On a class of starlike functions

by Sanford S. Miller* (Brockport, N. Y.)

Abstract. Let \( f(z) \) be a regular function defined in the unit disc for which

\[
\text{Re}(zf''(z)f(z)) = \mu (zf''(z)f(z) + 1)^* > 0,
\]

for \( \mu \) and \( \nu \) real. The author shows that for certain values of \( \mu \) and \( \nu \) the function is univalent and starlike.

We wish to define some new classes of regular functions which will prove to be starlike.

Definition. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be regular in the unit disc \( D \), with \( f(z)/z, f'(z), z f''(z)/f'(z) + 1 \neq 0 \) for \( z \in D \). If \( \mu \) and \( \nu \) are fixed real numbers and

\[
\text{Re} \left( \frac{zf''(z)}{f(z)} + 1 \right)^* > 0
\]

for \( z \in D \), where the powers appearing in (1) are meant as principal values, then we say that \( f(z) \) belongs to the class \( S(\mu, \nu) \).

This class of functions contains many classes of univalent functions. In fact, \( S(1, 0) = S^* \), \( S(0, 1) = C \), the class of convex functions, \( S(\mu, 0) \) with \( |\mu| \geq 1 \) corresponds to strongly starlike functions \( [1, 3] \), \( S(0, \nu) \) with \( |\nu| \geq 1 \) corresponds to strongly convex functions, and \( S(1 - \gamma, \gamma) \) with \( \gamma \) real corresponds to gamma-starlike functions \( [2] \). We will show that for many more values of \( \mu \) and \( \nu \) condition (1) implies univalence and starlikeness.

Note that condition (1) is equivalent to the following condition:

(2) \[
|\mu \arg \left( \frac{zf''(z)}{f(z)} \right) + \nu \arg \left( \frac{zf''(z)}{f'(z)} + 1 \right)| < \frac{\pi}{2}.
\]

* This work was carried out while the author was an I.R.E.X. Scholar in Poland.
In what follows we will have reference to the following region $K$ of the $(\mu, \nu)$-plane:

$$K = \{(\mu, \nu) | 4n + 1 \leq \mu + \nu \leq 4n + 3, n \in I \} \cup \{ (\mu, 0) | |\mu| \geq 1 \} \cup \{ (0, \nu) | |\nu| \geq 1 \},$$

where $I$ is the set of integers. Region $K$ is pictured below.

![Region K](image)

Fig. 1

We now show that if $(\mu, \nu) \in K$ and $f(z) \in S(\mu, \nu)$, then $f(z)$ is univalent and starlike.

**Theorem 1.** $S(\mu, \nu) \subset S^*$ if $(\mu, \nu) \in K$.

**Proof.** The cases $|\mu| \geq 1$, $\nu = 0$ and $|\nu| \geq 0$, $\mu = 0$ are trivial and we only need to prove the result for the region

$$4n + 1 \leq \mu + \nu \leq 4n + 3,$$

where $n$ is an integer. The technique we use is similar to that used in [2].

If $f(z) \in S(\mu, \nu)$ and if we set

$$\frac{1+w}{1-w} = \frac{zf'(z)}{f(z)} \tag{4}$$

for $z \in D$, then $w(0) = 0$, $w(z) \neq \pm 1$ and $w(z)$ is defined as a meromorphic function. To complete the proof of the theorem we need to show that
\[ |w(z)| < 1 \text{ for } z \in D. \text{ Let } \]
\[ w(z) = R(z)e^{i\varphi(z)} \quad \text{for } z = re^{i\theta}, \]
and suppose that \( z_0 = r_0e^{i\theta_0} \) is a point of \( D \) such that
\[ \max_{|z| < r_0} |w(z)| = |w(z_0)| = 1. \]

Then \( \frac{\partial R(z_0)}{\partial \theta} = 0 \), and since
\[ \frac{zw'(z)}{w(z)} = \frac{\partial \Phi}{\partial \theta} - i \frac{1}{R} \frac{\partial R}{\partial \theta}, \]
we must have
\[ \frac{z_0w'(z_0)}{w(z_0)} = \frac{\partial \Phi(z_0)}{\partial \theta}, \]
and hence \( z_0w'(z_0)/w(z_0) \) is a real number. A simple geometric argument can show even more. If we assume \( \frac{\partial \Phi(z_0)}{\partial \theta} < 0 \), then \( w(z) \) would be locally univalent at \( z_0 \) and this would lead to a contradiction of (5).

Thus we see that \( \frac{\partial \Phi(z_0)}{\partial \theta} \) must be non-negative and so we set
\[ \frac{z_0w'(z_0)}{w(z_0)} = B, \]
where \( B \geq 0 \).

Since \( |w(z_0)| = 1 \) and \( w(z_0) \neq \pm 1 \), we have
\[ \frac{1 + w(z_0)}{1 - w(z_0)} = A i, \]
where \( A \) is real and \( A \neq 0 \).

From (1) and (4) we obtain
\[ \text{Re} I(\mu, v, f(z)) = \text{Re} \left( \frac{zf'(z)}{f(z)} \right)^{\mu} \left( \frac{zf''(z)}{f'(z)} + 1 \right)^{\nu} \]
\[ = \text{Re} \left( \frac{1 + w}{1 - w} \right)^{\mu} \left( \frac{1 + w}{1 - w} + \frac{zw'}{w} \left[ \frac{w}{1 + w} + \frac{w}{1 - w} \right] \right)^{\nu}. \]

Thus at \( z = z_0 \), by using (6) and (7) we obtain
\[ \text{Re} I(\mu, v, f(z_0)) = \text{Re} (Ai)^{\mu} \left( Ai + \frac{B}{2} \left[ A + \frac{1}{A} \right] i \right)^{\nu}. \]

If we let \( C = A + B[A + 1/A]/2 \), then since \( B \geq 0 \) and \( A \neq 0 \), we have \( AC > 0 \) and we obtain
\[ \text{Re} I(\mu, v, f(z_0)) = |A|^{\mu}|C|^{\nu} \cos(\mu + v)\pi/2. \]
Since \( \mu + v \) is restricted by condition (3) we have \( \text{Re} I(\mu, v, f(z_0)) \leq 0 \), and since this contradicts \( f(z) \in S(\mu, v) \) we must have \( |w(z)| < 1 \), and thus \( f(z) \in S^* \).

In Theorem 1 we proved the inclusion relationship \( S(\mu, v) \subset S(1, 0) \). We can generalize this result as we do in the following theorem.

**Theorem 2.** If \( 0 \leq t \leq 1 \) and \( (\mu, v) \in K \), then \( S(\mu, v) \subset S((\mu - 1)t + 1, vt) \).

**Proof.** If \( f(z) \in S(\mu, v) \), then

\[
\left( \frac{zf'(z)}{f(z)} \right)^\mu \left( \frac{zf''(z)}{f'(z)} + 1 \right)^v = P_1(z),
\]

where \( P_1(0) = 1 \) and \( \text{Re} P_1(z) > 0 \). By Theorem 1

\[
\frac{zf'(z)}{f(z)} = P_2(z),
\]

where \( P_2(0) = 1 \) and \( \text{Re} P_2(z) > 0 \). If we raise both sides of (8) to the \( t \) power and both sides of (9) to the \((1-t)\) power and multiply these two equations we obtain

\[
\left( \frac{zf'(z)}{f(z)} \right)^{(\mu-1)t+1} \left( \frac{zf''(z)}{f'(z)} + 1 \right)^{(1-t)t} = [P_1(z)]^t[P_2(z)]^{1-t} = P_3(z).
\]

Now \( P_3(0) = 1 \) and since \( 0 \leq t \leq 1 \),

\[
|\arg P_3(z)| = t|\arg P_1(z)| + (1-t)|\arg P_2(z)| \leq \pi/2.
\]

Hence \( \text{Re} P_3(z) > 0 \) and \( f(z) \in S((\mu - 1)t + 1, vt) \).

**Remarks.** (i) Theorem 2 has the following geometric interpretation in terms of Fig. 1. If \( (\mu, v) \) is a point in the shaded region \( K \), then all points on the line \( L \) between \( (\mu, v) \) and \( (1, 0) \) will satisfy (1). Note that there may be points on \( L \) that are not in \( K \) and for such points, \( S((\mu - 1)t + 1, vt) \) need not be a subclass of \( S^* \). However, if we restrict \( (\mu, v) \) to the band \( 1 \leq \mu + v \leq 3 \), then \( L \subset K \) and \( S((\mu - 1)t + 1, vt) \subset S^* \).

(ii) If \( 0 \leq \delta \leq v \) (or \( v \leq \delta \leq 0 \)) and \( (1, v) \in K \), then we have \( S(1, v) \subset S(1, \delta) \).

(iii) If \( 0 \leq \delta \leq v \) (or \( v \leq \delta \leq 0 \)), then \( S(1 - v, v) \subset S(1 - \delta, \delta) \subset S^* \) for any real \( v \).

By comparing the coefficients of \( z \) and \( z^2 \) in equation (8) we obtain the following theorem:

**Theorem 3.** If \( f \in S(\mu, v) \), then

(i) \[ |a_2(\mu + 2v)| < 2, \]

and

(ii) \[ |a_2(4\mu + 12v) - a_2^2(3\mu - \mu^2 - 4\mu v + 12v - 4v^2)| < 4. \]
The author has not been able to prove that (i) and (ii) are sharp. They would be sharp if it is possible to show that the differential equation

\[
\left( \frac{zf'(z)}{f(z)} \right)' \left( \frac{zf''(z)}{f'(z)} + 1 \right) = \frac{1+z}{1-z}
\]

with initial conditions \( f(0) = 0, f'(0) = 1 \) has a solution that is regular in \( D \). This formidable task we leave to future work.

References