftrightarrow ba = cb$. The relation B may be considered as a relation of orthonormal (equi-orthogonal) base of Szmielew [5]. Now we shall prove it satisfies all the properties imposed by Szmielew on the relation of orthonormal base.

THEOREM 3. *If G satisfies* OS1–OS10 *and* OS12, *then B is the relation of orthonormal base in* $\langle \Pi(G), \|_G, B_G \rangle$.

Proof. The proof consists in analytical checking of appropriate axioms.

(i) $(\forall a, p)(\exists q) B(paq)$.

Take f with $f^2 = a$ and put $q = fpf^{-1}$.

(ii) $B(paq) \to B(qap)$.

If $f^2 = a$ and $fp = qf$, then $(f^{-1})^2 = g$ satisfies $g^2 = a$, $gq = pg$.

(iii) $B(paq) \wedge B(paq') \to q = q' \vee q \oplus q' = a$.

Assume $f^2 = g^2 = a$ and $fp = qf$, $gp = q'g$. Then $f = g$ or $f = g^{-1}$. In the first case, $q = q'$. In the second one, we calculate $f^2 q (f^2)^{-1} = f^3 (f^{-1} qf) f^{-3} = gpg^{-1} = q'$, so $a = f^2 = q \oplus q'$.

(iv) $B(paq) \wedge q \oplus q' = a \to B(paq')$.

Assume $f^2 = a$, $fp = qf$, and $aq = q'a$. Then

$$f^3 p = f^2 fp = aqf = q' af = qf^3 \quad \text{and} \quad (f^3)^2 = a.$$

(v) $B(paq) \wedge B(ras) \wedge a = q \oplus q' \wedge p \oplus r = a \oplus b \wedge q \oplus s = a \oplus c \wedge q' \oplus s = c' \oplus a \to B(cab) \vee B(c'ab)$.

First we prove that the following holds:

$$a \oplus b = c \oplus d \Leftrightarrow ac = db.$$

Take p such that $ap = pb$ and $cp = pd$. Then

$$ac = pbppdp = pbdp = dbpp = db.$$

Next assume $ap = pb$ and put $d' = pcp$. Consequently, $ac = d'b$. Thus if $ac = db$, then $d = d'$. Now we can prove (v). Assume $f^2 = a$. From the assumption $fp = qf$ we have $f^{-1} p = q' f^{-1}$. Moreover, $fr = sf$ or $f^{-1} r = sf^{-1}$. Assume $fr = sf$. Next, using the fact proved above, we get $b = par$, $c' = saq'$, $c = saq$. We have $fbf^{-1} = fparf^{-1} = gas = c$. Analogously we calculate when $f^{-1} r = sf^{-1}$.

(vi) $B(paq) \wedge B(pa'q) \to a = a' \vee q \oplus p = a \oplus a'$.

Assume $f^2 = a$, $g^2 = a'$, $fp = qf$, $gp = qg$. Then take $h = g^{-1}f$. We have $hp = ph$. By Lemma 7, $h \in V(G)$ or $h \in \Pi(G)$. But $h \in V(G)$ implies $h = 1$ and $a = a'$. If $h \in \Pi(G)$, then $h = p$. We should prove that $pa = a'q$. This is equivalent to $pf^2 = g^2 q = g^2 fpf^{-1}$. We have

$$pf^3 = g^2 fp \Leftrightarrow g^{-1} ff^3 = g^2 f^{-1} g = g^3.$$

(vii) $a \neq p \wedge B(paq) \wedge ap \| ap' \wedge aq \| aq' \wedge pq \| p'q' \to B(p'aq')$.

Take f with $f^2 = a$, $fp = qf$. By the homogeneity of the underlying affine space there exists g such that $g \in G^2$, $ga = ag$, $gp = p'g$, $gq = q'g$. By OS4, $fg = gf$. Therefore,

$$fp'f^{-1} = fgpg^{-1}f^{-1} = gfpf^{-1}g^{-1} = gqg^{-1} = q',$$

so $B(p'aq')$ holds.

(viii) $a \neq p \wedge B(paq) \wedge pa \| pa' \wedge pq \| pq' \wedge aq \| a'q' \to B(pa'q')$.

Let $f^2 = a$ and $fp = qf$. Analogously, there exists h such that $hp = ph$ and $ha = a'h$, $hq = q'h$, $h \in G^2$. Take $g = hfh^{-1}$. Then $g^2 = a'$. Moreover,

$$gp = hfh^{-1}p = hfph^{-1} = hqfh^{-1} = q'g.$$

This means $B(pa'q')$ holds.

Finally, assume the last axiom holds:

OS13: $(\forall g, p, \bar{p}, q)[p, \bar{p}, q \in \Pi(G), g \in P(G), \neq (g^2, p, \bar{p}, q)$

$$\to (\exists f, h)[fg^2 = g^2 f \wedge h^2 fpf^{-1} h^{-2} = f\bar{p}f^{-1} \wedge h^2 q = qh^2]].$$

LEMMA 8. *If* $G \models$ OS1–OS11, OS13, *then* $\langle \Pi(G), \|_G \rangle$ *is a two-dimensional affine plane.*

Proof. Fix $p, a \in \Pi(G)$, $p \neq a$. Take g with $g^2 = a$ and $\bar{p} = gpg^{-1}$. Let $q \in \Pi(G)$, $q \neq a, p, \bar{p}$. We are going to prove that $a, p, \bar{p}, q$ are coplanar. Take f and h as in OS13. Then $h^2 \in G^2 = \mathrm{Dil}(E(G))$. Write $p_1 = fpf^{-1}$, $\bar{p}_1 = f\bar{p}f^{-1}$. Assume first that $f \in G^2$. Then $p_1, \bar{p}_1, a, p, \bar{p}$ are coplanar. Next $h^2 qh^{-2} = q$ and $h^2 p_1 h^{-2} = \bar{p}_1$, so $p_1, \bar{p}_1, q$ are collinear. Thus $a, p, \bar{p}, q$ are coplanar. Next assume $f \notin G^2$. By OS11 there exists f_1 such that $f = f_1^2 g$. Let $p' = g\bar{p}g^{-1}$; then $p' \oplus p = a$, so $a, p, \bar{p}, p'$ are coplanar. Define $\bar{p}_2$ and p'_2 to be $f_1^2 \bar{p} f_1^{-2}$ and $f_1^2 p' f_1^{-2}$, respectively. An analogous consideration leads us to the conclusion that $a, \bar{p}, p', q$ are coplanar, which completes the proof.

Now we can correlate three notions: B, $\|$ and $\perp$. Namely, we have

LEMMA 9. $a, b, c, d \in \Pi(G) \to (ab \perp cd \Leftrightarrow (\exists p, q, r)\ [B(p, q, r) \wedge pq \| ab \wedge cd \| qr])$.

Proof. Let $a, b, c, d \in \Pi(G)$. The implication $\leftarrow$ is trivial since just by the definition $B(p, q, r)$ implies $pq \perp qr$; next we use properties of a half-equivalence. Assume $ab \perp cd$, $a \neq b$ (then also $c \neq d$). Take q such that $B(abq)$. So $ab \perp bq$. Thus $cd \| bq$, which completes the proof.

All the lemmas lead us to the main representation theorem:

THEOREM 4 [Representation Theorem]. *For every structure G, G is a model of* OS1–OS13 *if and only if, for some Euclidean plane $A = \langle U, \perp\rangle$ over a Euclidean 2-formally real field, G is isomorphic to* $\mathrm{OSim}(A) = \mathrm{Izt}(H^A)$.

Proof. The first part consists in analytical checking of our axioms in groups $\mathrm{OSim}(A)$. This will be omitted since it is easy but rather involved. Some of them were proved in Section 2. Second, we take any model G of OS1–OS13. We know that $\perp_G^2$ is a half-equivalence and $\|_G = (\perp_G^2)^2$ gives us the affine plane. Next, by Theorem 3, the structure $A' = \langle \Pi(G); \|_G, B_G\rangle$ is the Euclidean plane in the sense of Szmielew [5] – the affine plane with orthonormal base. As proved in [5], orthogonality in such a plane is defined by

$$xy \perp zt \Leftrightarrow (\exists\, u, v, w)\,[xy \,\|\, uv \wedge zt \,\|\, vw \wedge B(uvw)].$$

Thus $\perp^{A'} = \perp_G$ by Lemma 9. So we have proved that $E(G)$ is a Euclidean plane. Obviously, $\lambda(G) \subseteq \mathrm{OSim}(E(G))$; moreover, $\lambda_g = \mathrm{id} \Leftrightarrow g = 1$, so λ is an injection.

Now we should prove that $E(G)$ is a plane over a Euclidean 2-formally real field. Consider any dilatation φ in $E(G)$, let $\varphi(p) = p$ for some point p. Then there is f in G such that $\varphi = \lambda_{f^2}$ by Theorem 2 (ii). First assume $f \in G^2$; then $\psi = \lambda_f$ is a dilatation and $\psi^2 = \varphi$. Otherwise, there are g and ϱ such that $f = g^2\varrho$ and $\varrho^2 = p$; so $gp = pg$. Then for $\psi = \lambda_{g^2}$ we have $\psi^2 = \sigma_p\varphi^2$. Therefore, every homothety φ with center p is a square or its product with central symmetry is a square. This means that in the underlying field every element is a square or its opposite is a square, so the field is Euclidean. Next, there is no isotropic line in $E(G)$, since if $ab \perp_G ab$, then $fa = af$ and $fb = bf$ for some $f \notin G^2$. But this is impossible if $a \neq b$; the condition above is satisfied by $f = 1$ and $1 \in G^2$ (the group is 2-rigid). Therefore, the field is 2-formally real. Consequently, $\mathrm{Izt}(H^{E(G)})$ are 2-rigid as well. Take $\varphi \in \mathrm{OSim}(E(G))$ and $a, b \in \Pi(G)$, $a \neq b$. Let $c = \varphi(a)$ and $d = \varphi(b)$. Then $abHcd$; so $ab \,\|_G\, cd$ or $ab \perp_G cd$. Take f in G such that $fa = cf$ and $fb = df$. We have $\lambda_f(a) = \varphi(a)$ and $\lambda_f(b) = \varphi(b)$, $a \neq b$, $\lambda_f, \varphi \in \mathrm{OSim}(E(G))$. So $\varphi = \lambda_f$. Thus λ is the desired isomorphism of G and $\mathrm{OSim}(E(G))$.

THEOREM 5. *A is a Euclidean plane over a Euclidean 2-formally real field if and only if for some group G satisfying* OS1–OS13 *we have $A \cong E(G)$.*

Proof. First we notice that if A is such a plane, then $A \cong E(\mathrm{OSim}(A))$. To prove this it suffices to see that involutions in $\mathrm{OSim}(A)$ are exactly central symmetries. For every point p let σ_p denote the central symmetry with center p. Then for $\varphi \in \mathrm{OSim}(A)$ we have $\varphi\sigma_p\varphi^{-1} = \sigma_{\varphi(p)}$, i.e. $\lambda_\varphi(\sigma_p) = \sigma_{\varphi(p)}$. Thus and by Section 2, σ is an isomorphism of A and $E(\mathrm{OSim}(A))$. Now from Theorem 4 we obtain our assertion.

This means we can reconstruct the plane geometry of Euclidean orthogonality from the groups of orthogonalizations just like it can be done for the equidistance relation with the help of positive isometries or for parallelity with dilatation groups. It is worthwhile to notice that the system of notions consisting of parallelity and orthogonality is very natural for Euclidean geometry. A very elegant axiom system for such structures (of dimension 3 and greater) was presented by Kusak [3]; from this paper it follows that such an elegant system should exist also for 2-dimensional geometry.

Appendix. There is another more analytical approach to groups of orthogonalizations. Assume we have chosen in an affine space a coordinate system with e_1 and e_2 as unit vectors and origin p. We can assume $e_1 \perp e_2$. To extend this orthogonality to the whole plane we should find an "orthogonalization" φ such that, for every vector v, $\varphi(v) \perp v$. Clearly, we should obtain $\varphi(\varphi(v)) \| v$. Put $\varphi(e_1) = e_2$. Now we have two cases: $\varphi(e_2) = -e_1$ or $\varphi(e_2) = e_1$. The first leads us to the groups we have just considered and to the Euclidean orthogonality. The second one gives us the orthogonality of the Minkowski plane. Thus the same trick may be done here; the relation $\perp \cup \|$ is an equivalence and the class OSim may be introduced. From the Euclidean point of view the group we obtain may be described as follows. Let (k) and (m) be two families of parallel lines, let $k \perp m$. Denote by N the group generated by $\Sigma = \{\sigma_x\colon x \in (k) \cup (m)\}$. It consists of all translations, central symmetries and symmetries with axes from Σ. Then denote by NSim the group generated by all the dilatations and the group N (i.e., by $\mathrm{Dil} \cup \Sigma$). Then NSim is the group of orthogonalizations of the Minkowski plane, where Σ corresponds to the class of isotropic lines in this plane. In a very similar, though more involving way we can reconstruct the geometry from it.

First we see that involutions in NSim are elements of Σ and central symmetries. They are intrinsically distinguishable – a is a central symmetry ($a \in \Pi$) iff for any involutions b and c if $a \neq b, c$ and $ab = ba$, $ac = ca$, then $a = bc$ or $b = c$. The group satisfies the restricted 2-rigidness: $a, b \in \Pi$, $f \in \mathrm{NSim}$, $fa = af$, $fb = bf \to f = \mathrm{id} \vee (\exists n \in \Sigma)[an = na \wedge bn = nb]$.

Next NSim^2 consists of all positive dilatations. We obtain Dil as a group generated by $\Pi \cup \mathrm{NSim}^2$ ($\mathrm{Dil} = \mathrm{NSim}^2 \cup \Pi \cdot \mathrm{NSim}^2$). This allows us to reconstruct parallelity in NSim to be $\equiv^2_{\mathrm{Dil}}$.

Finally, we get the Minkowski orthogonality as $\equiv^2_{\mathrm{NSim} \setminus \mathrm{Dil}}$. We will not present any axiomatics for groups NSim here; from the above considerations it follows that the class of orthogonalizations of Minkowski planes is also axiomatizable.

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